R.K. Jain □ S.R.K. Iyengar

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R.K. Jain
Professor
S.R.K. Iyengar
Professor and Former Head
Department of Mathematics
Indian Institute of Technology
New Delhi 110 016, India

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To Our Parents

Bhagat Ram Jain and Sampati Devi Jain

&

S.T.V. Raghavacharya and Rajya Lakshmi

whose memories had always been an inspiration

Preface

This book is based on the experience and the lecture notes of the authors while teaching mathematics courses to engineering students at the Indian Institute of Technology, Delhi for more than three decades. A number of available textbooks have been a source of inspiration for introduction of concepts and formulation of problems. We are thankful to the authors of these books for their indirect help.

This comprehensive textbook covers syllabus for two courses in Mathematics for engineering students in various Institutes, Universities and Engineering Colleges. The emphasis is on the presentation of the fundamentals and theoretical concepts in an intelligible and easy to understand manner.

Each chapter in the book has been carefully planned to make it an effective tool to arouse interest in the study and application of mathematics to solve engineering and scientific problems. Simple and illustrative examples are used to explain each theoretical concept. Graded sets of examples and exercises are given in each chapter, which will help the students to understand every important concept. The book contains 682 solved examples and 2984 problems in the exercises. Answers to every problem and hints for difficult problems are given at the end of each chapter which will motivate the students for self-learning. While some problems emphasize the theoretical concepts, others provide enough practice and generate confidence to use these concepts in problem solving. This textbook offers a logical and lucid presentation of both the theory and problem solving techniques so that the student is not lost in unnecessary details.

We hope that this textbook will meet the requirements and the expectations of all the engineering students.

We will gratefully receive and acknowledge every comment, suggestions for inclusion/exclusion of topics and errors in the book, both from the faculty and the students.

We are grateful to our former teachers, colleagues and well wishers for their encouragement and valuable suggestions. We are also thankful to our students for their feed back. We are grateful to the authorities of IIT Delhi for providing us their support.

We extend our thanks to the editorial and the production staff of M/s Narosa Publishing House, in particular Mr. Mohinder Singh Sejwal, for their care and enthusiasm in the preparation of this book.

Last, but not the least, we owe a lot to our family members, in particular, our wives Vinod Jain and Seetha Lakshmi whose encouragement and support had always been inspiring and rejuvenating. We appreciate their patience during our long hours of work day and night.

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R.K. Jain
S.R.K. Iyengar

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Chapter 1

Functions of a Real Variable

1.1 Introduction

A real valued function $y = f(x)$ of a real variable x is a mapping whose domain S is a set of real numbers (in most of the applications S is an open interval (a, b) or a closed interval $[a, b]$) and whose codomain is $\mathbb{R}$, the set of real numbers. The range of the function is the set $\{y = f(x): x \in \mathbb{R}\}$, which is a subset of $\mathbb{R}$.

A real valued function f is said to be *bounded above* if $| f(x) | \le M$ and *bounded below* if $| f(x) | \ge m$ for all $x \in S_0$, $S_0 \subseteq S$.

The real positive finite numbers M and m are respectively called the upper bound and the lower bound of the function.

1.2 Limits, Continuity and Differentiability

Let f be a real valued function defined over $S \subseteq \mathbb{R}$. We define the distance function as

$$d(x_1, x_2) = | x_2 - x_1 |,\ x_1, x_2, \in \mathbb{R}. \tag{1.1}$$

Let a be any real number. Then, the open interval $N_\delta(a) = (a - \delta, a + \delta)$, $\delta > 0$ is called a δ-neighborhood of the point a. The interval $0 < | x - a | < \delta$ is called a deleted neighborhood of a.

1.2.1 Limit of a Function

The function f is said to tend to the limit l as $x \to a$, if for a given positive real number $\varepsilon > 0$, we can find a real number $\delta > 0$, such that

$$| f(x) - l | < \varepsilon, \quad \text{whenever} \quad 0 < | x - a | < \delta. \tag{1.2}$$

Symbolically, we write

$$\lim_{x \to a} f(x) = l. \tag{1.3}$$

Let f and g be two functions defined over S and let a be any point, not necessarily in S. Let

$$\lim_{x \to a} f(x) = l_1 \quad \text{and} \quad \lim_{x \to a} g(x) = l_2$$

exist. Then, we have the following properties:

(i) $\lim_{x \to a} [c f(x)] = c \lim_{x \to a} f(x) = c\, l_1$, c a real constant.

(ii) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = l_1 \pm l_2$.

(iii) $\lim_{x \to a} [f(x)\, g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] = l_1 l_2$.

(iv) $\lim_{x \to a} [f(x)/g(x)] = \left[\lim_{x \to a} f(x)\right] \Big/ \left[\lim_{x \to a} g(x)\right] = l_1 / l_2$, provided $l_2 \neq 0$.

(v) $\lim_{x \to a} [f(x)]^{g(x)} = (l_1)^{l_2}$.

Right hand limit Let $x > a$ and $x \to a$ from the right hand side. If

$$|f(x) - l_1| < \varepsilon, \quad a < x < a + \delta, \quad \text{or} \quad \lim_{x \to a^+} f(x) = l_1 \tag{1.4}$$

then, l_1 is called the *right hand limit*.

Left hand limit Let $x < a$ and $x \to a$ from the left hand side. If

$$|f(x) - l_2| < \varepsilon, \quad a - \delta < x < a, \quad \text{or} \quad \lim_{x \to a^-} f(x) = l_2 \tag{1.5}$$

then, l_2 is called the *left hand limit*.

If $l_1 = l_2$, then $\lim_{x \to a} f(x)$ exists. If the limit exists, then it is unique.

Example 1.1 Show that $\lim_{x \to 0} \sin(1/x)$ does not exist.

Solution For different values of x in the interval $0 < |x| < \delta$, the function $\sin(1/x)$ takes values between -1 and 1. Since $\lim_{x \to 0} \sin(1/x)$ is not unique, limit does not exist.

Example 1.2 Show that $\lim_{x \to 4} \lfloor x^2 + 1 \rfloor$ does not exist, where $\lfloor \cdot \rfloor$ is the greatest integer function.

Solution Let $h > 0$. We have

$$\lim_{h \to 0} f(4 + h) = \lfloor (4 + h)^2 + 1 \rfloor = \lfloor 17 + h(h + 8) \rfloor = 17 \quad \text{if} \quad h(h + 8) < 1$$

or
$$(h + 4)^2 < 17, \quad \text{or} \quad h < \sqrt{17} - 4$$

and
$$\lim_{h \to 0} f(4 - h) = \lfloor (4 - h)^2 + 1 \rfloor = \lfloor 17 + h(h - 8) \rfloor = 16 \quad \text{if} \quad h(h - 8) > -1$$

or
$$(h - 4)^2 > 15 \quad \text{or} \quad h > 4 - \sqrt{15}.$$

Therefore,
$$\lim_{x \to 4^+} f(x) = 17 \quad \text{and} \quad \lim_{x \to 4^-} f(x) = 16.$$

The limit does not exist.

1.2.2 Continuity of a Function

Let f be a real valued function of the real variable x. Let x_0 be a point in the domain of f and let f be defined in some neighborhood of the point x_0. The function f is said to be continuous at $x = x_0$, if

$$\text{(i)} \lim_{x \to x_0} f(x) = l \text{ exists and } \quad \text{(ii)} \lim_{x \to x_0} f(x) = f(x_0). \tag{1.6}$$

Alternately, f is said to be continuous at a point $x_0 \in I$, if given any real positive number $\varepsilon > 0$, there exists a real $\delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon, \quad \text{whenever} \quad |x - x_0| < \delta. \tag{1.7}$$

Note that δ depends on both ε and the point x_0. If a function f is continuous at every point in an interval I, then f is said to be continuous on I.

A point at which f is not continuous is called a point of discontinuity.

If $\lim_{x \to x_0} f(x) = l$ exists, but $\lim_{x \to x_0} f(x) \neq f(x_0)$, then the point x_0 is called a point of *removable discontinuity*. In this case, we can redefine $f(x)$, such that $f(x_0) = l$, so that the new function is continuous at $x = x_0$. For example, the function

$$f(x) = \begin{cases} (\sin x)/x, & x \neq 0 \\ 4 & , x = 0 \end{cases}$$

has a removable discontinuity at $x = 0$, since $\lim_{x \to 0} f(x) = 1$ and a new function can be defined as

$$f(x) = \begin{cases} (\sin x)/x, & x \neq 0 \\ 1 & , x = 0 \end{cases}$$

so that it is continuous at $x = 0$.

Continuous functions have the following properties:

1. Let the functions f and g be continuous at a point $x = x_0$. Then,

(i) cf, $f \pm g$ and $f \cdot g$ are continuous at $x = x_0$, where c is any constant.

(ii) f/g is continuous at $x = x_0$, if $g(x_0) \neq 0$.

2. If f is continuous at $x = x_0$ and g is continuous at $f(x_0)$, then the composite function $g(f(x))$ is continuous at $x = x_0$.

3. A function f is continuous in a closed interval $[a, b]$, if it is continuous at every point in (a, b), $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

4. If f is continuous at an interior point c of a closed interval $[a, b]$ and $f(c) \neq 0$, then there exists a neighborhood of c, throughout which $f(x)$ has the same sign as $f(c)$.

5. If f is continuous in a closed interval $[a, b]$, then it is bounded there and attains its bounds at least once in $[a, b]$.

6. If f is continuous in a closed interval $[a, b]$ and $f(a), f(b)$ are of opposite signs, then there exists at least one point $c \in [a, b]$ such that $f(c) = 0$.

7. If f is continuous in a closed interval $[a, b]$ and $f(a) \neq f(b)$, then it assumes every value between $f(a)$ and $f(b)$. (This result is known as *intermediate value theorem*).

Piecewise continuity A function $f(x)$ is said to be piecewise continuous in an interval I, if the

interval can be subdivided into a finite number of subintervals such that $f(x)$ is continuous in each of the subintervals and the limits of $f(x)$ as x approaches the end points of each subinterval are finite. Thus, a *piecewise* continuous function has finite jumps at one or more points in I.

For example, the function

$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 2, & 1 \le x < 2 \\ 3, & 2 \le x < 3 \\ 4, & 3 \le x \le 4 \end{cases}$$

is continuous in each of the subintervals and has finite jumps at the points $x = 1, 2, 3$. The magnitude of these jumps is 1.

Uniform continuity In the definition of continuity given in Eq. (1.7), the value of δ depends both on ε and the point x_0. However, if a value of δ can be obtained which depends only on ε and not on the choice of the point x_0 in I, then the function f is said to be *uniformly continuous*. Thus, a function f is uniformly continuous on an interval I, if for a given real positive number $\varepsilon > 0$, there exists a real $\delta > 0$ such that

$$|f(x_2) - f(x_1)| < \varepsilon, \text{ whenever } |x_2 - x_1| < \delta \tag{1.8}$$

for arbitrary points x_1, x_2 in I.

Obviously, a function which is uniformly continuous on an interval is also continuous on that interval.

Example 1.3 What value should be assigned to

$$f(x) = \frac{1 - x}{1 - \sqrt[3]{x}}, x \ne 1$$

at $x = 1$, so that it is continuous at $x = 1$.

Solution Let $x = 1 + h$. Then, $h \to 0$ as $x \to 1$. We have

$$f(x) = \frac{1 - (1 + h)}{1 - (1 + h)^{1/3}} = -\frac{h}{1 - \left[1 + \frac{1}{3}h + O(h^2)\right]} = 3 + O(h).$$

Therefore, $\lim\limits_{x \to 1} f(x) = 3$.

Hence, if we assign $f(1) = 3$, the function will be continuous at $x = 1$.

Example 1.4 Prove that the function f defined by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is irrational} \\ -1, & \text{when } x \text{ is rational} \end{cases}$$

is discontinuous at every point.

Solution Let $x = a$ be any rational number so that $f(a) = -1$. Now, in any given interval there lie an infinite number of rational and irrational numbers. Therefore, for each positive integer n, we can choose an irrational number a_n such that $|a_n - a| < 1/n$. Thus, the sequence $\{a_n\}$ converges to a. Now, $f(a_n) = 1$ for all n and $f(a) = -1$. Hence, $\lim\limits_{n \to \infty} f(a_n) \ne f(a)$. Therefore, the function is discontinuous at all rational points.

Now, let $x = b$ be any irrational number and $f(b) = 1$. For each positive integer n, we can choose a rational number b_n such that $|b_n - b| < 1/n$. Thus, the sequence $\{b_n\}$ converges to b. Now, $f(b_n) = -1$ for all n and $f(b) = 1$. Hence, $\lim_{n \to \infty} f(b_n) \neq f(b)$. Therefore, the function is discontinuous at all irrational points.

Hence, the given function ~~is~~ discontinuous at all points.

Example 1.5 Show that the function $f(x) = x^2$ is uniformly continuous on $[-1, 1]$.

Solution We have

$$|f(x_2) - f(x_1)| = |x_2^2 - x_1^2| = |x_2 - x_1||x_2 + x_1| < 2|x_2 - x_1| < \varepsilon,$$

whenever $|x_2 - x_1| < \varepsilon/2 = \delta$.

Thus, for $\delta \leq \varepsilon/2$, $|f(x_2) - f(x_1)| < \varepsilon$, whenever $|x_2 - x_1| < \delta$ for arbitrary $x_1, x_2 \in [-1, 1]$. Hence, the function x^2 is uniformly continuous on $[-1, 1]$.

Example 1.6 Show that the function $f(x) = \sin(1/x)$ is continuous and bounded on $(0, 2/\pi)$, but it is not uniformly continuous there.

Solution Since both the functions $1/x$ and $\sin(1/x)$ are continuous for all $x \neq 0$, the given function is continuous for all $x \neq 0$. Also, since $|\sin(1/x)| \leq 1$, it is bounded there. Thus, on $(0, 2/\pi)$, $f(x)$ is continuous and bounded. Let $\varepsilon > 0$ be given. Choose $\varepsilon = 1/2$. Consider the points $x_1 = 1/(n\pi)$ and $x_2 = 2/[2n + 1)\pi]$. Both the points lie in the given interval, that is $x_1, x_2 \in (0, 2/\pi)$. We have

$$|f(x_2) - f(x_1)| = \left| \sin \frac{(2n+1)\pi}{2} - \sin n\pi \right| = 1 > \varepsilon = \frac{1}{2}$$

Now,

$$|x_2 - x_1| = \left| \frac{1}{n\pi} - \frac{2}{(2n+1)\pi} \right| = \frac{1}{n(2n+1)\pi}$$

can be made arbitrarily small by choosing n sufficiently large. Therefore, $|x_2 - x_1| < \delta$. However, no matter how small $\delta > 0$ may be, $|x_2 - x_1| < \delta$ cannot ensure $|f(x_2) - f(x_1)| < \varepsilon$. Thus, the function $\sin(1/x)$ is not uniformly continuous on $(0, 2/\pi)$.

1.2.3 Derivative of a Function

Let a real valued function $f(x)$ be defined on an interval I and let x_0 be a point in I. Then, if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}, \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{1.9}$$

exists and is equal to l, then $f(x)$ is said to be differentiable at x_0 and l is called the derivative of $f(x)$ at $x = x_0$. If $f(x)$ is differentiable at every point in the interval (a, b), then $f(x)$ is said to be differentiable in (a, b). If the interval $[a, b]$ is closed, then at the end points a and b, we consider one-sided limits. Geometrically, the derivative of $f(x)$ at a given point P gives the slope of the tangent line to the curve $y = f(x)$ at the point P.

The following properties are satisfied by the differentiable functions.

1. Let the functions f and g be differentiable at a point x_0. Then,

   (i) $(cf')(x_0) = cf'(x_0)$, c any constant.

   (ii) $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$.

(iii) $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$.

(iv) $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g^2(x_0)},\ g(x_0) \neq 0.$

2. If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then the composite function $h = g(f(x))$, is differentiable at x_0 and $h'(x_0) = g'(f(x_0))\, f'(x_0)$.

3. If the function $y = f(x)$ is represented in the parametric form $x = \phi(t)$, $y = \psi(t)$, $t_0 \le t \le T$ and $\phi'(t)$, $\psi'(t)$ exist, then

$$f'(x) = \frac{dy/dt}{dx/dt} = \frac{\psi'(t)}{\phi'(t)},\ \phi'(t) \neq 0. \tag{1.10}$$

Higher order derivatives The derivative of $f(x)$ at any point x, if it exists, is again a function of x, say $f'(x) = g(x)$. If $g(x)$ is differentiable at x, then we define the second order derivative of $f(x)$ as

$$f''(x) = g'(x) = \frac{d^2 f}{dx^2}.$$

Similarly, we define the nth order derivative of f as

$$f^{(n)}(x) = \frac{d}{dx}\left[\frac{d^{n-1} f}{dx^{n-1}}\right] = \frac{d^n f}{dx^n}.$$

The existence of the nth order derivative $f^{(n)}(x)$, implies the existence and continuity of $f, f', f'', \ldots, f^{(n-1)}$ in a neighborhood of the point x.

Leibniz formula Let f and g be two differentiable functions. Then, the nth order derivative of the product fg is given by the Leibniz formula as

$$(f \cdot g)^{(n)} = {}^nC_0 f^{(n)}(x)\, g(x) + {}^nC_1 f^{(n-1)}(x)\, g'(x) + \ldots + {}^nC_r f^{(n-r)}(x)\, g^{(r)}(x) + \ldots + {}^nC_n f(x)\, g^{(n)}(x). \tag{1.11}$$

This formula can be proved by induction.

Example 1.7 Show that the function

$$f(x) = \begin{cases} x^2 \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ but $f'(x)$ is not continuous at $x = 0$.

Solution We have $\lim_{x \to 0} f(x) = 0 = f(0)$. Therefore, $f(x)$ is continuous at $x = 0$. Now,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} [x \cos(1/x)] = 0.$$

Hence, $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$. For $x \neq 0$, we have

$$f'(x) = 2x \cos\left(\frac{1}{x}\right) + x^2\left[-\sin\left(\frac{1}{x}\right)\right]\left[-\frac{1}{x^2}\right] = 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right).$$

Now, $\lim\limits_{x \to 0} f'(x)$ does not exist as $\lim\limits_{x \to 0} \sin(1/x)$ does not exist. Therefore, $f'(x)$ is not continuous at $x = 0$.

Example 1.8 Find the derivative of $f(x) = x\,|\,x\,|, -1 \le x \le 1$.

Solution We have

$$f(x) = \begin{cases} -x^2, & -1 \le x \le 0 \\ x^2, & 0 \le x \le 1. \end{cases}$$

The function is continuous for all x in $[-1, 1]$.

For $x \in [-1, 0]$, we get

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x - \Delta x) - f(x)}{-\Delta x} = \lim_{\Delta x \to 0} -\frac{1}{\Delta x}[-(x - \Delta x)^2 + x^2] = -2x.$$

Hence, $f'(0^-) = 0$.

For $x \in [0, 1]$, we get

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}[(x + \Delta x)^2 - x^2] = 2x.$$

Hence, $f'(0^+) = 0$.

We find that $f(x)$ is differentiable for all x in $[-1, 1]$ and the derivative function is given by

$$f'(x) = \begin{cases} -2x, & x \le 0 \\ 2x, & x \ge 0 \end{cases} \quad \text{or} \quad f'(x) = 2\,|\,x\,|.$$

Example 1.9 Find the equations of the tangent and the normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point (x_0, y_0) on the ellipse.

Solution The slope of the tangent to the ellipse at the point (x_0, y_0) is given by $m = (dy/dx)_{(x_0, y_0)}$ and the slope of the normal to the ellipse at the point (x_0, y_0) is given by $m_1 = -1/m$.

Differentiating the equation of the ellipse, we obtain

$$m = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} = -\frac{b^2 x_0}{a^2 y_0} \quad \text{and} \quad m_1 = -\frac{1}{m} = \frac{a^2 y_0}{b^2 x_0}.$$

Hence, equation of the tangent at (x_0, y_0) is given by

$$y - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0) \quad \text{or} \quad y y_0 a^2 - a^2 y_0^2 = -b^2 x x_0 + b^2 x_0^2$$

or

$$\frac{y y_0}{b^2} + \frac{x x_0}{a^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1.$$

Equation of the normal at (x_0, y_0) is given by

$$y - y_0 = \frac{a^2 y_0}{b^2 x_0}(x - x_0) \quad \text{or} \quad y x_0 b^2 - b^2 x_0 y_0 = a^2 x y_0 - a^2 x_0 y_0$$

or
$$b^2x_0y - a^2y_0x = (b^2 - a^2)x_0y_0.$$

Example 1.10 Find the fourth order derivative of e^{ax} sin bx at the point $x = 0$.

Solution Let $f(x) = e^{ax}$, $g(x) = \sin bx$ and $F(x) = f(x)g(x)$. Using the Leibniz formula, we obtain

$$F^{(4)}(x) = \frac{d^4}{dx^4}(e^{ax}\sin bx) = {}^4C_0(e^{ax})^{(4)}\sin bx + {}^4C_1(e^{ax})^{(3)}(\sin bx)'$$
$$+ {}^4C_2(e^{ax})''(\sin bx)'' + {}^4C_3(e^{ax})'(\sin bx)^{(3)} + {}^4C_4\, e^{ax}(\sin bx)^{(4)}$$
$$= e^{ax}[a^4\sin bx + 4a^3b\cos bx - 6a^2b^2\sin bx - 4ab^3\cos bx + b^4\sin bx]$$

Hence, $F^{(4)}(0) = 4a^3b - 4ab^3 = 4ab(a^2 - b^2)$.

Exercise 1.1

From the first principles, show the following.

1. $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0.$

2. $\lim_{x\to 0^+} \frac{1}{1 + e^{-1/x}} = 1.$

3. $\lim_{x\to 3} (x-3)^{1/5}\, sgn\,(x-3) = 0$, where sgn is the *sign function*

$$sgn\,(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

4. $\lim_{x\to 0} x\lfloor 1 + x\rfloor = 0$, where $\lfloor x\rfloor$ is the greatest integer function.

5. $\lim_{x\to 2} e^{1/(x-2)}$ does not exist.

Obtain $\lim_{x\to a} f(x)$, if it exists, in problems **6** to **16**.

6. $f(x) = \frac{xe^{1/x}}{1 + e^{1/x}}$, $a = 0$.

7. $f(x) = \frac{\sin^2(x-1)}{x-1}$, $a = 1$.

8. $f(x) = \sin(e^{-1/x})$, $a = 0$.

9. $f(x) = [x + |x|]/x$, $a = 0$.

10. $f(x) = \frac{\sqrt{1+x} - 1}{x}$, $a = 0$.

11. $f(x) = (x-2)^2 e^{-1/(x-2)^2}$, $a = 2$.

12. $f(x) = \frac{\tan^{-1}|x|}{x}$, $a = 0$.

13. $f(x) = \frac{\ln(1+px) - \ln(1+qx)}{x}$, $a = 0$.

14. $f(x) = \sqrt{\frac{x - \sin x}{x + \cos^2 x}}$, $a = \infty$.

15. $f(x) = \frac{a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_{m-1}x + b_m}$, $a = \infty$.

16. $f(n) = \frac{n(1^3 + 2^3 + \ldots + n^3)}{[1^2 + 2^2 + \ldots + n^2]^2}$, $a = \infty$.

17. Show that the function f defined as $f(x) = \begin{cases} x, & x \text{ rational} \\ -x, & x \text{ irrational} \end{cases}$

is not continuous at any point $x \neq 0$. Is it continuous at $x = 0$?

18. Find the points of discontinuity of the function $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer function.

19. Discuss the continuity of the function $f(x) = \lfloor x \rfloor + \sqrt{x - |x|}$ at $x = 1$.

20. Show that the function $f(x) = \begin{cases} |x|/x, & x \neq 0 \\ 1 & , x = 0 \end{cases}$ is bounded in $\mathbb{R}$, though it is not continuous over any interval containing $x = 0$.

21. Determine the values of a, b, c so that the function

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & x < 0 \\ c & , x = 0 \\ \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}} & , x > 0 \end{cases}$$

is continuous for all x.

22. Show that the function $f(x) = x^3$ is uniformly continuous on $[0, 1]$ but not on $[0, \infty)$.

23. Show that the function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$.

24. Show that the function $f(x) = \dfrac{x+2}{x-2}$ is not uniformly continuous on $(2, 3)$.

25. Show that $f(x) = \cos x$ is uniformly continuous on $[0, \infty)$.

26. Show that $f(x) = \sqrt{1 - x^2}$ is uniformly continuous on $[0, 1]$.

In problems **27** to **31**, find the derivative of the given function at the given point.

27. $x \tan^{-1} x + \sec^{-1}(1/x)$, at $x = 0$.

28. $x \cosh x - \sinh x$, at $x = 2$.

29. $(x+1)^2 x^{-1/2}$, at $x = 4$.

30. $e^x \ln(\operatorname{cosec} x)$, at $x = \pi/6$.

31. $\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}$, at any point x.

In problems **32** to **37**, find dy/dx, where y is defined implicitly.

32. $xy + xe^{-y} + ye^x - x^2 = 0$ at any point (x, y).

33. $x^3 + y^3 - 3axy = 0$, a is a constant, at $x = a$ and $y \neq a$.

34. $y - \cos(x - y) = 0$ at $x = \pi/2$ and $y \neq 0$.

35. $x^y y^x = 1$ at any point (x, y).

36. $x^y + y^x = (x + y)^{x+y}$ at $x = 1$, $y = 1$.

37. $(\tan^{-1}x)^y + (y)^{\cot x} = 1$ at any point (x, y).

38. Find dy/dx when $x = a(t - \sin t)$, $y = a(1 - \cos t)$ at $t = \pi/2$.

39. Find dy/dx when $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

40. Find the derivative of $\sin x^2$ with respect to x^2.

41. Find the derivative of $\sin^{-1}\left(\dfrac{1-x}{1+x}\right)$ with respect to $\sqrt{x}$.

42. Find the derivative of $(x)^{\sin x}$ with respect to $(\sin x)^x$.

43. Show that the function $f(x) = x^2 \sin(1/x)$, $x \neq 0$, $f(0) = 0$ is differentiable for all $x \in \mathbb{R}$. Also, show that $f'(x)$ is not continuous at $x = 0$.

44. Find all values of a and b, so that the function

$$f(x) = \begin{cases} \tan x, & x < \pi \\ ax + b, & x \geq \pi \end{cases}$$

and its derivative $f'(x)$ are continuous at $x = \pi$.

45. Show that the function

$$f(x) = \begin{cases} (x-1)\tan(\pi x/2), & x \neq 1 \\ -1 & , \ x = 1 \end{cases}$$

is not differentiable at $x = 1$.

46. Show that the function $f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}}, & x \neq 0 \\ 0 & , \ x = 0 \end{cases}$

is not differentiable at $x = 0$.

47. If $y = \tan x + \sec x$, $x \neq \pi/2$, then show that $\dfrac{d^2y}{dx^2} = \dfrac{\cos x}{(1-\sin x)^2}$.

48. If $x = (1 + \ln t)/t^2$ and $y = (3 + 2\ln t)/t$, then find d^2y/dx^2.

49. Find the fourth order derivative of $f(x) = e^{-x}\cos x$.

50. Find the nth order derivative of $f(x) = (ax + b)^m$, $m > n$.

51. If $y = x^3e^{2x}$, then find d^ny/dx^n at $x = 0$.

52. Find the nth order derivative of $f(x) = \sqrt{ax+b}$.

53. Find the nth order derivative of $f(x) = e^{ax}\sin(bx + c)$.

54. If $y = \cos^{-1}x$, $-(\pi/2) \leq x \leq (\pi/2)$, then find d^ny/dx^n at $x = 0$.

55. If $y = e^{a\sin^{-1}x}$, then find d^ny/dx^n at $x = 0$.

1.3 Application of Derivatives and Taylor Series

We now discuss some applications of derivatives like finding approximate values of a function, mean value theorems, increasing and decreasing functions, maximum and minimum values of a function and series representation of a function.

1.3.1 Differentials and Approximations

Let $y = f(x)$ be a real valued differentiable function and x_0 be a point in its domain. Let $x_0 + \Delta x$ be a point in the neighborhood of x_0. Then, Δx may be considered as an increment in x. The corresponding increment in $f(x)$ is given by

$$\Delta f_0 = \Delta f(x_0) = f(x_0 + \Delta x) - f(x_0).$$

From the definition of derivative, we have (see Eq. (1.9))

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f_0}{\Delta x}. \tag{1.12}$$

Since $f'(x_0)$ exists, we can write from Eq. (1.12) that

$$\frac{\Delta f_0}{\Delta x} = f'(x) + \alpha \quad \text{or} \quad \Delta f_0 = f'(x_0)\,\Delta x + \alpha\,\Delta x \tag{1.13}$$

where α is an infinitesimal quantity dependent on Δx and tends to zero as $\Delta x \to 0$. Thus, the increment Δf_0 consists of the following two parts.

(i) Principle part $f'(x_0)\,\Delta x$, which is called the *differential* of f.

(ii) Residual part $\alpha\,\Delta x$ which tends to zero as $\Delta x \to 0$.

In the limit, the differential is also written as

$$df\,(x_0) = dy_0 = f'(x_0)\,dx. \tag{1.14}$$

Hence, an approximation to $f(x + \Delta x)$ can be written as

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)dx. \tag{1.15}$$

Differentials have application in calculating errors in functions due to small errors in the independent variable. We define $|\,dy\,|$ as the *absolute error*; dy/y as the *relative error* and $(dy/y) \times 100$ as the *percentage error* in computations.

Example 1.11 Find an approximate value of

$$y = 3(4.02)^2 - 2(4.02)^{3/2} + 8/\sqrt{4.02}\,.$$

Solution Let a function be defined as

$$y = f(x) = 3x^2 - 2x^{3/2} + 8/\sqrt{x}\,.$$

Let $x_0 = 4$ and $\Delta x = 0.02$. Then, we need an approximation to $f\,(x_0 + \Delta x) = f\,(4.02)$. The approximate value is given by (see Eq. (1.15)).

$$f\,(4.02) \approx f\,(4) + (0.02)\,f'(4)$$

We have
$$f\,(4) = 48 - 2(8) + 8/2 = 36,$$
$$f'(x) = 6x - 3x^{1/2} - 4x^{-3/2} \quad \text{and} \quad f'(4) = 24 - 6 - 4/8 = 35/2.$$

Therefore, the required approximation is

$$f\,(4.02) \approx 36 + 0.02\,(35/2) = 36.35.$$

Example 1.12 If there is a possible error of 0.02 cm in the measurement of the diameter of a sphere, then find the possible percentage error in its volume, when the radius is 10 cm.

Solution Let the radius of the sphere be r cm. Volume of the sphere $= V = 4\pi r^3/3$ and $dr = \pm\, 0.01$ when $r = 10$ cm.
Differentiating V, we obtain $dV = 4\pi\, r^2 dr$.
When $r = 10$, we get from Eq. (1.14), $dV = 4\pi(10)^2\,(\pm\, 0.01) = \pm\, 4\pi$.
Hence, the percentage error in volume is

$$\left(\frac{dV}{V}\right) \times 100 = 100\left[\frac{\pm\, 12\,\pi}{4\pi(10)^3}\right] = \pm\, 0.3 \text{ cubic cm}\,.$$

1.3.2 Mean Value Theorems

We now state some important results.

Theorem 1.1 **(Rolle's theorem)** Let a real valued function f be continuous on a closed interval $[a, b]$ and differentiable in the open interval (a, b). If $f\,(a) = f\,(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.
(See Appendix 1 for proof).

Remark 1

(a) Differentiability of $f(x)$ in an open interval (a, b) is a necessary condition for the applicability of the Rolle's theorem.
For example, consider the function $f(x) = |x|, -1 \le x \le 1$. Now, $f(x)$ is continuous on $[-1, 1]$ and is differentiable at all points in the interval $(-1, 1)$ except at the point $x = 0$. Now,

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

does not vanish at any point in the interval $(-1, 1)$. This shows that the Rolle's theorem cannot be applied as the function $f(x)$ is not differentiable in $(-1, 1)$.

(b) Rolle's theorem gives sufficient conditions for the existence of a point c such that $f'(c) = 0$. For example, the function

$$f(x) = \begin{cases} 0, & 1 \le x \le 2 \\ 2, & 2 < x \le 3 \end{cases}$$

is not continuous on $[1, 3]$, but $f'(c) = 0$ for all c in $[1, 3]$.

(c) Geometrically, the theorem states that if a function satisfies the conditions of Rolle's theorem and has the same value at the end points of an interval $[a, b]$, then there exists at least one point c, $a < c < b$ where the tangent to the curve $y = f(x)$, $a \le x \le b$ is parallel to the x-axis.

Theorem 1.2 (Lagrange mean value theorem) Let f be a real valued function which is continuous on a closed interval $[a, b]$ and differentiable in the open interval (a, b). Then there exists a point c, $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1.16}$$

(See Appendix 1 for proof).

Remark 2

(a) If $f(a) = f(b)$, then Lagrange mean value theorem reduces to the Rolle's theorem.

(b) Geometrically, Lagrange mean value theorem states that there exists a point $(c, f(c))$ on the curve C: $y = f(x)$, $a \le x \le b$, such that the tangent to the curve C at this point is parallel to the chord joining the points $(a, f(a))$ and $(b, f(b))$ on the curve.

(c) Using Eq. (1.16), we can write

$$\min_{a \le x \le b} f'(x) \le \frac{f(b) - f(a)}{b - a} \le \max_{a \le x \le b} f'(x). \tag{1.17}$$

Theorem 1.3 (Cauchy mean value theorem) Let $f(x)$ and $g(x)$ be two real valued functions defined on a closed interval $[a, b]$ such that (i) they are continuous on $[a, b]$, (ii) they are differentiable in (a, b) and (iii) $g'(x) \ne 0$ for every x in (a, b). Then, there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \quad a < c < b. \tag{1.18}$$

(See Appendix 1 for proof).

Remark 3

(a) For $g(x) = x$, Cauchy mean value theorem reduces to Lagrange mean value theorem.

(b) Let a curve C be represented parametrically as $x = f(t)$, $y = g(t)$, $a \le t \le b$. Then, Cauchy mean value theorem states that there exists a point $(f(c), g(c))$, $c \in (a, b)$ on the curve such that the slope $g'(c)/f'(c)$ of the tangent to the curve at this point is equal to the slope of the chord joining the end points of the curve. Hence, Cauchy mean value theorem has the same geometrical interpretation as the Lagrange mean value theorem.

(c) Cauchy mean value theorem cannot be proved by applying the Lagrange mean value theorem separately to the numerator and denominator on the left side of Eq. (1.18). If we apply the Lagrange mean value theorem to the numerator and the denominator separately, we obtain

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c_1)}{g'(c_2)}, a < c_1 < b, a < c_2 < b, c_1 \neq c_2.$$

Example 1.13 A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. Prove that there is at least one value ξ, $a < \xi < b$ for which $f''(\xi) < 0$.

Solution Consider the function $f(x)$ defined on $[a, b]$. Since $f''(x)$ exists, both f and f' exist and are continuous on $[a, b]$. Let c be any point in (a, b). Applying the Lagrange mean value theorem to $f(x)$ on $[a, c]$ and $[c, b]$ separately, we get

$$\frac{f(c) - f(a)}{c - a} = f'(\xi_1), a < \xi_1 < c, \text{ and } \frac{f(b) - f(c)}{b - c} = f'(\xi_2), c < \xi_2 < b.$$

Using $f(a) = f(b) = 0$, we obtain from the above equations

$$f'(\xi_1) = \frac{f(c)}{c - a} \text{ and } f'(\xi_2) = -\frac{f(c)}{b - c}.$$

Now, $f'(x)$ is continuous and differentiable on $[\xi_1, \xi_2]$. Using the Lagrange mean value theorem again, we obtain

$$\frac{f'(\xi_2) - f'(\xi_1)}{\xi_2 - \xi_1} = f''(\xi), \xi_1 < \xi < \xi_2.$$

Substituting the values of $f'(\xi_1)$ and $f'(\xi_2)$, we get

$$f''(\xi) = -\frac{f(c)}{\xi_2 - \xi_1}\left[\frac{1}{b - c} + \frac{1}{c - a}\right] = -\frac{(b - a) f(c)}{(b - c)(c - a)(\xi_2 - \xi_1)} < 0.$$

Example 1.14 Using the Lagrange mean value theorem, show that

$$|\cos b - \cos a| \le |b - a|.$$

Solution Let $f(x) = \cos x$, $a \le x \le b$. Using the Lagrange mean value theorem to $f(x)$, we obtain

$$\frac{\cos b - \cos a}{b - a} = f'(c) = -\sin c, \text{ or } \left|\frac{\cos b - \cos a}{b - a}\right| = |-\sin c| < 1.$$

Hence, the result.

Example 1.15 Let $f'(x) = 1/(3 - x^2)$ and $f(0) = 1$. Find an interval in which $f(1)$ lies.

Solution Using Eq. (1.17), we obtain for $a = 0$ and $b = 1$

$$\min_{0 \le x \le 1} f'(x) \le \frac{f(1) - f(0)}{1 - 0} \le \max_{0 \le x \le 1} f'(x)$$

or
$$\min_{0\le x\le 1}\left[\frac{1}{3-x^2}\right]\le f(1)-1\le \max_{0\le x\le 1}\left[\frac{1}{3-x^2}\right]$$

or
$$\frac{1}{3}\le f(1)-1\le\frac{1}{2},\quad\text{or}\quad \frac{4}{3}\le f(1)\le\frac{3}{2}.$$

Example 1.16 Let C be a curve defined parametrically as $x = a\cos^3\theta$, $y = a\sin^3\theta$, $0\le\theta\le\pi/2$. Determine a point P on C, where the tangent to C is parallel to the chord joining the points $(a, 0)$ and $(0, a)$.

Solution We have $x = f(\theta) = a\cos^3\theta$ and $y = g(\theta) = a\sin^3\theta$. Using the Cauchy mean value theorem, we have at some point θ

slope of tangent to C = slope of the chord joining the points $(a, 0)$ and $(0, a)$

or
$$\frac{g'(\theta)}{f'(\theta)}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=\frac{a-0}{0-a}$$

or
$$-\tan\theta=-1,\quad\text{or}\quad\theta=\pi/4.$$

Therefore, the required point is $(a/2\sqrt{2}, a/2\sqrt{2})$.

1.3.3 Indeterminate Forms

Consider the ratio $f(x)/g(x)$ of two functions $f(x)$ and $g(x)$. If at any point $x = a$, $f(a) = g(a) = 0$, then the ratio $f(x)/g(x)$ takes the form 0/0 and it is called an *indeterminate form*. The problem is to determine $\lim_{x\to a}[f(x)/g(x)]$, if it exists. Since $f(a) = g(a) = 0$, we can write

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=\lim_{x\to a}\frac{[f(x)-f(a)]/(x-a)}{[g(x)-g(a)]/(x-a)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the limit on the right hand side exists. This result is known as *L' Hospital's rule*.

L' Hospital's rule Suppose that the real valued functions f and g are differentiable in some open interval containing the point $x = a$ (except may be at the point $x = a$) and $f(a) = 0 = g(a)$. Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}\tag{1.19}$$

Suppose now that $f'(a) = 0 = g'(a)$. Then, we repeat the application of L' Hospital's rule on $f'(x)/g'(x)$ and obtain

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}=\frac{f''(a)}{g''(a)}$$

provided the limits exist. This application of the rule can be continued as long as the indeterminate form is obtained.

When both $f(a)=\pm\infty$ and $g(a)=\pm\infty$ we get another indeterminate form. In this case also *L'* Hospital's rule can be applied. We write

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{[1/g(x)]}{[1/f(x)]}$$

which is of 0/0 from.

Remark 4

(a) L'Hospital's rule can be used only when the ratio is of indeterminate form, that is, either it is

of the form 0/0 or ∞/∞.

(b) The other indeterminate forms are $0 \cdot \infty$, 0^0, ∞^0, 1^∞ and $\infty - \infty$. In each of these cases, we can reduce the ratio function to the form 0/0 or ∞/∞ and use this rule. For the indeterminate forms 0^0, ∞^0 and 1^∞, we take logarithm of the given function and then take the limits.

(c) When the function is0 of the form 0^∞, $\infty \cdot \infty$, $\infty + \infty$, ∞^∞ or $\infty^{-\infty}$, it is not of indeterminate form and we cannot apply L'Hospital's rule. We note that $0^\infty = 0$, $\infty \cdot \infty = \infty$, $\infty + \infty = \infty$, $\infty^\infty = \infty$ and $\infty^{-\infty} = 0$.

(d) L' Hospital's rule can also be applied to find the limits as $x \to \pm\infty$.

Example 1.17 Evaluate the following limits

(i) $\lim\limits_{x\to 0}\left[\dfrac{\ln(1+x)}{\sin x}\right]$, (ii) $\lim\limits_{x\to 0}[x^n(\ln x)]$, (iii) $\lim\limits_{x\to\infty}\left[\dfrac{e^x}{x}\right]$.

Solution Using L'Hospital's rule, we get

(i)
$$\lim_{x\to 0}\left[\frac{\ln(1+x)}{\sin x}\right] = \lim_{x\to 0}\frac{1/(1+x)}{\cos x} = 1.$$

(ii)
$$\lim_{x\to 0}[x^n(\ln x)] = \lim_{x\to 0}\frac{[\ln x]}{[1/x^n]} = \lim_{x\to 0}\frac{[1/x]}{[-n/x^{n+1}]} = \lim_{x\to 0}\frac{-x^n}{n} = 0.$$

(iii)
$$\lim_{x\to\infty}\left[\frac{e^x}{x}\right] = \lim_{x\to\infty}\left[\frac{e^x}{1}\right] = \infty.$$

Example 1.18 Evaluate $\lim\limits_{x\to 0} x^x$.

Solution The given limit is of the form 0^0 which is an indeterminate form. Let $y = x^x$. Then, $\ln y = x \ln x$. Now,

$$\lim_{x\to 0}[\ln y] = \ln\left[\lim_{x\to 0} y\right] = \lim_{x\to 0}[x\ln x] = \lim_{x\to 0}\left[\frac{\ln x}{1/x}\right]$$

$$= \lim_{x\to 0}\frac{[1/x]}{[-1/x^2]} = -\lim_{x\to 0} x = 0.$$

Therefore, $\lim\limits_{x\to 0} y = e^0 = 1$.

Example 1.19 Evaluate $\lim\limits_{x\to\infty} x\tan(1/x)$.

Solution As $x \to \infty$, the function takes the form $\infty \cdot 0$. We first write it as $\lim\limits_{x\to\infty}\dfrac{x}{\cot(1/x)}$ which is of the form ∞/∞. Applying the L'Hospital's rule, we obtain

$$\lim_{x\to\infty} x\tan(1/x) = \lim_{x\to\infty}\frac{x}{\cot(1/x)} = \lim_{x\to\infty}\frac{1}{(1/x^2)\operatorname{cosec}^2(1/x)}$$

$$= \lim_{x\to\infty}\frac{\sin^2(1/x)}{(1/x)^2} = \lim_{y\to 0}\frac{\sin^2 y}{y^2} = \lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2 = 1.$$

1.3.4 Increasing and Decreasing Functions

Let $y = f(x)$ be a function defined on an interval I contained in the domain of the function $f(x)$. Let

x_1, x_2 be any two points in I, where x_1, x_2 are not the end points of the interval. On the interval I, the function $f(x)$ is said to be

(*i*) an *increasing* function, if $f(x_1) \le f(x_2)$ whenever $x_1 \le x_2$.

(ii) a *strictly increasing* function, if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(iii) a *decreasing* function, if $f(x_1) \ge f(x_2)$ whenever $x_1 < x_2$.

(iv) a *strictly* decreasing function, if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

A function which is either increasing or decreasing in the entire interval I is called a *monotonic* function.

Let a real valued function f defined on an interval I, have a derivative at every point x in I. Then, using the Lagrange mean value theorem, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), x_1 < c < x_2.$$

Therefore, we conclude that

(i) f increases in I if $f'(x) > 0$ for all x in I,
(ii) f decreases in I if $f'(x) < 0$ for all x in I.

Thus, a differentiable function increases when its graph has positive slopes and decreases when its graph has negative slopes. Now, if $f'(x)$ is continuous, then $f'(x)$ can go from positive to negative values or from negative to positive values only by going through the value 0. The values of x for which $f'(x) = 0$ are called the *turning points* or the *critical points*. At a turning point, the tangent to the curve is parallel to the x-axis. On the left and right of a turning point, tangents to the curve have different directions.

Example 1.20 Find the intervals in which the function $f(x) = \sin 3x$, $0 \le x \le \pi/2$ is increasing or decreasing.

Solution We have $f'(x) = 3 \cos 3x$. Now, $f'(x) = 0$ when $3x = \pi/2, 3\pi/2, \ldots$ for positive x. Hence, $x = \pi/6$ is the only turning point in $(0, \pi/2)$. We consider the intervals $(0, \pi/6)$ and $(\pi/6, \pi/2)$. We have in

$0 < x < \pi/6$: $f'(x) = 3 \cos 3x > 0$, $f(x)$ is an increasing function,

$\pi/6 < x < \pi/2$: $f'(x) = 3 \cos 3x < 0$, $f(x)$ is a decreasing function.

Example 1.21 Show that for all $x > 0$

$$1 - x < e^{-x} < 1 - x + \frac{x^2}{2}.$$

Solution Let $f(x) = e^{-x} + x - 1$. Now,

$$f'(x) = 1 - e^{-x} > 0 \text{ for all } x > 0.$$

Hence, $f(x)$ is an increasing function for all $x > 0$. Therefore,

$$f(x) > f(0) = 0, \text{ or } e^{-x} + x - 1 > 0 \text{ or } e^{-x} > 1 - x.$$

Now, consider $g(x) = e^{-x} - 1 + x - \dfrac{x^2}{2}$.

We have
$$g'(x) = 1 - x - e^{-x} < 0 \text{ for all } x > 0.$$

Hence, $g(x)$ is a decreasing function for all $x > 0$. Therefore,

$$g(x) < g(0) = 0, \text{ or } e^{-x} < 1 - x + \frac{x^2}{2}.$$

Combining the above two results, we obtain

$$1 - x < e^{-x} < 1 - x + \frac{x^2}{2}, x > 0.$$

1.3.5 Maximum and Minimum Values of a Function

Let a real valued function $f(x)$ be continuous on a closed interval $[a, b]$. Since a continuous function in a closed interval is bounded and attains these bounds at least once in the interval, we wish to determine the points where $f(x)$ attains these bounds. Let x_0 be a point in (a, b) and $I = (x_0 - h, x_0 + h)$ be an infinitesimal interval around x_0. Then, the function $f(x)$ is said to have a

local maximum (or a *relative maximum*) at the point x_0, if $f(x_0) \geq f(x)$, for all x in I.

local minimum (or a *relative minimum*) at the point x_0, if $f(x_0) \leq f(x)$ for all x in I.

The points of local maximum and local minimum are called the *critical points* or the *stationary points*. The values of the function at these points are called the *extreme values*.

The following theorem gives the necessary condition for the existence of a local maximum/minimum.

Theorem 1.4 **(First derivative test)** Let $f(x)$ be differentiable at $x_0 \in (a, b)$. Then, a necessary condition for the function $f(x)$ to have a local maximum or a local minimum at x_0 is that $f'(x_0) = 0$.

At a critical point, $f'(x)$ changes direction. Thus, to find the local maximum/minimum values of the function in an interval I, we find the critical points in I by solving $f'(x) = 0$. By studying the sign of $f'(x)$ as it passes through the critical point, we decide whether it is a point of local maximum ($f'(x)$ changes sign from positive to negative) or a point of local minimum ($f'(x)$ changes sign from negative to positive).

Example 1.22 Examine the functions

(i) $f(x) = x^3 - 3x + 3, x \in \mathbb{R}$, (ii) $f(x) = \sin^2 x, 0 < x < \pi$

for maximum and minimum values.

Solution We have

(i) $f'(x) = 3x^2 - 3$. Now, $f'(x) = 0$ gives $x = 1, -1$.

For $x < 1, f'(x) < 0$ and for $x > 1, f'(x) > 0$. Since $f'(x)$ changes sign from negative to positive as it passes through the critical point $x = 1$, the function has a local minimum value $f(1) = 1$ at $x = 1$. For $x < -1, f'(x) > 0$ and for $x > -1, f'(x) < 0$. Since $f'(x)$ changes sign from positive to negative as it passes through the critical point $x = -1$, the function has a local maximum value $f(-1) = 5$ at $x = -1$.

(ii) $f'(x) = 2 \sin x \cos x = \sin 2x = 0$ at $x = \pi/2$.

For $x < \pi/2, f'(x) > 0$ and for $x > \pi/2, f'(x) < 0$. Since $f'(x)$ changes sign from positive to negative as it passes through the critical point $x = \pi/2$, the function has a local maximum value $f(\pi/2) = 1$ at $x = \pi/2$.

Theorem 1.5 **(Second derivative test)** Let $f(x)$ be differentiable at x_0, $a \leq x_0 \leq b$ and let $f'(x_0) = 0$. If $f''(x)$ exists and is continuous in a neighborhood of x_0, then

$$f''(x_0) = \lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0)}{h} = \lim_{h\to 0} \frac{f'(x_0-h) - f'(x_0)}{-h},\quad h > 0$$

Therefore,

(i) $f(x)$ has a maximum value at $x = x_0$, when $f''(x_0) < 0$,

(ii) $f(x)$ has a minimum value at $x = x_0$, when $f''(x_0) > 0$.

When $f''(x_0) = 0$, further investigation is needed to decide whether $x = x_0$ is a point of local maximum or minimum. In this direction, we have the following result.

Theorem 1.6 Let $f^{(n)}(x)$ exist for x in (a, b) and be continuous there. Let

$$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0 \quad \text{and} \quad f^{(n)}(x_0) \neq 0$$

Then,

(i) when n is even, $f(x)$ has a maximum if $f^{(n)}(x_0) < 0$ and a minimum if $f^{(n)}(x_0) > 0$.

(ii) when n is odd, $f(x)$ has neither a maximum, nor a minimum.

Absolute maximum/minimum values of a function $f(x)$ in an interval $[a, b]$ are defined as follows:

Absolute maximum value = max $\{f(a), f(b)$, all local maximum values$\}$.

Absolute minimum value = min $\{f(a), f(b)$, all local minimum values$\}$.

Example 1.23 Find the absolute maximum/minimum values of the function

$$f(x) = \sin x\,(1 + \cos x),\ 0 \le x \le 2\pi.$$

Solution We have

$$f(x) = \sin x\,(1 + \cos x) = \sin x + \frac{1}{2}\sin 2x,\quad f'(x) = \cos x + \cos 2x.$$

Setting $f'(x) = 0$, we get

$$\cos x + \cos 2x = 0, \quad \text{or} \quad \cos x + 2\cos^2 x - 1 = 0, \quad \text{or} \quad \cos x = -1,\ 1/2.$$

Therefore, the critical points are $x = \pi/3$, π and $5\pi/3$.

Now, $$f''(x) = -\sin x - 2\sin 2x.$$

At $x = \pi/3$, $f''(\pi/3) = -\ 3\sqrt{3}/2 < 0$. Hence, $f(x)$ has a local maximum at $x = \pi/3$ and the local maximum value is $f(\pi/3) = 3\sqrt{3}/4$.

At $x = \pi$, $f''(\pi) = 0$. We find that

$$f'''(x) = -\cos x - 4\cos 2x \quad \text{and} \quad f'''(\pi) = -3 \neq 0.$$

Since, $f^{(n)}(\pi) \neq 0$ and $n = 3$ is odd, the function has neither maximum nor minimum at $x = \pi$.

At $x = 5\pi/3$, $f''(5\pi/3) = 3\sqrt{3}/2 > 0$. Hence, $f(x)$ has a local minimum at $x = 5\pi/3$. The local minimum value is $f(5\pi/3) = -3\sqrt{3}/4$.

We also have $f(0) = f(2\pi) = 0$. Therefore,

absolute maximum value of $f(x)$ = max $\{f(0), f(2\pi)$, local maximum value at $x = \pi/3\}$

$= \max\{0, 0, 3\sqrt{3}/4\} = 3\sqrt{3}/4.$

absolute minimum value of $f(x)$ = min $\{f(0), f(2\pi)$, local minimum value at $x = 5\pi/3\}$

$$= \min\{0, 0, -3\sqrt{3}/4\} = -3\sqrt{3}/4.$$

Example 1.24 Find a right angled triangle of maximum area with hypotenuse h.

Solution Let x be the base of the right angled triangle. The area of the right angled triangle is

$$A(x) = \frac{1}{2}x\sqrt{h^2 - x^2}, \quad 0 < x < h.$$

Now,
$$A'(x) = \frac{1}{2}\left[\sqrt{h^2 - x^2} - \frac{x^2}{\sqrt{h^2 - x^2}}\right] = \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}}.$$

Setting $A'(x) = 0$ we obtain the critical point as $x = h/\sqrt{2}$.

Now, $A'(x) > 0$ for $x < h/\sqrt{2}$ and $A'(x) < 0$ for $x > h/\sqrt{2}$.

Therefore, $A(x)$ is maximum when $x = h/\sqrt{2}$ and the maximum area is $A(h/\sqrt{2}) = h^2/4$.

1.3.6 Taylor's Theorem and Taylor's Series

A very useful technique in the analysis of real valued functions is the approximation of continuous functions by polynomials. Taylor's theorem (Taylor's formula) is an important tool which provides such an approximation by polynomials. Taylor's theorem can be regarded as an extension of the mean value theorems to higher order derivatives. Mean value theorems relate the value of the function and its first order derivative, whereas the Taylor's theorem relates the value of the function and its higher order derivatives.

Theorem 1.7 (Taylor's theorem with remainder) Let $f(x)$ be defined and have continuous derivatives upto $(n + 1)$th order in some interval I, containing a point a. Then, Taylor's expansion of the function $f(x)$ about the point $x = a$ is given by

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x) \quad (1.20)$$

where
$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(c), \quad a < c < x \quad (1.21)$$

is the *remainder* or the error term of the expansion.

Proof We first find a polynomial $P_n(x)$, of degree n, which satisfies the conditions

$$P_n(a) = f(a), P_n^{(k)}(a) = f^{(k)}(a), k = 1, 2, \ldots, n.$$

In a certain sense, $P_n(x)$ is a polynomial approximation to $f(x)$. Write the required polynomial as

$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n.$$

Substituting $P_n(x)$ in the given conditions, we obtain

$$P_n(a) = f(a) = c_0, P_n'(a) = f'(a) = c_1, P_n''(a) = f''(a) = 2c_2, \ldots,$$

$$P_n^{(n)}(a) = f^{(n)}(a) = (n!)\, c_n.$$

Hence, we have
$$c_k = \frac{1}{k!}f^{(k)}(a), k = 0, 1, 2, \ldots, n.$$

Therefore, $f(x) \approx P_n(x) = f(a) + \dfrac{(x-a)}{1!}f'(a) + \dfrac{(x-a)^2}{2!}f''(a) + \dots + \dfrac{(x-a)^n}{n!}f^{(n)}(a).$

The error of approximation is given by $R_n(x) = f(x) - P_n(x)$. Therefore,

$$f(x) = P_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + R_n(x).$$

Now, we derive a form of $R_n(x)$. Write $R_n(x)$ as

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} h(x)$$

where $h(x)$ is to be determined.

Consider the auxiliary function

$$F(t) = f(x) - \left[f(t) + (x-t)f'(t) + \dots + \frac{(x-t)^n}{n!} f^{(n)}(t) + \frac{(x-t)^{n+1}}{(n+1)!} h(x) \right], a < t < x.$$

We have t as a variable and x is fixed. The function $F(t)$ has the following properties

(i) $F(t)$ is continuous in $a \le t \le x$ and differentiable in $a < t < x$,

(ii) $F(x) = 0$,

(iii) $F(a) = f(x) - \left[f(a) + (x-a)f'(a) + \dots + \dfrac{(x-a)^n}{n!} f^{(n)}(a) + \dfrac{(x-a)^{n+1}}{(n+1)!} h(x) \right]$

$= f(x) - f(x) = 0.$

Hence, $F(t)$ satisfies the hypothesis of the Rolle's theorem on $[a, x]$. Therefore, there exists a point c, $a < c < x$ such that $F'(c) = 0$. Now,

$$F'(t) = 0 - \left[f'(t) - f'(t) + (x-t)f''(t) - \frac{2(x-t)}{2!} f''(t) + \dots \right.$$

$$\left. + \frac{(x-t)^n}{n!} f^{(n+1)}(t) - \frac{(n+1)(x-t)^n}{(n+1)!} h(x) \right] = \frac{(x-t)^n}{n!} [h(x) - f^{(n+1)}(t)]$$

and
$$F'(c) = 0 = \frac{(x-c)^n}{n!} [h(x) - f^{(n+1)}(c)]$$

We obtain $h(x) = f^{(n+1)}(c)$. Therefore,

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c), a < c < x.$$

The error term can also be written as

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(a + \theta(x-a)), 0 < \theta < 1 \tag{1.22}$$

which is called the *Lagrange form of the remainder*.

If $a = 0$, we get

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c), 0 < c < x \quad (1.23)$$

which is called the *Maclaurin's theorem* with remainder.

If we neglect the error term in Eq. (1.20), we obtain

$$f(x) \approx P_n(x) = \sum_{m=0}^{n} \frac{(x-a)^m}{m!} f^{(m)}(a) \quad (1.24)$$

which is called the nth degree Taylor's polynomial approximation to $f(x)$.

Since c or θ in the remainder term (see Eqs. (1.21), (1.22)) is not known, we cannot evaluate $R_n(x)$ exactly for a given x in the interval I. However, a bound on the error can be obtained as

$$|R_n(x)| = \left| \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c) \right| \le \max_{x \in I} \frac{|x-a|^{n+1}}{(n+1)!} \left[\max_{x \in I} |f^{(n+1)}(x)| \right] \quad (1.25)$$

For a given error bound ε, we can use Eq. (1.25) to determine

(i) n for a given x and a,

or (ii) $x = x^*$ for a given n and a such that $|R_n(x^*)| < \varepsilon$.

Remark 5

(a) Writing $x = a + h$ in Eq. (1.20), we obtain

$$f(a+h) \approx f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^{(n)}(a). \quad (1.26)$$

The error of approximation simplifies as

$$R_n(x) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(c), \quad a < c < a + h. \quad (1.27)$$

(b) Another form of the remainder $R_n(x)$ is the *Cauchy form* of *remainder* which is given by

$$R_n(x) = \frac{h^{n+1}}{n!}(1-\theta)^n f^{(n+1)}(a + \theta h), \quad 0 < \theta < 1. \quad (1.28)$$

(see Appendix 1 for proof).

(c) Another form of the remainder $R_n(x)$ is the *integral form of remainder*, which is given by

$$R_n(x) = \frac{1}{n!} \int_a^{a+h} (a+h-s)^n f^{n+1}(s)\, ds \quad (1.29)$$

(see Appendix 1 for proof).

Example 1.25 The function $f(x) = \sin x$ is approximated by Taylor's polynomial of degree three about the point $x = 0$. Find c such that the error satisfies $|R_3(x)| \le 0.001$ for all x in the interval $[0, c]$.

Solution We have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0).$$

For $f(x) = \sin x$, we obtain

$$f'(x) = \cos x, f''(x) = -\sin x, \; f'''(x) = -\cos x \text{ and } f^{(4)}(x) = \sin x.$$

Hence, $\quad f'(0) = 1, f''(0) = 0, f'''(0) = -1 \text{ and } f^{(4)}(\xi) = \sin \xi.$

The required approximation is $f(x) = \sin x \approx x - \dfrac{x^3}{6}$.

The maximum error in the interval $[0, c]$ is given by

$$| R_3(x) | = \left| \frac{x^4}{4!} \sin \xi \right| \le \max_{0 \le x \le c} \left[\frac{x^4}{24} \right] \max_{0 \le x \le c} | \sin x | \le \frac{c^4}{24}.$$

Now, c is to be determined such that

$$\frac{c^4}{24} \le 0.001 \quad \text{or} \quad c^4 \le 0.024.$$

We obtain $c \approx 0.3936$. Hence, for all x in the interval $[0, 0.3936]$, this error criterion is satisfied.

Taylor's Series

In the Taylor's formula with remainder (Eqs. (1.20), (1.21)), if the remainder $R_n(x) \to 0$ as $n \to \infty$, then we obtain

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \ldots \tag{1.30}$$

which is called the *Taylor's series*. When $a = 0$, we obtain the *Maclaurin's series*

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots \tag{1.31}$$

Since it is assumed that $f(x)$ has continuous derivatives upto $(n + 1)$th order, $f^{(n+1)}(x)$ is bounded in the interval (a, x). Hence, to establish that $\lim_{n \to \infty} | R_n(x) | = 0$, it is sufficient to show that $\lim_{n \to \infty} \dfrac{| x - a |^{n+1}}{(n+1)!} = 0$ for any fixed numbers x and a. Now, for any fixed numbers x and a, we can always find a finite positive integer N such that $| x - a | < N$. Denote $q = | x - a |/N$. Then,

$$\left| \frac{(x-a)^{n+1}}{(n+1!} \right| = \left| \frac{x-a}{1} \right| \left| \frac{x-a}{2} \right| \cdots \left| \frac{x-a}{N-1} \right| \left| \frac{x-a}{N} \right| \cdots \left| \frac{x-a}{n+1} \right|$$

$$< \left| \frac{(x-a)^{N-1}}{(N-1)!} \right| q \cdot q \ldots q = \left| \frac{(x-a)^{N-1}}{(N-1)!} \right| q^{n-N+2}$$

Now, $\left| \dfrac{(x-a)^{N-1}}{(N-1)!} \right|$ is a finite quantity and is independent of n. Also $q < 1$. Hence,

$\lim_{n \to \infty} \left| \dfrac{(x-a)^{n+1}}{(n+1)!} \right| = 0$ for any fixed x and a, and $\lim_{n \to \infty} | R_n(x) | = 0$.

Example 1.26 Obtain the Taylor's polynomial expansion of the function $f(x) = \sin x$ about the point $x = \pi/4$. Show that the error term tends to zero as $n \to \infty$ for any real x. Hence, write the Taylor's series expansion of $f(x)$.

Solution For $f(x) = \sin x$, we have

$$f^{(2n)}(x) = (-1)^n \sin x \quad \text{and} \quad f^{(2n+1)}(x) = (-1)^n \cos x$$

for any integer n. Therefore,

$$f^{(2n)}(\pi/4) = (-1)^n/\sqrt{2} \quad \text{and} \quad f^{(2n+1)}(\pi/4) = (-1)^n/\sqrt{2}.$$

Hence, the Taylor's expansion of $f(x) = \sin x$ about $x = \pi/4$ is given by

$$f(x) = f\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \ldots + \frac{1}{n!}\left(x - \frac{\pi}{4}\right)^n f^{(n)}\left(\frac{\pi}{4}\right) + R_n(x).$$

Now,

$$|R_{2n}(x)| = \left| \frac{1}{(2n+1)!}\left(x - \frac{\pi}{4}\right)^{2n+1} f^{(2n+1)}(\xi) \right| \le \frac{1}{(2n+1)!}\left(x - \frac{\pi}{4}\right)^{2n+1}$$

since $f^{(2n+1)}(c) = |(-1)^n \cos c| < 1$. Hence, $R_{2n}(x) \to 0$ as $n \to \infty$.

Similarly, we find that $R_{2n+1}(x) \to 0$ as $n \to \infty$.

Therefore, the required Taylor's series expansion is given by

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{(2!)\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{(3!)\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 + \ldots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \ldots\right]$$

1.3.7 Exponential, Logarithmic and Binomial Series

Exponential series

Consider the Taylor's polynomial approximation of degree $\le n$ about the point $x = 0$ for the function $f(x) = e^x$. The Taylor's polynomial approximation is given by

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0).$$

For $f(x) = e^x$, we obtain

$$f^{(r)}(x) = e^x, f^{(r)}(0) = 1, r = 0, 1, \ldots, n \text{ and } f^{(n+1)}(x) = e^x.$$

Hence,
$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}.$$

Using the Lagrange form of the remainder, we get

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c) = \frac{x^{n+1}}{(n+1)!}e^c$$

or as
$$R_n(x) = \frac{x^{n+1}}{(n+1)!}e^{\theta x}, \quad 0 < \theta < 1.$$

Therefore,
$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty}\left| \frac{x^{n+1}}{(n+1)!}e^{\theta x} \right| = \lim_{n\to\infty}\left[\frac{|x|^{n+1}}{(n+1)!}\right]e^{\theta x} = 0$$

for all x, since $e^{\theta x}$ is bounded for a given x.
Hence, we obtain the *exponential series*

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \tag{1.32}$$

Example 1.27 For the Taylor's polynomial approximation of degree $\le n$ about the point $x = 0$ for the function $f(x) = e^x$, determine the value of n such that the error satisfies $| R_n(x) | \le 0.005$, when $-1 \le x \le 1$.

Solution We have the Taylor's polynomial approximation of e^x as

$$f(x) = e^x \approx 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

The maximum error in the interval $[-1, 1]$ is given by

$$| R_n(x) | = \left| \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c) \right| \le \max_{-1 \le x \le 1} \left[\frac{| x |^{n+1}}{(n+1)!} \right] \max_{-1 \le x \le 1} [| e^x |] \le \frac{e}{(n+1)!}$$

Now, n is to be determined such that

$$\frac{e}{(n+1)!} \le 0.005 \quad \text{or} \quad (n+1)! \ge 200\, e$$

We find that $n \ge 5$. Hence, we will require at least 6 terms in the Taylor's polynomial approximation to achieve the given accuracy.

Example 1.28 Obtain the fourth degree Taylor's polynomial approximation to $f(x) = e^{2x}$ about $x = 0$. Find the maximum error when $0 \le x \le 0.5$.

Solution We have

$$f(x) \approx f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0).$$

For $f(x) = e^{2x}$, we obtain $f^{(r)}(x) = 2^r e^{2x}$, $f^{(r)}(0) = 2^r$, $r = 0, 1, 2, \ldots$ and $f^{(5)}(c) = 32\, e^{2c}$.

Therefore, $f(x) = e^{2x} \approx 1 + 2x + \dfrac{4x^2}{2!} + \dfrac{8x^3}{3!} + \dfrac{16x^4}{4!} = 1 + 2x + 2x^2 + \dfrac{4}{3}x^3 + \dfrac{2}{3}x^4.$

The error term is given by

$$R_4(x) = \frac{x^5}{5!} f^{(5)}(c) = \frac{32}{5!} x^5 e^{2c}, \quad 0 < c < x.$$

and

$$| R_4(x) | \le \frac{32}{120} \left[\max_{0 \le x \le 0.5} x^5 \right] \left[\max_{0 \le x \le 0.5} e^{2x} \right] \le \frac{e}{120}.$$

Logarithmic series

Consider the Taylor's polynomial approximation of degree $\le n$ about the point $x = 0$ for the function $f(x) = \ln(1 + x)$. The Taylor's polynomial approximation is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0).$$

For $f(x) = \ln(1 + x)$, we obtain

$$f(x) = \ln(1 + x), \; f(0) = 0; \; f'(x) = \frac{1}{1 + x}, \; f'(0) = 1;$$

$$f^{(r)}(x) = \frac{(-1)^{r-1}(r-1)!}{(1+x)^r}, \; f^{(r)}(0) = (-1)^{r-1}(r-1)!, \quad r = 2, 3, \ldots, n;$$

$$f^{(n+1)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}.$$

Hence,
$$f(x) = \ln(1 + x) = x + \frac{(-1)(1!)}{2!}x^2 + \frac{(-1)^2(2!)}{3!}x^3 + \ldots + \frac{(-1)^{n-1}(n-1)!}{n!}x^n$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n}. \tag{1.33}$$

We note that $f(x)$ and all its derivatives exist and are continuous for $-1 < x \le 1$. Using the Lagrange form of the remainder, we get

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x) = \frac{x^{n+1}}{(n+1)!}\left[\frac{(-1)^n n!}{(1+\theta x)^{n+1}}\right]$$

$$= \frac{(-1)^n}{(n+1)}\left[\frac{x}{1+\theta x}\right]^{n+1}, \quad 0 < \theta < 1.$$

We consider the following two cases:

Case 1 Let $0 \le x \le 1$. Since $0 < \theta < 1$, we have

$$0 < \theta x < x \le 1 \quad \text{and} \quad \frac{x}{1+\theta x} < 1.$$

Therefore, we obtain

$$\lim_{n \to \infty} |R_n(x)| = \lim_{n \to \infty} \frac{1}{(n+1)} \left|\frac{x}{1+\theta x}\right|^{n+1} = 0.$$

Case 2 Let $-1 < x < 0$. Since $0 < \theta < 1$, $| x / (1 + \theta x) |$ may or may not be less than 1. Hence, we cannot use the Lagrange form of the remainder to find $\lim_{n \to \infty} | R_n(x) |$.

Now, consider the Cauchy's form of the remainder. We have

$$R_n(x) = \frac{x^{n+1}}{n!}(1-\theta)^n f^{(n+1)}(\theta x) = \frac{(-1)^n(1-\theta)^n}{(1+\theta x)^{n+1}} x^{n+1}$$

$$= \frac{(-1)^n}{(1+\theta x)}\left(\frac{1-\theta}{1+\theta x}\right)^n x^{n+1}, \quad 0 < \theta < 1.$$

Since $$\left|\frac{1-\theta}{1+\theta x}\right| < 1 \quad \text{and} \quad \left|\frac{1}{1+\theta x}\right| < \frac{1}{1-|x|},$$

we have $$\lim_{n\to\infty} |R_n(x)| < \lim_{n\to\infty}\left[\frac{|x|^{n+1}}{(1-|x|)}\left|\frac{1-\theta}{1+\theta x}\right|^n\right] = 0.$$

Hence, we obtain

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots, \quad -1 < x \le 1. \tag{1.34}$$

Note that for $|x| > 1$, $\lim_{n\to\infty} |R_n(x)| = \infty$.

Writing $1 + x = y$ or $x = y - 1$ in Eq. (1.34), we get

$$\ln y = (y-1) - \frac{1}{2}(y-1)^2 + \ldots + \frac{(-1)^{n-1}}{n}(y-1)^n + \ldots, \quad 0 < y \le 2. \tag{1.35}$$

The series given in Eqs. (1.34) and (1.35) are called the *logarithmic series*.

Binomial series

Consider the expansion of the function $(x + y)^m$. We can write

$$(x+y)^m = x^m[1 + (y/x)]^m = x^m(1+z)^m, \text{ where } z = y/x.$$

Therefore, it is sufficient to obtain the expansion of the function $f(z) = (1 + z)^m$ or $f(x) = (1 + x)^m$. Consider the Taylor's polynomial approximation for the function $f(x) = (1 + x)^m$ about the point $x = 0$. We consider the following two cases.

Case 1 When m is a positive integer, $f(x) = (1 + x)^m$ possesses continuous derivatives of all orders and $f^{(r)}(x) = 0$, $r \ge m + 1$ for all x. We have

$$f(x) = (1+x)^m, \; f(0) = 1; \; f'(x) = m(1+x)^{m-1}, \; f'(0) = m;$$

$$f''(x) = m(m-1)(1+x)^{m-2}; \; f''(0) = m(m-1); \ldots$$

$$f^{(m)}(x) = m(m-1)\ldots 2.1, \; f^{(m)}(0) = m! \; \text{ and } \; f^{(r)}(x) = 0, \; r > m.$$

Therefore, we obtain

$$f(x) = (1+x)^m = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^m}{m!}f^{(m)}(0)$$

$$= 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + x^m$$

$$= \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \ldots + \binom{m}{m}x^m. \tag{1.36}$$

We also have

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c) = 0, \quad n \geq m.$$

Case 2 When m is not a positive integer, $f(x) = (1 + x)^m$ possesses continuous derivatives of all orders provided $x \neq -1$.

Let $-1 < x < 1$. We have

$$f^{(n+1)}(x) = m(m-1)\ldots(m-n)(1+x)^{m-n-1}$$

Using the Cauchy's form of the remainder, we get

$$R_n(x) = \frac{x^{n+1}}{n!}(1-\theta)^n f^{(n+1)}(\theta x)$$

$$= \frac{x^{n+1}}{n!}(1-\theta)^n m(m-1)\ldots(m-n)(1+\theta x)^{m-n-1}$$

$$= \frac{m(m-1)\ldots(m-n)}{n!}\left(\frac{1-\theta}{1+\theta x}\right)^n (1+\theta x)^{m-1} x^{n+1}, \quad 0 < \theta < 1. \tag{1.37}$$

Now, for $|x| < 1$, $0 < \frac{1-\theta}{1+\theta x} < 1$ and

$$\lim_{n\to\infty}\left(\frac{1-\theta}{1+\theta x}\right)^n = 0, \quad \lim_{n\to\infty} |x|^{n+1} = 0, \quad \lim_{n\to\infty}\left|\frac{m(m-1)\ldots(m-n)}{n!}\right| = a$$

where a is a finite quantity. Since $(1 + \theta x)^{m-1}$ is independent of n and bounded, we obtain from Eq. (1.37)

$$\lim_{n\to\infty} |R_n(x)| = 0.$$

Therefore, when m is not a positive integer and $|x| < 1$, we obtain

$$f(x) = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + \frac{m(m-1)\ldots(m-n)}{n!}x^n + \ldots \tag{1.38}$$

Alternative proof

Consider the series $\sum a_n$ where $a_n = \frac{m(m-1)\ldots(m-n)}{n!}x^n$.

Using the ratio test, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(m-n-1)}{(n+1)}x\right| = |x|.$$

Hence, the series (1.38) converges for $-1 < x < 1$.
Further, it can be shown that the binomial series (1.38) converges at $x = 1$ when $m > -1$.
For example, consider the series

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

For $x = 1$, the series on the right hand side has two limit points 0 and 1 and hence the series is not convergent.
We have the following binomial series

$$(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots, \quad -1 < x < 1.$$

$$(1 + x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \dots, \quad -1 < x \le 1.$$

$$(1 + x)^{1/3} = 1 + \frac{1}{3}x - \frac{2}{3 \cdot 6}x^2 + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9}x^3 - \dots, \quad -1 < x \le 1.$$

Exercise 1.2

Find the approximate values of the following quantities using differentials.

1. $(1005)^{1/3}$.
2. $(999)^{1/3}$.
3. $(1.001)^3 + 2(1.001)^{4/3} + 5$.
4. $\sin 60° \, 10'$.
5. $\tan 45° \, 5' \, 30''$.
6. State why Rolle's theorem cannot be applied to the following functions.
   (i) $f(x) = \tan x$ in the interval $[0, \pi]$, (ii) $f(x) = \lfloor x \rfloor$ in the interval $[-1/2, 3/2]$,
   (iii) $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2. \end{cases}$
7. It is given that the Rolle's theorem holds for the function $f(x) = x^3 + bx^2 + cx$, $1 \le x \le 2$ at the point $x = 4/3$. Find the values of b and c.
8. The functions $f(x)$ and $g(x)$ are continuous on $[a, b]$ and differentiable in (a, b) such that $f(a) = 4$, $f(b) = 10$, $g(a) = 1$ and $g(b) = 3$. Then, show that $f'(c) = 3g'(c)$, $a < c < b$.
9. Prove that between any two real roots of $e^x \sin x = 1$, there exists atleast one root of $e^x \cos x + 1 = 0$.
10. Let $f'(x)$, $g'(x)$ be continuous and differentiable functions on $[a, b]$. Then, show that for $a < c < b$
$$\frac{f(b) - f(a) - (b-a)f'(a)}{g(b) - g(a) - (b-a)g'(a)} = \frac{f''(c)}{g''(c)}, \; g''(c) \ne 0.$$
11. Let $f(x)$ be continuous on $[a - 1, a + 1]$ and differentiable in $(a - 1, a + 1)$. Show that there exists a θ, $0 < \theta < 1$ such that $f(a - 1) - 2f(a) + f(a + 1) = f'(a + \theta) - f'(a - \theta)$.
12. Using the Lagrange mean value theorem, show that
   (i) $e^x > 1 + x$, $x > 0$; (ii) $\ln(1 + x) < x$, $x > 0$;
   (iii) $x < \sin^{-1}x < x/\sqrt{1 - x^2}$, $0 < x < 1$.
13. Suppose that $f(x)$ is differentiable for all values of x such that $f(a) = a$, $f(-a) = -a$ and $|f'(x)| \le 1$ for all x. Show that $f(0) = 0$.
14. Let $F(x)$ and $G(x)$ be two functions defined on $[a, b]$ satisfying the hypothesis of the mean value theorem with $G(x) \ne 0$ for any x in $[a, b]$. Show that there exists a point c in (a, b) such that

$$\frac{F(b)-F(a)}{G(b)-G(a)}=\frac{F'(c)}{G'(c)}\left[\frac{G^2(c)}{G(a)G(b)}\right]$$

Evaluate the limits in problems **15** to **28**.

15. $\lim_{x\to1}\frac{x-1}{x^n-1}$.

16. $\lim_{x\to0}\frac{e^x-2\cos x+e^{-x}}{x\sin x}$.

17. $\lim_{x\to\pi/2}\frac{\ln(\sin x)}{(\pi-2x)^2}$.

18. $\lim_{x\to0}\frac{\sin^2x-x^2}{x^2\sin^2x}$.

19. $\lim_{x\to1}(1-x)\tan(\pi x/2)$.

20. $\lim_{x\to2}\left[\frac{x-1}{x-2}-\frac{1}{\ln(x-1)}\right]$.

21. $\lim_{x\to1}x^{1/(x-1)}$.

22. $\lim_{x\to\pi/2}(\sin x)^{\tan x}$.

23. $\lim_{x\to0}\frac{e^{f(x)}-1}{f(x)}$, $f(0)=0$.

24. $\lim_{x\to\infty}\frac{x^2}{e^x}$.

25. $\lim_{x\to\infty}[1+f(x)]^{1/f(x)}$, $\lim_{x\to\infty}f(\infty)=0$.

26. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

27. $\lim_{x\to\infty}\sqrt{\frac{x+\sin x}{x-\cos^2x}}$.

28. $\lim_{x\to\infty}\left(\frac{x+4}{x+2}\right)^{x+3}$.

In problems **29** to **36**, find the intervals in which $f(x)$ is increasing or decreasing.

29. $\ln(2+x)-2x/(2+x)$, $x\in\mathbb{R}$.

30. $x\,|x|$, $x\in\mathbb{R}$.

31. $\tan^{-1}x+x$, $x\in\mathbb{R}$.

32. $\sin x+|\sin x|$, $0<x\le2\pi$.

33. $\ln(\sin x)$, $0<x<\pi$.

34. $(\ln x)/x$, $x>0$.

35. $\sin x\,(1+\cos x)$, $0<x<\pi/2$.

36. x^x, $x>0$.

37. Let $a>b>0$ and n be a positive integer satisfying $n\ge2$. Prove that $a^{1/n}-b^{1/n}<(a-b)^{1/n}$.

In problems **38** to **43**, find the extreme values of the given function $f(x)$.

38. $(x-1)^2(x+1)^3$.

39. $\sin x+\cos x$.

40. $x^{1/x}$.

41. $(\sin x)^{\sin x}$.

42. $2\sin x+\cos 2x$, $0\le x\le2\pi$.

43. $\sin^2x\sin 2x+\cos^2x\cos 2x$, $0<x<\pi$.

44. Show that the function $f(x)=(ax+b)/(cx+d)$ has no extreme value regardless of the values of a, b, c, d.

45. Let $f(x)=\begin{cases}-x^3+[(b^3-b^2+b-1)/(b^2+3b+2)], & 0\le x<1\\ 2x-3, & 1\le x\le3.\end{cases}$

Find all possible real values of b such that $f(x)$ has minimum value at $x=1$.

In problems **46** to **50**, obtain the Taylor's polynomial approximation of degree n to the function $f(x)$ about the point $x=a$. Estimate the error in the given interval.

46. $f(x)=\sqrt{x}$, $n=3$, $a=1$, $1\le x\le1.5$.

47. $f(x)=e^{-x^2}$, $n=3$, $a=0$, $-1\le x\le1$.

48. $f(x)=x\sin x$, $n=4$, $a=0$, $-1\le x\le1$.

49. $f(x)=x^2e^{-x}$, $n=4$, $a=1$, $0.5\le x\le1.5$.

50. $f(x)=1/(1-x)$, $n=3$, $a=0$, $0\le x\le0.25$.

In problems **51** to **54**, obtain the Taylor's polynomial approximation of degree n to the function $f(x)$ about the point $x=a$. Find the error term and show that it tends to zero as $n\to\infty$. Hence, write its Taylor's series.

51. $f(x)=\sin 3x$, $a=0$.

52. $f(x)=\sin^2x$, $a=0$.

53. $f(x)=x^2\ln x$, $a=1$.

54. $f(x)=2^x$, $a=1$.

55. Show that the number θ which occurs in the Taylor's formula with Lagrange form of remainder (given in Eq. 1.22)) after n terms approaches the limit $1/(n+1)$ as $h \to 0$ provided $f^{(n+1)}(x)$ is continuous and not zero at $x = a$.

56. Find the number of terms that must be retained in the Taylor's polynomial approximation about the point $x = 0$ for the function $\cosh x$ in the interval $[0, 1]$ such that $|\text{ Error }| < 0.001$.

57. Find the number of terms that must be retained in the Taylor's polynomial approximation about the point $x = 0$ for the function $\sin x \cos x$ in the interval $[0, 1]$ such that $|\text{ Error }| < 0.0005$.

58. The function $\ln(1 - x^2)$ is approximated about $x = 0$ by an nth degree Taylor's polynomial. Find n such that $|\text{ Error }| < 0.1$ on $0 \le x \le 0.5$.

59. The function $\sin^2 x$ is approximated by the first two non-zero terms in the Taylor's polynomial expansion about the point $x = 0$. Find c such that $|\text{ Error }| < 0.005$, when $0 < x < c$.

60. The function $\tan^{-1} x$ is approximated by the first two non-zero terms in the Taylor's polynomial expansion about the point $x = 0$. Find c such that $|\text{ Error }| < 0.005$, when $0 < x < c$.

Obtain the Taylor's series expansions as given in Problems **61** to **65**.

61. $a^x = 1 + x \ln a + \dfrac{(x \ln a)^2}{2!} + \dfrac{(x \ln a)^3}{3!} + \dots, -\infty < x < \infty.$

62. $\ln(1 - x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dots, -1 \le x < 1.$

63. $\ln\left(\dfrac{1+x}{1-x}\right) = 2\left[x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dots\right], -1 < x < 1.$

64. $\ln x = 2\left[\left(\dfrac{x-1}{x+1}\right) + \dfrac{1}{3}\left(\dfrac{x-1}{x+1}\right)^3 + \dfrac{1}{5}\left(\dfrac{x-1}{x+1}\right)^5 + \dots\right], \quad x > 0.$

65. $\ln x = \left(\dfrac{x-1}{x}\right) + \dfrac{1}{2}\left(\dfrac{x-1}{x}\right)^2 + \dfrac{1}{3}\left(\dfrac{x-1}{x}\right)^3 + \dots, \quad x \ge 1/2.$

1.4 Integration and Its Applications

Let $f(x)$ be defined and continuous on a closed interval $[a, b]$. Let there exist a function $F(x)$ such that $F'(x) = f(x)$, $a \le x \le b$. Then, the function $F(x)$ is called the *anti-derivative* of $f(x)$. We observe that if $F(x)$ is an anti-derivative of $f(x)$, then $F(x) + c$, where c is an arbitrary constant, is also an anti-derivative of $f(x)$.

1.4.1 Indefinite Integrals

If $F(x)$ is an anti-derivative of $f(x)$ on $[a, b]$, then for any arbitrary constant c, $F(x) + c$ is called the indefinite integral of $f(x)$ on $[a, b]$ and is written as

$$\int f(x)dx = F(x) + c. \tag{1.39}$$

In this case, we say that the function $f(x)$ is integrable on $[a, b]$. We note that, not every function is integrable. For example, the function $f(x)$ defined as

$$f(x) = \begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$$

on [0, 1] does not have an anti-derivative and hence is not integrable. We have the following result.

Theorem 1.8 Every function which is continuous on a closed and bounded interval is integrable.

We can evaluate an indefinite integral directly or by using the method of substitution, or integration by parts etc.

1.4.2 Definite Integrals

Let $f(x)$ be a continuous function on $[a, b]$. Let

$$m = \min_{a \le x \le b} f(x) \text{ and } M = \max_{a \le x \le b} f(x).$$

Divide the interval $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, where $x_0 = a$, $x_n = b$ and $a = x_0 < x_1 < x_2 \ldots < x_n = b$ (Fig. 1.1). Let $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$. Let

$$m_i = \min_{x_{i-1} \le x \le x_i} f(x) \text{ and } M_i = \max_{x_{i-1} \le x \le x_i} f(x), i = 1, 2, \ldots, n$$

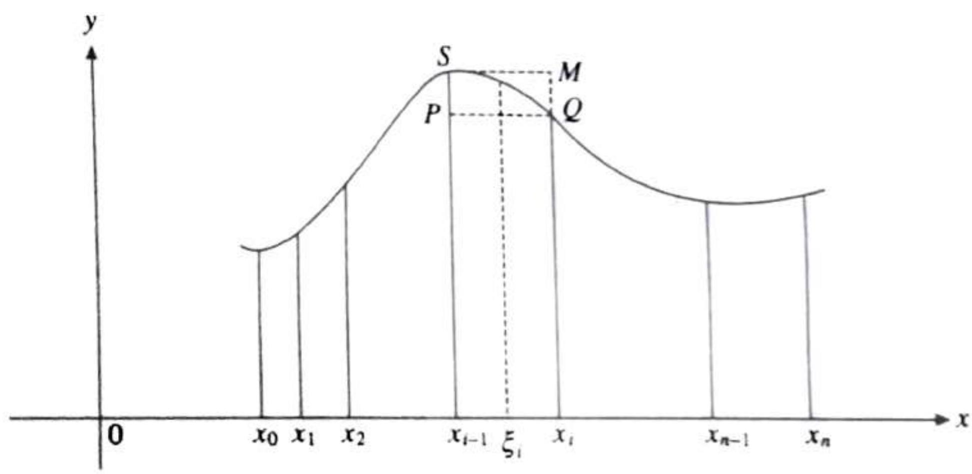


Fig. 1.1. Definite integrals.

and ξ_i be any point in the interval $[x_{i-1}, x_i]$. Corresponding to this partition, define

$$\text{lower sum} = l_n(f) = \sum_{i=1}^{n} m_i \Delta x_i \tag{1.40}$$

$$\text{upper sum} = u_n(f) = \sum_{i=1}^{n} M_i \Delta x_i \tag{1.41}$$

and
$$S_n(f) = \sum_{i=1}^{n} f(\xi_i) \Delta x_i. \tag{1.42}$$

Using these definitions, we obtain

$$l_n(f) \le S_n(f) \le u_n(f).$$

The sum $S_n(f)$ depends upon the way in which the interval $[a, b]$ is sub-divided and also upon the choice of the points ξ_i inside the corresponding subintervals. Let $n \to \infty$ such that $\max(\Delta x_i) \to 0$. If

$$\lim_{n \to \infty} l_n(f) = \lim_{n \to \infty} S_n(f) = \lim_{n \to \infty} u_n(f)$$

for any choice of the sequence of subdivisions of the interval $[a, b]$ and any ξ_i in the interval $[x_{i-1}, x_i]$, then this limit is called the definite integral of $f(x)$ over the interval $[a, b]$ and is written as $\int_a^b f(x)dx$.

Remark 6

(a) For integrability, the condition that $f(x)$ is continuous on $[a, b]$ can be relaxed. The function $f(x)$ may only be piecewise continuous on $[a, b]$.

(b) The choice of points $x_0, x_1, \ldots, x_n$ is arbitrary. One may choose such that they form an arithmetic progression or a geometric progression.

(c) Let m and M be the minimum and maximum values of $f(x)$ on $[a, b]$. Then,

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a). \tag{1.43}$$

(d)

$$\int_a^b f(x)dx = (b-a)f(\xi), a < \xi < b \tag{1.44}$$

(*mean value theorem of integrals*).

(e) If $f(x)$ is bounded and integrable on $[a, b]$, then $|f(x)|$ is also bounded and integrable on $[a, b]$, and

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|\, dx. \tag{1.45}$$

(f) The *average value* of an integrable function $f(x)$ defined on $[a, b]$ is given by $\frac{1}{(b-a)} \int_a^b f(x)dx$.

(g) Let $f(x)$ and $g(x)$ be integrable functions on $[a, b]$ and let $f(x) \le g(x)$, $a \le x \le b$. Then,

$$\int_a^b f(x)dx \le \int_a^b g(x)dx. \tag{1.46}$$

The method of evaluating a definite integral as a limit of sum can be used only when $f(x)$ is a simple function. It may not always be possible to find the limit of sum for every integrable function $f(x)$. We use the following result to evaluate definite integrals.

Theorem 1.9 **(First fundamental theorem of integrals)** Let $f(x)$ be a continuous function on a closed and bounded interval $[a, b]$. Then the function

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$, differentiable in (a, b) and $F'(x) = f(x)$.

Theorem 1.10 (Second fundamental theorem of integrals, Newton-Leibniz formula) Let $F(x)$ be an anti-derivative of a continuous function $f(x)$ on $[a, b]$. Then,

$$\int_a^b f(x) = F(b) - F(a). \tag{1.47}$$

Definite integrals have many applications. In particular, they can be used to find the (i) areas of bounded regions, (iii) lengths of curves, (iii) volumes of solids, (iv) areas of surfaces of revolution etc.

1.4.3 Areas of Bounded Regions

C_1 The area of the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$, $x = b$ is given by (Fig. 1.2)

$$\text{Area} = \int_a^b y\,dx = \int_a^b f(x)dx. \tag{1.48}$$

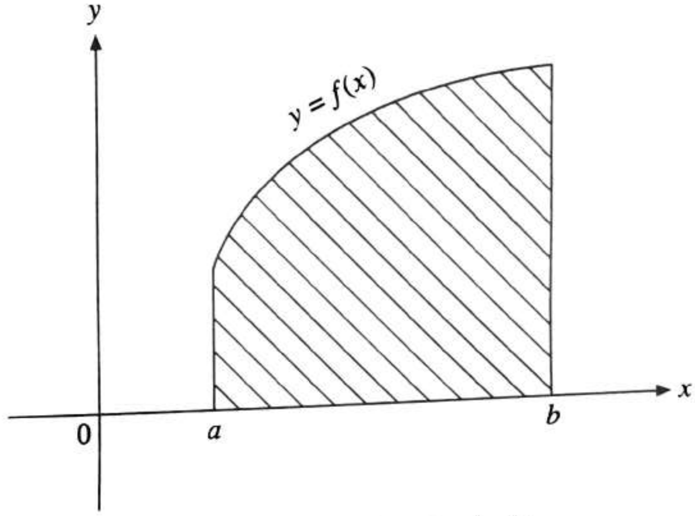


Fig. 1.2. Area of region in C_1.

We note the following:

(i) If the curve $y = f(x)$, $a \le x \le b$ is above the x–axis, then the value of the integral given in Eq. (1.48) is positive. If the curve $y = f(x)$, $a \le x \le b$ is below the x-axis, then the value of the integral given in Eq. (1.48) is negative. Since area is a positive quantity, we take the magnitude of this value.

(ii) If the curve $y = f(x)$ is above the x-axis in the interval $a \le x \le c$ and below the x-axis in the interval $c \le x \le b$, then we write

$$\text{Area} = \int_a^c f(x)dx + \left| \int_c^b f(x)dx \right|$$

C_2 The area of the region bounded by the curve $x = \phi(y)$, the y-axis and the lines $y = c$, $y = d$ is given by (Fig. 1.3)

$$\text{Area} = \int_c^d x\,dy = \int_c^d \phi(y)dy. \tag{1.49}$$

C_3 The area of the region enclosed between the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by (Fig. 1.4)

$$\text{Area} = \int_a^b [f(x) - g(x)]dx, \text{ where } f(x) \ge g(x) \text{ in } [a, b]. \tag{1.50}$$

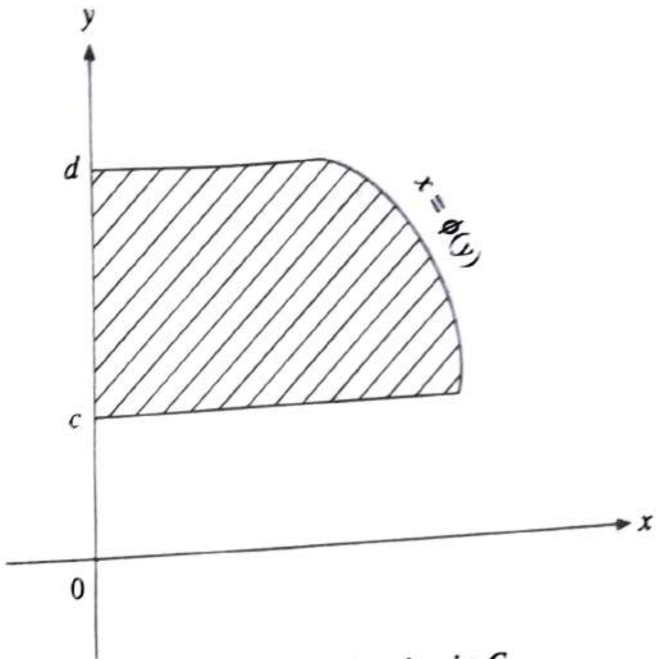


Fig. 1.3. Area of region in C_2.

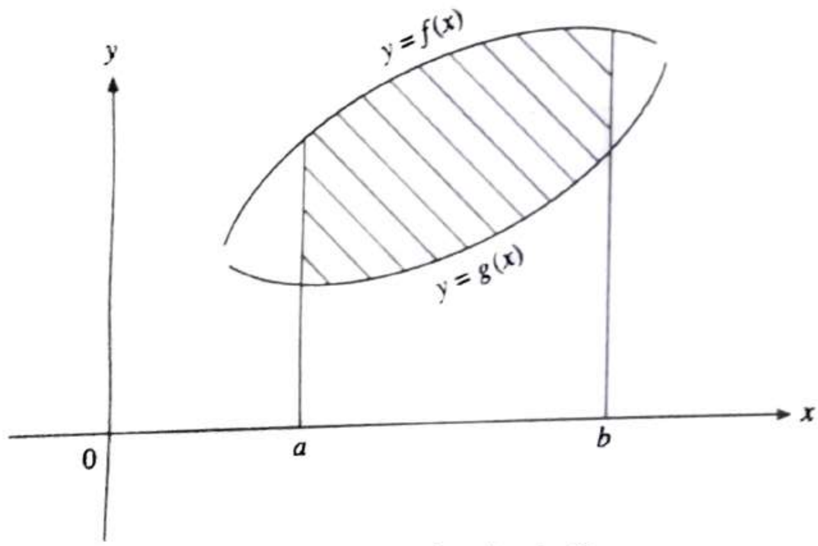


Fig. 1.4. Area of region in C_3.

C_4 If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then we write the area as (Fig. 1.5)

$$\text{Area} = \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx. \tag{1.51}$$

Area bounded by a curve represented in parametric form

Let the curve $y = f(x)$ be defined in parametric form as

$$x = \phi(t),\ y = \psi(t),\ a \leq t \leq b$$

where $\phi(t)$ and $\psi(t)$ are continuous functions of t in the interval $[a, b]$. Let $x_0 = \phi(a)$ and $x_1 = \phi(b)$. Then, from Eq. (1.48), the area is given by

$$\text{Area} = \int_{x_0}^{x_1} y\, dx = \int_a^b \psi(t)\, \phi'(t) dt. \tag{1.52}$$

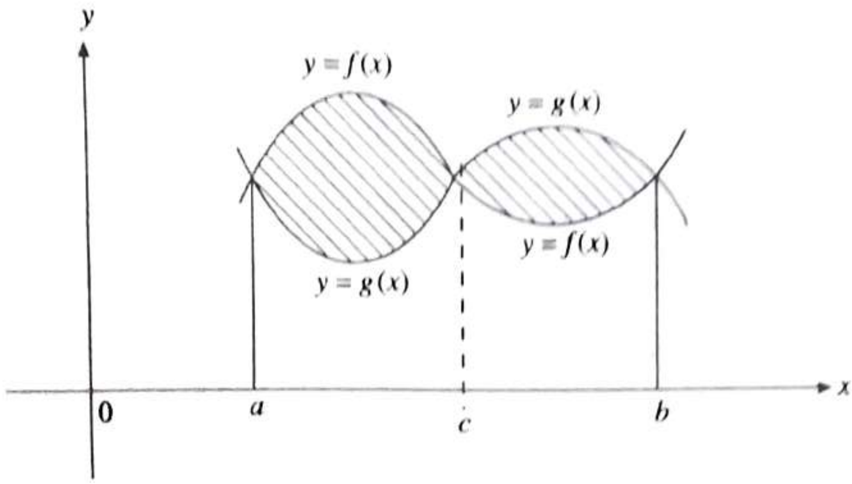


Fig. 1.5. Area of region in C_4.

Area of a sector

Let the curve be defined in polar form as

$$r = f(\theta),\ \alpha \le \theta \le \beta \tag{1.53}$$

where $f(\theta)$ is a continuous function in $[\alpha, \beta]$. Let A be the area of the sector bounded by the curve and the radial lines $\theta = \alpha$ and $\theta = \beta$ (Fig. 1.6).

In an element area, we approximate area of the sector OPQ, by the area of the triangle OPN, with base PN = $r d\theta$ and height ON $\approx$ OP = r. (PN is perpendicular to OQ). Then

$$dA = \frac{1}{2} r^2 d\theta, \text{ and Area} = A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta. \tag{1.54}$$

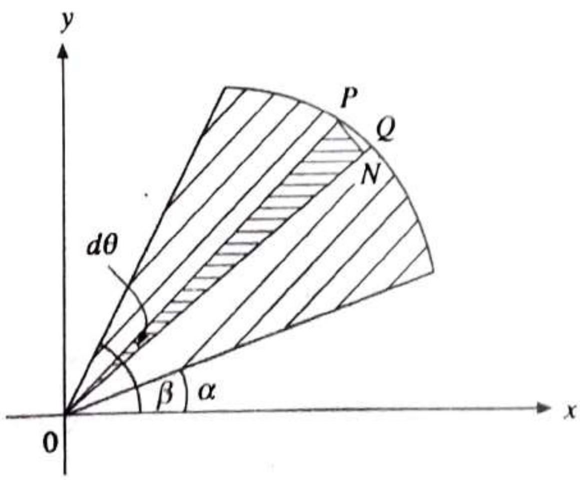


Fig. 1.6. Area of sector.

Example 1.29 Find the area of the region enclosed between the curves $y = \sqrt{x}$ and $y = x^2$.

Solution The curves intersect at the points where $\sqrt{x} = x^2$, or $x^4 - x = 0$, that is at $x = 0$ and $x = 1$. Since $\sqrt{x} \ge x^2$ when $0 \le x \le 1$, we obtain the area as

$$\text{Area} = \int_0^1 \left[\sqrt{x} - x^2\right] dx = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ square units.}$$

Example 1.30 Find the area of the region enclosed by the curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \le t \le 2\pi$.

Solution As t varies from 0 to 2π, x varies from 0 to $2\pi a$. Hence,

$$\text{Area} = \int_0^{2\pi a} [a(1 - \cos t)]\,[a(1 - \cos t)]dt$$

$$= a^2 \int_0^{2\pi a} (1 - 2\cos t + \cos^2 t)dt = a^2 \int_0^{2\pi a} \left[1 - 2\cos t + \frac{1}{2}(1 + \cos 2t)\right]dt$$

$$= \frac{a^2}{2} \int_0^{2\pi a} (3 - 4\cos t + \cos 2t)dt = \frac{a^2}{2}\left[3t - 4\sin t + \frac{1}{2}\sin 2t\right]_0^{2\pi a}$$

$$= 3\pi a^2 \text{ square units.}$$

Example 1.31 Find the area of the region that lies inside the circle $r = a\cos\theta$ and outside the cardioid $r = a(1 - \cos\theta)$.

Solution The region is given in Fig. 1.7. The curves intersect at $\theta = \pm\pi/3$. Let $r_1 = a\cos\theta$ and $r_2 = a(1 - \cos\theta)$. Therefore, the required area is given by

$$\text{Area} = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (r_1^2 - r_2^2)d\theta = \frac{1}{2}a^2 \int_{-\pi/3}^{\pi/3} [\cos^2\theta - (1 - \cos\theta)^2]d\theta$$

$$= a^2 \int_0^{\pi/3} (2\cos\theta - 1)d\theta = a^2[2\sin\theta - \theta]_0^{\pi/3} = a^2\left[2\left(\frac{\sqrt{3}}{2}\right) - \frac{\pi}{3}\right]$$

$$= \frac{a^2}{3}[3\sqrt{3} - \pi] \text{ square units.}$$

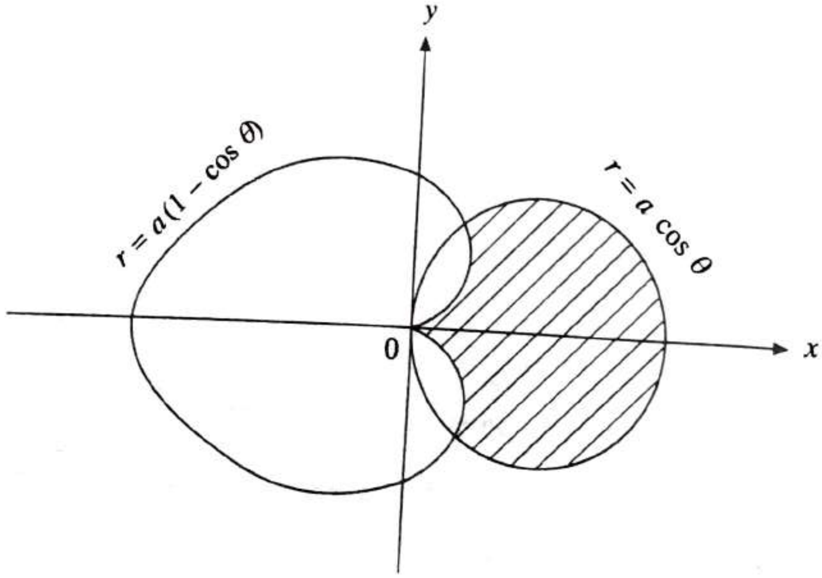


Fig. 1.7. Example 1.31.

1.4.4 Arc Length of a Plane Curve

Consider a portion of the curve $y = f(x)$ between $x = a$ and $x = b$. Then, the length of the arc of the curve between $x = a$ and $x = b$ is given by

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{1.55}$$

If the curve is defined by $x = \phi(y)$, $c \le y \le d$, then the length of the arc is given by

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \tag{1.56}$$

Arc length of a curve represented in parametric form

Let the parametric form of the curve be given by

$$x = \phi(t),\ y = \psi(t),\ t_0 \le t \le t_1$$

where $\phi(t)$ and $\psi(t)$ are continuously differentiable functions on $[t_0, t_1]$. If $\phi(t_0) = a$ and $\phi(t_1) = b$, then from Eq. (1.55), the arc length is given by

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{t_0}^{t_1} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{1.57}$$

Arc length of a curve represented in polar form

Consider the portion of the curve defined by $r = f(\theta)$, $\alpha \le \theta \le \beta$, where $f(\theta)$ is continuously differentiable on $[\alpha, \beta]$. The curve can be represented in parametric form as

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta,\ \alpha \le \theta \le \beta.$$

Therefore,

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta,\ \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta,$$

and

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f'(\theta)]^2 + f^2(\theta).$$

Using Eq. (1.57), we obtain the length of the portion of the curve between $\theta = \alpha$ and $\theta = \beta$ as

$$s = \int_\alpha^\beta \sqrt{f^2(\theta) + [f'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{1.58}$$

or

$$s = \int_{f(\alpha)}^{f(\beta)} \sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + 1}\, dr. \tag{1.59}$$

Example 1.32 Find the total length of the curve $r = a \sin^3(\theta/3)$.

Solution The curve is defined when $0 \le \theta \le 3\pi$. We have

$$f(\theta) = a \sin^3\left(\frac{\theta}{3}\right), f'(\theta) = a \sin^2\left(\frac{\theta}{3}\right) \cos\left(\frac{\theta}{3}\right) \text{ and } f^2(\theta) + [f'(\theta)]^2 = a^2 \sin^4\left(\frac{\theta}{3}\right).$$

Therefore,

$$s = \int_0^{3\pi} \sqrt{f^2(\theta) + [f'(\theta)]^2}\, d\theta = a \int_0^{3\pi} \sin^2\left(\frac{\theta}{3}\right) d\theta = 3a \int_0^{\pi} \sin^2\phi\, d\phi$$

where $\theta = 3\phi$. Integrating, we obtain

$$s = \frac{3a}{2} \int_0^{\pi} (1 - \cos 2\phi) d\phi = \frac{3a}{2}\left[\phi - \frac{1}{2} \sin 2\phi\right]_0^{\pi} = \frac{3a\pi}{2}.$$

1.4.5 Volume of Solids

In this section we discuss methods for finding the volume of solids.

Method of slicing

Let a solid be bounded by two parallel planes $x = a$ and $x = b$ (Fig. 1.8). Divide the interval $[a, b]$ for x into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, where $a = x_0 < x_1 < x_2 \ldots < x_n = b$. Let $\Delta x_k = x_k - x_{k-1}$, $k = 1, 2, \ldots, n$. Draw the planes $x = x_0, x = x_1, \ldots, x = x_n$. This will cut the solid into slices of thickness Δx_k. We now approximate the volume of the sliced solid part S_k by the volume of a cylinder with base as a cross section of the sliced solid S_k and the height as Δx_k. Therefore, an approximation to the volume of the sliced solid between $x = x_{k-1}$ and $x = x_k$ is given by

$$V_k = A(\xi_k)\, \Delta x_k, \quad x_{k-1} < \xi_k \le x_k$$

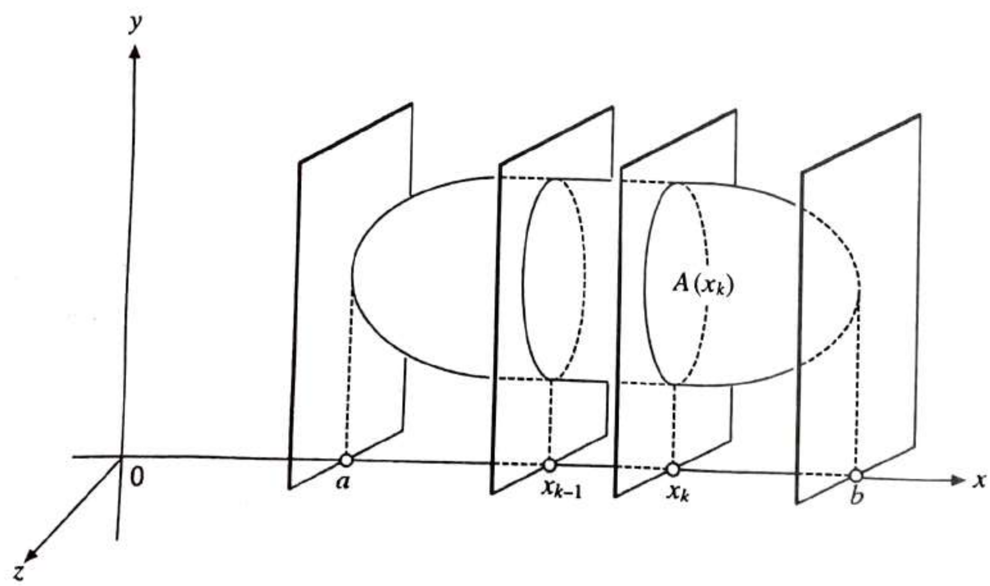


Fig. 1.8. Method of slicing.

where $A(\xi_k)$ is the cross-sectional area of the sliced solid. Now, consider the sum of all the approximate volumes of the sliced solids. We obtain

$$V_n = \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(\xi_k)\, \Delta x_k.$$

Let $n \to \infty$ such that max $\Delta x_k \to 0$. In the limit $V_n \to V$, volume of the solid and the summation reduces to an integral. Therefore, volume of the solid is given by

$$V = \int_a^b A(x) dx. \tag{1.60}$$

If the solid is bounded by the planes $y = c$ and $y = d$, then volume of the solid can be written as

$$V = \int_c^d A(y) dy$$

where $A(y)$ is the cross-sectional area.

Example 1.33 The cross sections of a certain solid made by planes perpendicular to the x-axis are circles with diameters extending from the curve $y = 3x^2$ to the curve $y = 16 - x^2$. Find the volume of the solid which lies between the points of intersection of these curves.

Solution At the points of intersection of the curves, we have $3x^2 = 16 - x^2$, or $x^2 = 4$, that is $x = \pm 2$. Therefore, the points of intersection of the curves are $(-2, 12)$ and $(2, 12)$ (Fig. 1.9).

Any point on the curve $y = 16 - x^2$ is $R(x, 16 - x^2)$.

Any point on the curve $y = 3x^2$ is $S(x, 3x^2)$.

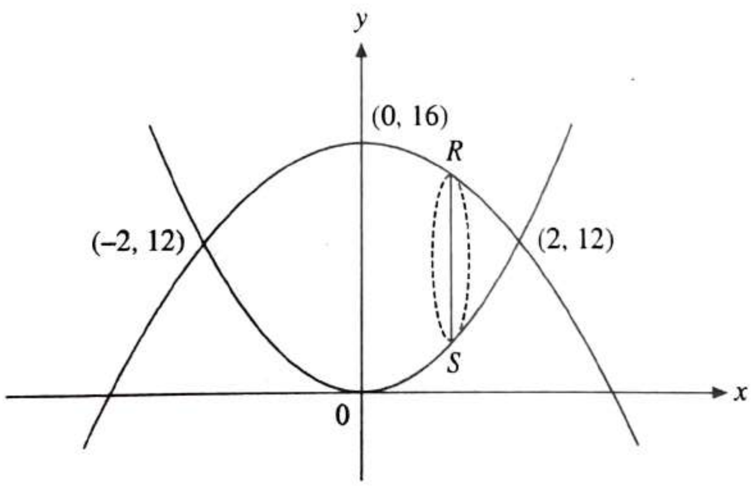


Fig. 1.9. Problem 1.33.

Diameter of the circle = $RS = 16 - 4x^2$.

Area of the circle = $A(x) = \frac{\pi}{4}(RS)^2 = 4\pi(4 - x^2)^2$.

Since the solid is symmetric about the y-axis, the required volume is obtained as

$$V = 2\int_0^2 A(x) dx = 8\pi \int_0^2 (4 - x^2)^2\, dx = 8\pi \left[16x - \frac{8}{3}x^3 + \frac{x^5}{5}\right]_0^2$$

$$= 8\pi\left[32 - \frac{64}{3} + \frac{32}{5}\right] = \frac{2048}{15}\pi \text{ cubic units.}$$

Volume of a solid of revolution

Let AB be the portion of a curve $y = f(x), f(x) > 0$, between $x = a$ and $x = b$. Consider the area bounded by the arc AB of the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$. A solid is generated by revolving this area about the x-axis (Fig 1.10).

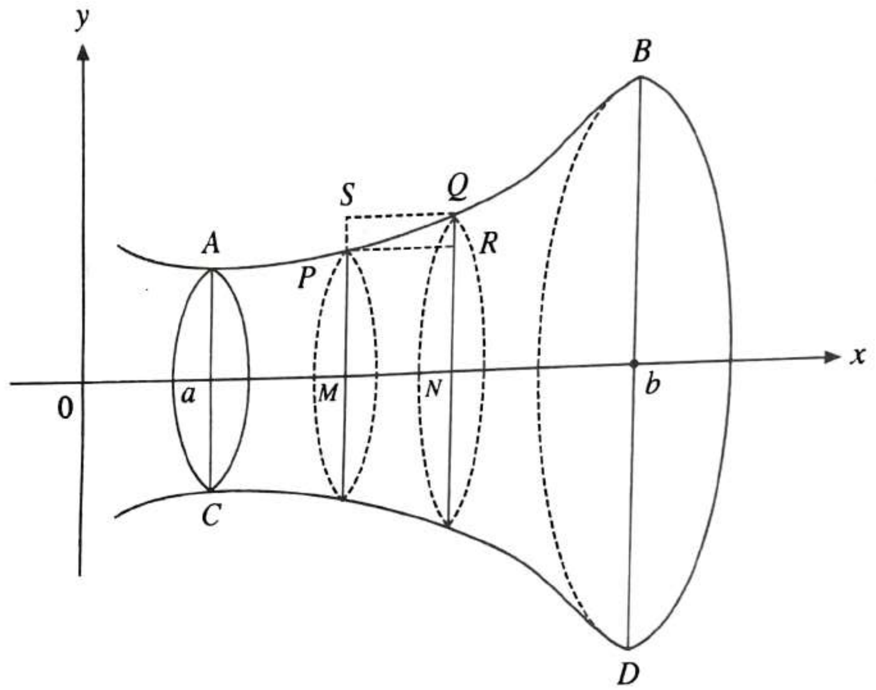


Fig. 1.10. Solid of revolution.

Divide the arc AB into n parts by considering the subintervals $[x_0, x_1]$, $[x_1, x_2]$, . . ., $[x_{n-1}, x_n]$, where $a = x_0 < x_1 < x_2 \ldots < x_n = b$. Let $\Delta x_k = x_k - x_{k-1}$, $k = 1, 2, \ldots, n$. Consider a typical subinterval $[x_{k-1}, x_k]$ of length $\Delta x_k = MN$. A solid is generated by rotating the area $MNQP$ about the x-axis. The volume V_k of this solid lies in magnitude between the volumes generated by revolving the rectangular areas $MNRP$ and $MNQS$ about the x-axis. Now,

$$MP = y_{k-1} = f(x_{k-1}) \quad \text{and} \quad NQ = y_k = f(x_k).$$

Hence, the volume V_k of the typical solid is bounded as

$$\pi y_{k-1}^2 \, \Delta x_k \le V_k \le \pi y_k^2 \, \Delta x_k .$$

Adding the inequalities corresponding to all the subintervals, we get

$$\pi \sum_{k=1}^{n} y_{k-1}^2 \, \Delta x_k \le \sum_{k=1}^{n} V_k \le \pi \sum_{k=1}^{n} y_k^2 \, \Delta x_k .$$

Let $n \to \infty$ such that $\max \Delta x_k \to 0$. In the limit, we obtain the volume of the solid of revolution as

$$V = \int_a^b \pi y^2 dx. \tag{1.61}$$

Similarly, if the area bounded by the arc AB of the curve $x = \phi(y)$, the y-axis, and the lines $y = c$ and $y = d$ is revolved about the y-axis, then the volume of the solid of revolution can be written as

$$V = \int_c^d \pi x^2 dy. \tag{1.62}$$

Remark 7

(a) If the area bounded by the curve $y = f(x)$, the line $y = p$ and the lines $x = a$, $x = b$ is revolved about the line $y = p$ (a line parallel to the x-axis), then the volume of the solid of revolution is given by

$$V = \pi \int_a^b (y - p)^2 dx. \tag{1.63}$$

(b) If the area bounded by the curve $x = g(y)$, the line $x = q$ and the lines $y = c$, $y = d$ is revolved about the line $x = q$ (a line parallel to the y-axis), then the volume of solid of revolution is given by

$$V = \pi \int_c^d (x - q)^2 dy. \tag{1.64}$$

Example 1.34 Find the volume of the solid generated by revolving the finite region bounded by the curves $y = x^2 + 1$, $y = 5$ about the line $x = 3$.

Solution The required region is given in Fig. 1.11.

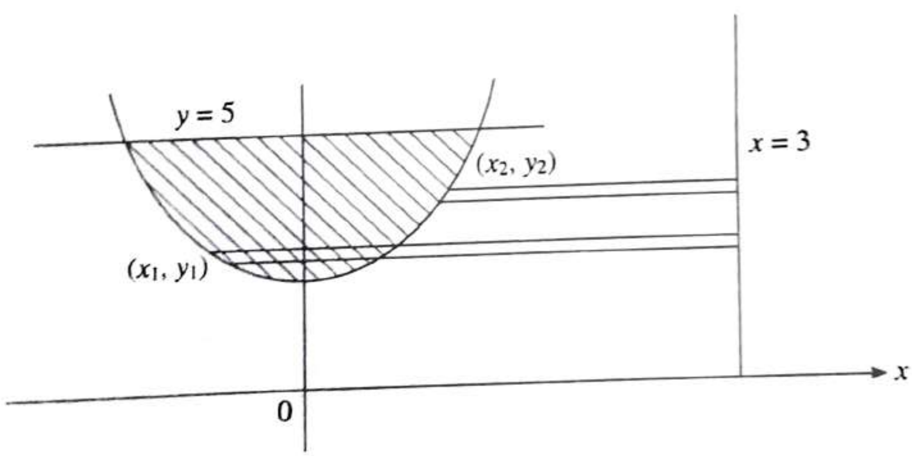


Fig. 1.11. Region of revolution in Example 1.34.

The volume is given by

$$V = \pi \int_1^5 (x_1^2 - x_2^2) dy = \pi \int_1^5 \left[(3 + \sqrt{y-1})^2 - (3 - \sqrt{y-1})^2\right] dy$$

$$= 12\pi \int_1^5 \sqrt{y-1}\, dy = 12\pi \left(\frac{2}{3}\right) [(y-1)^{3/2}]_1^5 = 8\pi(8) = 64\pi \text{ cubic units.}$$

Example 1.35 Find the volume of the solid generated by revolving the region bounded by the curves $y = 3 - x^2$ and $y = -1$ about the line $y = -1$.

Solution The required region is given in Fig. 1.12. The region PAQ is revolved about the line $y = -1$. Since the region is symmetrical about the y-axis, the volume is obtained as

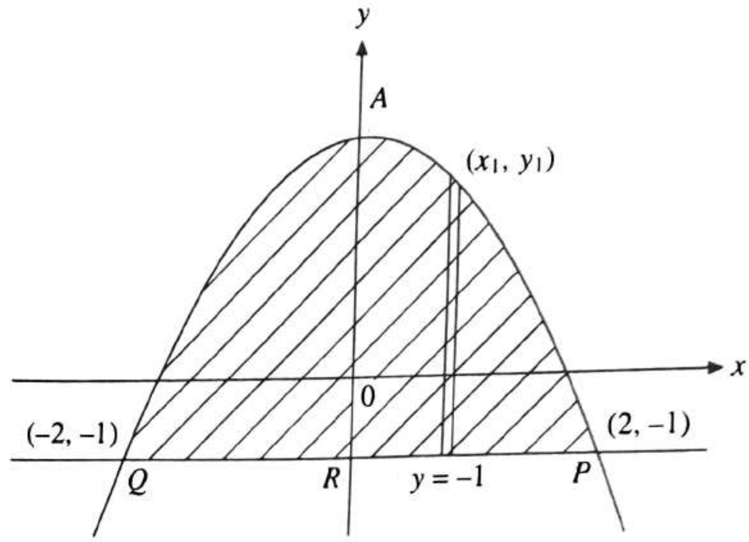


Fig. 1.12. Example 1.35.

$V = 2$ [volume of the solid obtained by revolving the region PAR about the line $y = -1$]

$$= 2\pi \int_0^2 (1 + y)^2\, dx = 2\pi \int_0^2 (1 + 3 - x^2)^2\, dx$$

$$= 2\pi \int_0^2 (16 - 8x^2 + x^4)dx = 2\pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2 = \frac{512}{15}\pi \text{ cubic units.}$$

Example 1.36 Find the volume of the solid generated by revolving an arch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ and x-axis about the x-axis.

Solution Setting $y = 0$, we obtain $\cos t = 1$, or $t = 0$, and $t = 2\pi$. Hence, one arch of the cycloid intersects the x-axis at the points (0, 0) and $(2\pi a, 0)$. Therefore, the required volume is given by

$$V = \pi \int_0^{2\pi a} y^2 dx = \pi \int_0^{2\pi} a^2 (1 - \cos t)^2\ [a(1 - \cos t)]dt$$

$$= \pi a^3 \int_0^{2\pi} 8 \sin^6\left(\frac{t}{2}\right) dt = 16\pi a^3 \int_0^{\pi} \sin^6 T\, dT$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^6 T\, dT = 32\pi a^3 \left[\frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right] = 5\pi^2 a^3 \text{ cubic units.}$$

Volume of solid of revolution by the method of cylindrical shells

Suppose that a region in the x-y plane bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$, $x = b$ is revolved about the y-axis. Divide the interval $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, . . ., $[x_{n-1}, x_n]$, where $a = x_0 < x_1 < x_2 \ldots < x_n = b$. Let $\Delta x_k = x_k - x_{k-1}$, $k = 1, 2, \ldots, n$. Consider a typical

region $MNQP$ (Fig. 1.13), where $MN = \Delta x_k$ and the coordinates of M, N are $M(x_{k-1}, 0)$ and $N(x_k, 0)$. When we revolve this strip about the y-axis, it generates a hollow thin walled shell of inner radius x_{k-1} and outer radius x_k and volume ΔV_k. The base of this shell is a ring bounded by the concentric circles with inner radius x_{k-1} and outer radius $x_k = x_{k-1} + \Delta x_k$. A cross-section of this solid is given in Fig. 1.14.

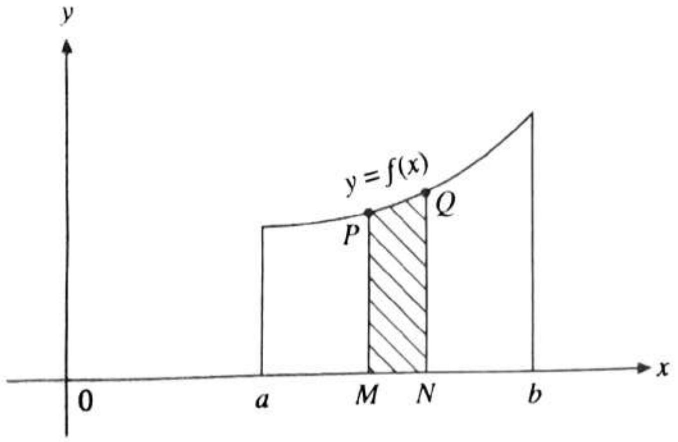


Fig. 1.13. Region of revolution.

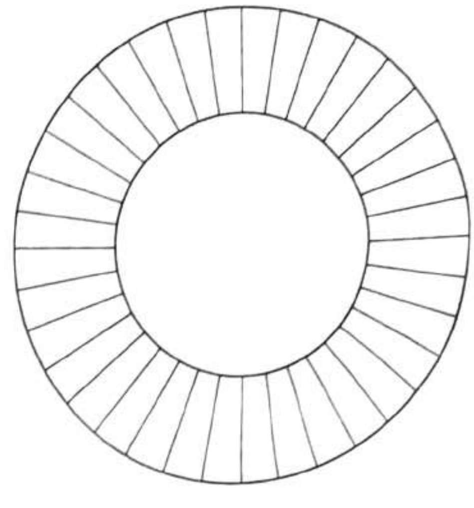


Fig. 1.14. Cross-section of solid of revolution.

The area of this ring is given by

$$\Delta A_k = \pi x_k^2 - \pi x_{k-1}^2 = \pi(x_k + x_{k-1})(x_k - x_{k-1}) = 2\pi\, \xi_k\, \Delta x_k$$

where $\xi_k = (x_k + x_{k-1})/2$ is the radius of the circle midway between the inner and outer boundaries of the ring and $2\pi\, \xi_k$ is its circumference. Now, if we take a cylindrical shell of constant height y standing on this base, we obtain volume as $\Delta V_k = (\Delta A_k)y$. Since $f(x)$ is continuous, y can take any value between the minimum and maximum values of $f(x)$ on $[x_{k-1}, x_k]$. If we take $y = f(\eta_k)$, $x_{k-1} \le \eta_k \le x_k$, then we can write approximately the volume of the shell as

$$\Delta V_k = (2\pi\, \xi_k)\, f(\eta_k)\, \Delta x_k,\ x = 1, 2, \ldots, n.$$

Adding the volumes corresponding to all the subintervals, we obtain

$$V_n = \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} (2\pi\, \xi_k) f(\eta_k)\, \Delta x_k.$$

Let $n \to \infty$ such that $\max \Delta x_k \to 0$. In the limit, we obtain the volume of the solid as

$$V = \int_a^b 2\pi x f(x) dx. \tag{1.65}$$

If the region given in Fig. 1.15 is revolved about the x- axis, the volume of the solid of revolution is obtained as

$$V = \int_c^d 2\pi y\, g(y) dy \tag{1.66}$$

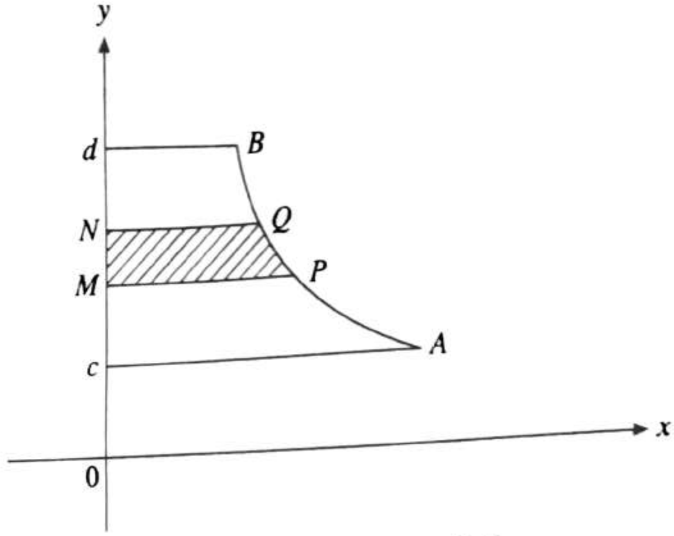


Fig. 1.15. Region of revolution.

where $x = g(y)$ is the equation of the bounding curve $APQB$.

Example 1.37 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 9$ about the y-axis.

Solution The region is plotted in Fig. 1.16. When we revolve the vertical strip of the area between the lines at distances x and $x + \Delta x$ from the y-axis, we generate a cylindrical shell of inner circumference $2\pi x$, inner radius x, inner height y and wall thickness Δx (Fig. 1.16(a)). We obtain the volume as

$$V = \int_0^9 2\pi xy\, dx = 2\pi \int_0^9 x\sqrt{x}\, dx = 2\pi\left(\frac{2}{5}\right)[x^{5/2}]_0^9 = \frac{972\,\pi}{5} \text{ cubic units.}$$

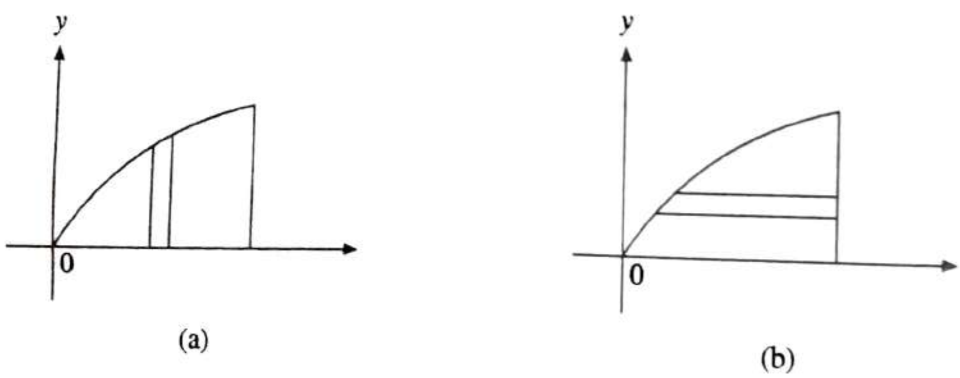


Fig. 1.16. Example 1.37.

Alternative If we revolve the horizontal strip about the y-axis (Fig. 1.16(b)), we obtain the volume as

$$V = \int_0^3 \pi x_2^2\, dy - \int_0^3 \pi x_1^2\, dy$$

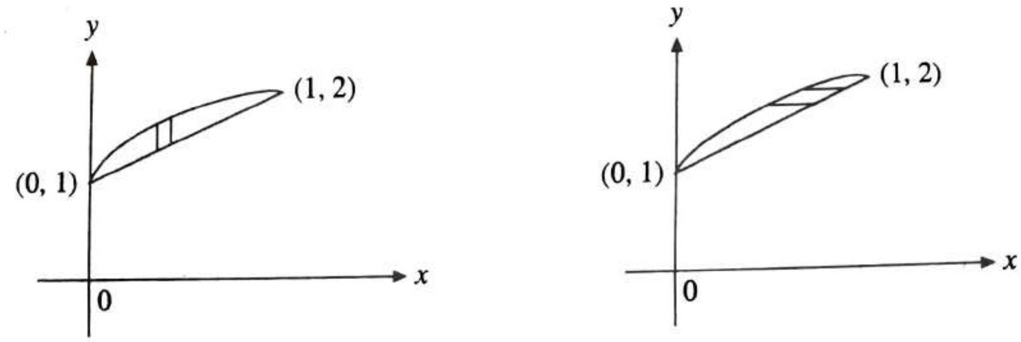
where $x_2 = 9$, $x_1 = y^2$. Therefore,

$$V = \pi\int_0^3 81\, dy - \pi\int_0^3 y^4 dy = \pi\left[243 - \frac{243}{5}\right] = \frac{972\pi}{5} \text{ cubic units.}$$

Example 1.38 Find the volume of the solid generated by revolving the region bounded by the curves $y = 1 + \sqrt{x}$ and $y = 1 + x$ about the y-axis.

Solution The curves intersect when $1 + x = 1 + \sqrt{x}$, or when $x = 0$ and $x = 1$. The points of intersection are (0, 1) and (1, 2). The region is plotted in Fig. 1.17. When we revolve the vertical strip of the area between the lines at distances x and $x + \Delta x$ from the y-axis, we generate a cylindrical shell of inner circumference $2\pi x$, inner radius x, inner height $y^* = (1 + \sqrt{x}) - (1 + x) = \sqrt{x} - x$ and wall thickness Δx (Fig. 1.17(a)). We obtain the volume as

$$V = \int_0^1 2\pi x\, y^*\, dx = 2\pi \int_0^1 x(\sqrt{x} - x)dx = 2\pi \left[\frac{x^{5/2}}{5/2} - \frac{x^3}{3}\right]_0^1 = \frac{2\pi}{15} \text{ cubic units.}$$



(a) (b)

Fig. 1.17. Example 1.38.

Alternative If we revolve the horizontal strip about the y-axis (Fig. 1.17(b)), we obtain the volume as

$$V = \pi \int_1^2 x_1^2\, dy - \pi \int_1^2 x_2^2\, dy$$

where $x_1 = y - 1$ and $x_2 = (y - 1)^2$. Therefore,

$$V = \pi \int_1^2 [(y-1)^2 - (y-1)^4]dy = \pi \left[\frac{(y-1)^3}{3} - \frac{(y-1)^5}{5}\right]_1^2 = \pi\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{2\pi}{15} \text{ cubic units.}$$

1.4.6 Surface Area of a Solid of Revolution

Let $y = f(x), f(x) \geq 0$ between $x = a$ and $x = b$ define a curve. Let this curve be revolved about the x-axis to generate a surface S (Fig. 1.18). Divide the interval $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2], \ldots, [x_{n-1}, x_n]$, where $a = x_0 < x_1 < x_2 \ldots < x_n = b$. Let $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1} = f(x_k) - f(x_{k-1}), x = 1, 2, \ldots, n$. Consider the portion of the curve PQ in the interval $[x_{k-1}, x_k]$. Let S_k be the area of the surface generated by revolving this portion of curve about the x-axis. In this interval $[x_{k-1}, x_k]$, we approximate arc $(PQ) \approx$ chord (PQ). If we revolve the chord PQ about the x-axis, we obtain a frustum of a cone (Fig. 1.19). Now, the area of the surface S_k is approximated by the area of the surface of the frustum of the cone. We have

$$PM = y_{k-1},\ QN = y_k,\ PQ = l = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

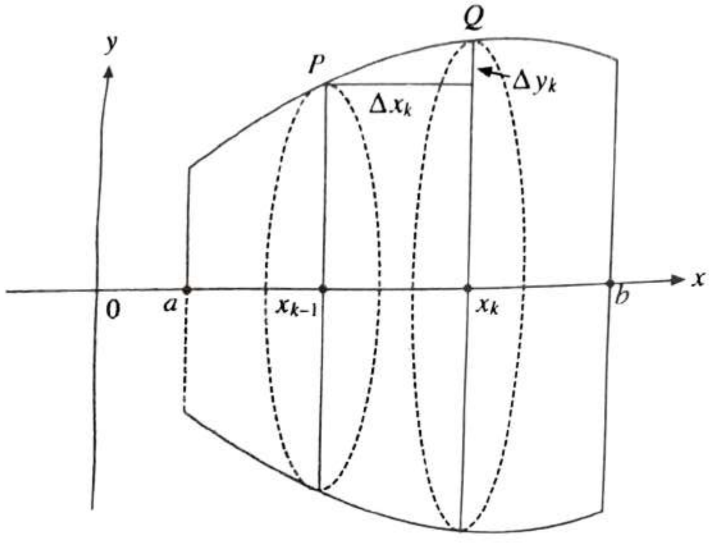


Fig. 1.18. Surface of revolution.

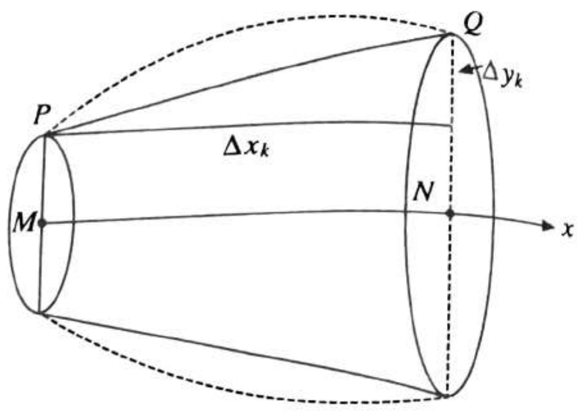


Fig. 1.19. Surface of revolution.

and
$$S_k \approx \pi\,(y_{k-1} + y_k)l = \pi(y_{k-1} + y_k)\ \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$
$$= \pi(y_{k-1} + y_k)\sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\ \Delta x_k$$

Adding the approximations corresponding to each of the subintervals $[x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$, we obtain

$$S \approx \sum_{k=1}^{n} S_k = \sum_{k=1}^{n} \pi(y_{k-1} + y_k)\sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\ \Delta x_k.$$

Let $n \to \infty$ such that $\max \Delta x_k \to 0$. In the limit, we obtain the surface area of the solid of revolution as

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi\, y\, ds \tag{1.67}$$

where $ds = \sqrt{1 + (dy/dx)^2}\,dx$.

If the given region is revolved about the y-axis, then we obtain the surface area as

$$S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d 2\pi\, x\, ds \tag{1.68}$$

where $ds = \sqrt{1 + (dx/dy)^2}\,dy$.

If the curve is given in parametric form as $x = \phi(t)$, $y = \psi(t)$, $t_0 \le t \le t_1$, then we have

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt. \tag{1.69}$$

If the curve is given in polar form $r = f(\theta)$, $\theta_0 \le \theta \le \theta_1$, then we have

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{1.70}$$

Example 1.39 Find the surface area of the solid generated by revolving the circle $x^2 + (y - b)^2 = a^2$, $b \ge a$ about the x-axis.

Solution The equation of the circle can be written in parametric form as

$$x = a\cos t,\ y = b + a\sin t,\ 0 \le t \le 2\pi.$$

We obtain

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = a.$$

Therefore,

$$S = \int_0^{2\pi} 2\pi(b + a\sin t)a\, dt = 2\pi a\left[bt - a\cos t\right]_0^{2\pi} = 4\,\pi^2 ab \text{ square units.}$$

Example 1.40 The part of the lemniscate $r^2 = 2a^2\cos 2\theta$, $0 \le \theta \le \pi/4$, is revolved about the x-axis. Find the surface area of the solid generated.

Solution We have $x = r\cos\theta$, and $y = r\sin\theta$. We find that

$$ds^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = r^2 + \frac{4a^4\sin^2 2\theta}{r^2} = \frac{1}{r^2}\left[4a^4\cos^2 2\theta + 4a^4\sin^2 2\theta\right] = \frac{4a^4}{r^2}$$

Therefore, $$s = \int_0^{\pi/4} 2\pi y\, ds = \int_0^{\pi/4} 2\pi r\sin\theta\left(\frac{2a^2}{r}\right)d\theta$$

$$= 4\pi a^2\left[-\cos\theta\right]_0^{\pi/4} = 4\pi a^2\left[1 - \frac{1}{\sqrt{2}}\right] = 2\pi a^2(2 - \sqrt{2}) \text{ square units.}$$

Example 1.41 The line segment $x = \sin^2 t$, $y = \cos^2 t$, $0 \le t \le \pi/2$ is revolved about the y-axis. Find the surface area of the solid generated.

Solution The surface area is given by

$$S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

We have

$$\frac{dx}{dt} = 2\sin t\cos t,\ \frac{dy}{dt} = -2a\cos t\sin t \quad \text{and} \quad \frac{dx}{dy} = -1.$$

As t varies from 0 to $\pi/2$, y varies from 1 to 0. Therefore,

$$S = \int_0^1 2\sqrt{2}\pi x\, dy = -\int_0^{\pi/2} 2\sqrt{2}\sin^2 t\,(-2\sin t\cos t)dt$$

$$= 4\sqrt{2}\,\pi \int_0^{\pi/2} \sin^3 t \cos t \, dt = 4\sqrt{2}\pi \left[\frac{1}{4}\sin^4 t\right]_0^{\pi/2} = \pi\sqrt{2} \text{ square units.}$$

Exercise 1.3

In problems **1** to **9**, find the area of the region bounded by the given curves.

1. $y = x^2 - 5x + 6$, the x-axis and the lines $x = 0$, $x = 3$.

2. $y = x$, $y = \sqrt{x}$ and the lines $x = 0$, $x = 1$.

3. $y^2 = x + 1$ and $y = x + 1$.

4. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, $0 < a < b$.

5. $\sqrt{x} + \sqrt{y} = 1$ and the coordinate axes.

6. $y = ex \ln x$ and $y = \ln x/(ex)$.

7. $x = 2t + 1$, $y = 4t^2 - 1$, $-1/2 \le t \le 1/2$ and the x-axis.

8. $x = a\cos^3 t$, $y = b\sin^3 t$, $0 \le t \le 2\pi$.

9. $r = a(2 - \cos 2\theta)$, $0 \le \theta \le 2\pi$.

10. Find the area of the region enclosed between the curve $y = 2x^4 - x^2$, the x-axis and the ordinates of the points where the curve has local minimum.

11. Find the area of a loop of the curve $x(x^2 + y^2) = a(x^2 - y^2)$.

12. Find the area of the region inside the curve $r^2 = 2a^2 \cos 2\theta$ and outside the circle $r = a$.

13. Find the area that is inside the circle $r = a$ and outside the cardioid $r = a(1 - \cos\theta)$.

In problems **14** to **26**, find the length of the indicated portion of the curve.

14. $9x^2 = y^3$, from $x = 0$ to $x = 9$.

15. $x^{2/3} + y^{2/3} = a^{2/3}$, from $x = 0$ to $x = a$ in the first quadrant.

16. $x = \dfrac{1}{4}y^4 + \dfrac{1}{8}y^{-2}$, from $y = 1$ to $y = 2$.

17. $x^2 + y^2 - 2ax = 0$ and above the line $y = a/2$, $a > 0$.

18. $y = \int_0^x \sqrt{\cos t}\, dt$, from $x = 0$ to $x = \pi/2$.

19. $y = \ln[(e^x + 1)/(e^x - 1)]$, from $x = 1$ to $x = 2$.

20. $x = 3at^2$, $y = a(t - 3t^3)$, from $t = 0$ to $t = 1$.

21. $x = a(t - \sin t)$, $y = a(1 - \cos t)$, from $t = 0$ to $t = 2\pi$.

22. $x = e^{2t}\cos t$, $y = e^{2t}\sin t$, from $t = 0$ to $t = 1$.

23. $x = [\ln(a^2 + t^2)]/2$, $y = \tan^{-1}(t/a)$, from $t = 0$ to $t = a$.

24. $x = 2\cos t + \cos 2t + 1$, $y = 2\sin t + \sin 2t$, from $t = 0$ to $t = \pi$.

25. $r = a\theta$, from $r = r_1$ to $r = r_2$.

26. $r = ae^{b\theta}$, from $r = r_1$ to $r = r_2$.

27. Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

28. The base of a certain solid is the circle $x^2 + y^2 = a^2$. Each cross-section of the solid cut out by a plane perpendicular to the x-axis is a square with one side of the square in the base of the solid. Find the volume of the solid.

29. The base of a certain solid is the circle $x^2 + y^2 = a^2$. Each cross-section of the solid cut out by a plane perpendicular to the x-axis is an isosceles right triangle with one of the equal sides in the base of the solid. Find its volume.

30. The base of a certain solid is the region between the x-axis and the curve $y = \cos x$ between $x = 0$ and

$x = \pi/2$. Each cross-section of the solid cut out by a plane perpendicular to the x-axis is an equilateral triangle with one side in the plane of the solid. Find its volume.

In problems **31** to **36**, find the volume of the solid of revolution generated by revolving the specified region about the given axis.

31. Region bounded by $y = \cos x$, $y = 0$ from $x = 0$ to $x = \pi/2$ about the x-axis.

32. Region bounded by $y = \sqrt{x}$, $y = 0$ from $x = 0$ to $x = 4$ about the x-axis.

33. Region bounded by $y = \sqrt{x}$, $y = 0$ from $x = 0$ to $x = 4$ about the line $y = 2$.

34. Region bounded by $y = x^2 + 1$ and $y = 3 - x$ about the x-axis.

35. Region bounded by $x = a\sin^3 t$, $y = a\cos^3 t$, $0 \le t \le \pi/2$, $x = 0$, $y = 0$ about the x-axis.

36. Region bounded by $x = 2t + 1$, $y = 4t^2 - 1$, $-1/2 \le t \le 0$, $y = 0$ about the line $x = 1$.

In problems **37** to **41**, use the method of cylindrical shells to find the volume of the solid generated by revolving the specified region about the given axis.

37. Region bounded by $y = x$, $y = 2$ and $x = 0$ about the y-axis.

38. Region bounded by $y = 2x - x^2$ and $y = x$ about the y-axis.

39. Region bounded by $y = x^2$ and $y = x$ about the y-axis.

40. Region inside the triangle with vertices at $(0, 0)$, $(a, 0)$ and $(0, b)$ about the y-axis.

41. Region inside the circle $x^2 + y^2 = a^2$ about the line $y = b$, $b > a > 0$.

In problems **42** to **50**, find the surface area of the solid generated by revolving the curve C about the given line.

42. $(x - b)^2 + y^2 = a^2$, $b \ge a$ about the y-axis.

43. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a \ge b$, $y \ge 0$ about the x-axis.

44. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a \ge b$, $x \ge 0$ about the y-axis.

45. $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$, $1 \le x \le 2$ about the line $y = -1$.

46. $x = \dfrac{y^3}{3} + \dfrac{1}{4y}$, $1 \le y \le 2$ about the line $x = -1$.

47. $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \le t \le 2\pi$ about the x-axis.

48. $x = a\cos^3 t$, $y = a\sin^3 t$, $0 \le t \le \pi/2$ about the x-axis.

49. $x = e^t\cos t$, $y = e^t\sin t$, $0 \le t \le \pi/2$ about the y-axis.

50. $r = a(1 + \cos\theta)$, $0 \le \theta \le \pi$ about the initial line.

1.5 Improper Integrals

While defining the definite integral $\int_a^b f(x)dx$, we had assumed that

(i) a and b are finite constants.
(ii) $f(x)$ is bounded for all x in $[a, b]$.

If in the above integral, (i) a or b or both a and b are infinite, or (ii) a, b are finite but $f(x)$ becomes infinite at $x = a$ or $x = b$ or at one or more points within the interval (a, b), then the definite integral is respectively called

(i) *improper integral of the first kind.*
(ii) *improper integral of the second kind.*

To define the improper integrals, we assume the following:

(i) The integrand $f(x)$ is of the same sign within its range of integration. Without any loss of generality, we assume that $f(x) \geq 0$ (when $f(x) \leq 0$, we can write $g(x) = -f(x)$ so that $g(x) \geq 0$). We shall discuss later, the case when $f(x)$ changes sign within its range of integration.

(ii) $f(x)$ is continuous over each finite subinterval $[\alpha, \beta]$ contained in the range of integration. Hence, there exists a positive constant K independent of α and β such that

$$\int_{\alpha}^{\beta} f(x)dx < K.$$

The improper integrals are evaluated by a limiting process.

1.5.1 Improper Integrals of the First Kind (Range of Integration is Infinite)

We shall now discuss methods to evaluate improper integrals of the form

$$\text{(i)} \int_{a}^{\infty} f(x)dx, \quad \text{(ii)} \int_{-\infty}^{b} f(x)dx, \quad \text{and} \quad \text{(iii)} \int_{-\infty}^{\infty} f(x)dx$$

if they exist. We define these improper integrals as follows:

(i)
$$\int_{a}^{\infty} f(x)dx = \lim_{p\to\infty} \int_{a}^{p} f(x)dx. \tag{1.71}$$

If the limit exists and is finite, say equal to l_1, then the improper integral converges and has the value l_1. Otherwise, the improper integral diverges.

(ii)
$$\int_{-\infty}^{b} f(x)dx = \lim_{p\to-\infty} \int_{p}^{b} f(x)dx. \tag{1.72}$$

If the limit exists and is finite, say equal to l_2, then the improper integral converges and has the value l_2. Otherwise, the improper integral diverges.

(iii)
$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty} \int_{a}^{c} f(x)dx + \lim_{b\to\infty} \int_{c}^{b} f(x)dx \tag{1.73}$$

where c is any finite constant including zero. If both the limits on the right hand side exist separately and are finite, say equal to l_3 and l_4 respectively, then the improper integral converges and has the value $l_3 + l_4$. If one or both the limits do not exist or are infinite, then the improper integral diverges.

Example 1.42 Evaluate the following improper integrals, if they exist.

$$\text{(i)} \int_{0}^{\infty} \frac{dx}{a^2 + x^2}, a > 0, \quad \text{(ii)} \int_{1}^{\infty} \frac{dx}{x\sqrt{x^2 - 1}}, \quad \text{(iii)} \int_{-\infty}^{0} e^{x} dx,$$

$$\text{(iv)} \int_{0}^{\infty} x \sin x \, dx, \qquad \text{(v)} \int_{0}^{\infty} e^{-ax} \cos px \, dx, \ a > 0, p \text{ constant.}$$

Solution

(i) $$\int_{0}^{\infty} \frac{dx}{a^2 + x^2} = \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{a^2 + x^2} = \lim_{b\to\infty} \left[\frac{1}{a} \tan^{-1}\left(\frac{b}{a}\right)\right] = \frac{\pi}{2a}.$$

Therefore, the improper integral converges to $\pi/(2a)$.

(ii) $$\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \int_1^{c} \frac{dx}{x\sqrt{x^2-1}} + \int_c^{\infty} \frac{dx}{x\sqrt{x^2-1}}.$$

$$= \lim_{\varepsilon \to 0} \int_{1+\varepsilon}^{c} \frac{dx}{x\sqrt{x^2-1}} + \lim_{b\to\infty} \int_c^{b} \frac{dx}{x\sqrt{x^2-1}} = \lim_{\varepsilon\to 0} \left[\sec^{-1} x\right]_{1+\varepsilon}^{c} + \lim_{b\to\infty} \left[\sec^{-1} x\right]_c^b$$

$$= \lim_{\varepsilon\to 0} \left[\sec^{-1} c - \sec^{-1}(1+\varepsilon)\right] + \lim_{b\to\infty} \left[\sec^{-1} b - \sec^{-1} c\right]$$

$$= \sec^{-1} c - \sec^{-1} 1 + \frac{\pi}{2} - \sec^{-1} c = \frac{\pi}{2}$$

Therefore, the improper integral converges to $\pi/2$.

(iii) $$\int_{-\infty}^{0} e^{x}\, dx = \int_0^{\infty} e^{-x}\, dx = \lim_{b\to\infty} \int_0^{b} e^{-x}\, dx = \lim_{b\to\infty} (1 - e^{-b}) = 1.$$

Therefore, the improper integral converges to 1.

(iv) $$\int_0^{\infty} x \sin x\; dx = \lim_{b\to\infty} \int_0^{b} x\sin x\, dx = \lim_{b\to\infty} (\sin b - b\cos b).$$

Since this limit does not exist, the improper integral diverges.

(v) Using the result

$$\int e^{-ax} \cos px\, dx = \frac{e^{-ax}}{a^2+p^2}(p\sin px - a\cos px),$$

we obtain

$$\int_0^{b} e^{-ax}\cos px\, dx = \left[\frac{e^{-ax}}{a^2+p^2}(p\sin px - a\cos px)\right]_0^b$$

$$= \frac{1}{a^2+p^2}\left[e^{-ab}(p\sin bp - a\cos bp) + a\right]$$

Now, $\sin bp$ and $\cos bp$ have finite values and $\lim\limits_{b\to\infty} e^{-ab} = 0$. Hence,

$$\int_0^{\infty} e^{-ax} \cos px\, dx = \lim_{b\to\infty} \int_0^{b} e^{-ax}\cos px\, dx = \frac{a}{a^2+p^2}.$$

Therefore, the improper integral converges to $a/(a^2+p^2)$.

Example 1.43 Discuss the convergence of the improper integral

$$\int_1^{\infty} \frac{dx}{x^p}$$

Solution We have

$$\int_1^{b} \frac{dx}{x^p} = \frac{1}{1-p}\left[x^{1-p}\right]_1^b = \frac{1}{1-p}\left[b^{1-p} - 1\right].$$

Now, $$\lim_{b\to\infty}\left[b^{1-p}\right] = \begin{cases} \infty, & \text{if } p < 1 \\ 0, & \text{if } p > 1. \end{cases}$$

Therefore, the improper integral converges if $p > 1$ and diverges if $p < 1$.

For $p = 1$, we have

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} \ln b.$$

Since the limit does not exist, the improper integral diverges. Hence, the given improper integral converges to $1/(p-1)$ for $p > 1$ and diverges for $p \le 1$.

Example 1.44 Discuss the convergence of the integral $\displaystyle\int_{-\infty}^{\infty} x\, e^{-x^2}\, dx$.

Solution We write

$$I = \int_{-\infty}^{\infty} x\, e^{-x^2}\, dx = \int_{-\infty}^{c} x\, e^{-x^2}\, dx + \int_{c}^{\infty} x\, e^{-x^2}\, dx$$

where c is any finite constant. We have

$$I = \lim_{a\to-\infty} \int_a^c x\, e^{-x^2}\, dx + \lim_{b\to\infty} \int_c^b x\, e^{-x^2}\, dx$$

$$= \lim_{a\to-\infty} \left[\frac{1}{2}(e^{-a^2} - e^{-c^2})\right] + \lim_{b\to\infty} \left[\frac{1}{2}(e^{-c^2} - e^{-b^2})\right]$$

$$= \frac{1}{2}(-e^{-c^2} + e^{-c^2}) = 0.$$

Therefore, the given improper integral converges to 0.

It is not always possible to study the convergence/divergence of an improper integral by evaluating it as was done in the previous examples. A simple example is the integral $\displaystyle\int_0^\infty e^{-x^2}\, dx$ which cannot be evaluated directly. We now present some results which can be used to discuss the convergence or divergence of improper integrals. In this case, we cannot find the value of the improper integral, that is the value to which it converges. However, we may be able to find a bound of the integral.

Theorem 1.11 **(Comparison Test 1)** If $0 \le f(x) \le g(x)$ for all x, then

(i) $\displaystyle\int_a^\infty f(x)dx$ converges if $\displaystyle\int_a^\infty g(x)dx$ converges.

(ii) $\displaystyle\int_a^\infty g(x)dx$ diverges if $\displaystyle\int_a^\infty f(x)dx$ diverges.

Theorem 1.12 (Comparison Test 2) Suppose that $f(x)$ and $g(x)$ are positive functions and let

$$\lim_{x\to\infty} \left[\frac{f(x)}{g(x)}\right] = L, \quad 0 < L < \infty. \tag{1.74}$$

Then, the improper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge or diverge together.

Example 1.45 Discuss the convergence of the following improper integrals

(i) $\int_1^\infty e^{-x^2}dx$, (ii) $\int_1^\infty \frac{dx}{(e^{-x}+1)x^2}$, (iii) $\int_2^\infty \frac{dx}{\ln x}$,

(iv) $\int_2^\infty \frac{dx}{x(\ln x)^p}$, (v) $\int_1^\infty \frac{x\tan^{-1}x}{\sqrt{4+x^3}}dx$.

Solution

(i) We have $e^{-x^2} < e^{-x}$ for all $x \ge 1$. Consider the improper integral $\int_1^\infty e^{-x}\,dx$.

We have $$\int_1^\infty e^{-x}dx = \lim_{b\to\infty}\int_1^b e^{-x}dx = \lim_{b\to\infty}[1 - e^{-b}] \doteq 1.$$

Therefore, the integral $\int_1^\infty e^{-x}\,dx$ is convergent. By Comparison Test 1 (i), the given integral is also convergent. Further, its value is less than 1.

(ii) Let $f(x) = \frac{1}{(e^{-x}+1)x^2}$ and $g(x) = \frac{1}{x^2}$.

Now $$\lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to\infty}\left[\frac{1}{(e^{-x}+1)x^2}\right]\left[\frac{x^2}{1}\right] = \lim_{x\to\infty}\frac{1}{e^{-x}+1} = 1.$$

Also, $\int_1^\infty g(x)dx = \int_1^\infty \frac{dx}{x^2}$ converges to 1 (see Example 1.43). Therefore, by Comparison Test 2, the given improper integral is also convergent. Its value is less than 1.

Alternative We have $\frac{1}{(e^{-x}+1)x^2} < \frac{1}{x^2}$ for all $x \ge 1$. The improper integral $\int_1^\infty \frac{dx}{x^2}$ is convergent. Therefore, by Comparison Test 1 (i), the given improper integral converges.

(iii) We have $\ln x < x$ for all $x > 0$. Hence,

$$\frac{1}{\ln x} > \frac{1}{x} \quad\text{and}\quad \int_2^\infty \frac{dx}{\ln x} > \int_2^\infty \frac{dx}{x}.$$

Let $g(x) = 1/(\ln x)$ and $f(x) = 1/x$. We have $g(x) > f(x)$. Now, the integral

$$\int_2^\infty f(x)dx = \int_2^\infty \frac{dx}{x} \text{ is divergent (see Example 1.43).}$$

Therefore, by Comparison Test 1 (ii), the integral $\int_2^\infty g(x)dx = \int_2^\infty \frac{dx}{\ln x}$ is also divergent.

(iv) Substitute $\ln x = t$. We get

$$I = \int_2^{\infty} \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\infty} \frac{dt}{t^p}$$

which is convergent for $p > 1$ and divergent for $p \le 1$ (see Example 1.43).

(v) Let $$f(x) = \frac{x \tan^{-1} x}{\sqrt{4 + x^3}} = \frac{\tan^{-1} x}{\sqrt{x}\sqrt{1 + 4x^{-3}}} \text{ and } g(x) = \frac{1}{\sqrt{x}}.$$

We find that $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\tan^{-1} x}{\sqrt{1 + 4x^{-3}}} = \frac{\pi}{2}.$$

Hence, by Comparison Test 2, the integrals $\int_1^{\infty} f(x)dx$ and $\int_1^{\infty} g(x)dx$ converge or diverge together. Now, $\int_1^{\infty} g(x)dx$ is divergent. Therefore, $\int_1^{\infty} f(x)dx$ is also divergent.

1.5.2 Improper Integral of the Second Kind

Now consider an improper integral of the form $\int_a^b f(x)dx$, where a, b are finite constants, $f(x)$ is continuous in (a, b) and has infinite discontinuity (becomes infinite) at (i) $x = a$, or (ii) $x = b$, or (iii) $x = a$ and $x = b$, or (iv) $f(x)$ is continuous in (a, b) except at $x = c$, $a < c < b$, where $f(x)$ has an infinite discontinuity.

If $f(x)$ has a finite number of points of discontinuity, $c_1, c_2, \ldots, c_m$, $a \le c_1 < c_2 < \ldots < c_m \le b$, then we write the integral as

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots + \int_{c_m}^{b} f(x)dx \tag{1.75}$$

and consider each integral on the right hand side separately.

Infinite discontinuity at $x = a$ Since the function $f(x)$ is continuous at all points except at $x = a$, the integral $\int_{a+\varepsilon}^{b} f(x)dx$ is a proper integral and exists for every ε, $0 < \varepsilon < b - a$. We evaluate the improper integral as

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0} \int_{a+\varepsilon}^{b} f(x)dx.$$

If this limit exists and is finite, say equal to l_1, then the improper integral converges to l_1. Otherwise, it diverges.

Infinite discontinuity at $x = b$ Since the function $f(x)$ is continuous at all points except at $x = b$, the integral $\int_a^{b-\varepsilon} f(x)dx$ is a proper integral and exists for every ε, $0 < \varepsilon < b - a$. We evaluate the improper integral as

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0} \int_a^{b-\varepsilon} f(x)dx.$$

Infinite discontinuity at $x = a$ and $x = b$ We write the improper integral as

$$\int_a^b f(x)dx = \int_a^{\alpha} f(x)dx + \int_{\alpha}^b f(x)dx$$

where α is any finite constant between a and b at which f is defined. We evaluate the improper integral as

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{\alpha} f(x)dx + \lim_{\xi \to 0} \int_{\alpha}^{b-\xi} f(x)dx.$$

If both the limits exist and are finite, then the improper integral converges. Otherwise, it diverges.

Infinite discontinuity at $x = c$, $a < c < b$ We write the improper integral as

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)dx + \lim_{\xi \to 0} \int_{c+\xi}^b f(x)dx.$$

The given improper integral converges, if both the integrals on the right hand side converge. If one or both the integrals on the right hand side diverge, then the given improper integral diverges.

The following tests can be used to discuss the convergence or divergence of the above improper integrals. In this case, we cannot find the value of the improper integral, that is the value to which it converges. However, we may be able to find a bound of the integral.

Theorem 1.13 (Comparison Test 3) If $0 \le f(x) \le g(x)$ for all x in $[a, b]$, then

(i) $\int_a^b f(x)dx$ converges if $\int_a^b g(x)dx$ converges.

(ii) $\int_a^b g(x)dx$ diverges if $\int_a^b f(x)dx$ diverges.

Theorem 1.14 (Comparison Test 4) If $f(x)$ and $g(x)$ are two positive functions and

(i) a is a point of infinite discontinuity such that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)} = l_1,\ 0 < l_1 < \infty$$

or (ii) b is a point of infinite discontinuity such that

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(b-h)}{g(b-h)} = l_2,\ 0 < l_2 < \infty$$

then, the improper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or diverge together.

Example 1.46 Evaluate the following improper integrals, if they exist:

(i) $\int_0^4 \frac{dx}{\sqrt{x}}$, (ii) $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$,

(iii) $\displaystyle\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx$ (iv) $\displaystyle\int_{0}^{3} \frac{dx}{3x-x^2}$,

(v) $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$, (vi) $\displaystyle\int_{0}^{3} \frac{dx}{x^2-3x+2}$.

Solution

(i) $$\int_0^4 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{4} \frac{dx}{\sqrt{x}} = 2\lim_{\varepsilon\to 0}[2-\sqrt{\varepsilon}] = 4.$$

Therefore, the improper integral converges to 4.

(ii) $$\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon\to 0}\int_0^{2-\varepsilon} \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon\to 0}\sin^{-1}\left(1-\frac{\varepsilon}{2}\right) = \sin^{-1}1 = \frac{\pi}{2}.$$

Therefore, the improper integral converges to $\pi/2$.

(iii) $$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,dx = \lim_{\varepsilon\to 0}\int_{-1}^{1-\varepsilon}\sqrt{\frac{1+x}{1-x}}\,dx = \lim_{\varepsilon\to 0}\left[\int_{-1}^{1-\varepsilon}\frac{dx}{\sqrt{1-x^2}} - \frac{1}{2}\int_{-1}^{1-\varepsilon}\frac{-2x}{\sqrt{1-x^2}}\,dx\right]$$

$$= \lim_{\varepsilon\to 0}[\{\sin^{-1}(1-\varepsilon) - \sin^{-1}(-1)\} - \{\sqrt{1-(1-\varepsilon)^2} - \sqrt{1-1}\}]$$

$$= \sin^{-1}(1) - \sin^{-1}(-1) = 2\sin^{-1}(1) = \pi.$$

Therefore, the improper integral converges to π.

(iv) Here, the integrand $f(x)$ has infinite discontinuity, at both the end points $x = 0$ and $x = 3$. We take any point, say $x = c$, inside the interval of integration, at which $f(x)$ is defined. We write

$$\int_0^3 \frac{dx}{3x-x^2} = \int_0^c \frac{dx}{3x-x^2} + \int_c^3 \frac{dx}{3x-x^2} = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{c}\frac{dx}{x(3-x)} + \lim_{\xi\to 0}\int_c^{3-\xi}\frac{dx}{x(3-x)}$$

$$= \frac{1}{3}\lim_{\varepsilon\to 0}\left[\ln\left(\frac{x}{3-x}\right)\right]_{\varepsilon}^{c} + \frac{1}{3}\lim_{\xi\to 0}\left[\ln\left(\frac{x}{3-x}\right)\right]_{c}^{3-\xi}$$

$$= \frac{1}{3}\lim_{\varepsilon\to 0}\left[\ln\left(\frac{c}{3-c}\right) - \ln\left(\frac{\varepsilon}{3-\varepsilon}\right)\right] + \frac{1}{3}\lim_{\xi\to 0}\left[\ln\left(\frac{3-\xi}{\xi}\right) - \ln\left(\frac{c}{3-c}\right)\right]$$

Since the limits do not exist, the improper integral diverges.

(v) The integrand has infinite discontinuity at $x = 0$ which lies inside the interval of integration. We write

$$\int_{-1}^{1}\frac{dx}{x^2} = \int_{-1}^{0}\frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon}\frac{dx}{x^2} + \lim_{\xi\to 0}\int_{\xi}^{1}\frac{dx}{x^2}$$

$$= \lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon} - 1\right] + \lim_{\xi\to 0}\left[\frac{1}{\xi} - 1\right] \to \infty.$$

Therefore, the improper integral diverges.

(vi) The integrand has infinite discontinuities at $x = 1$ and $x = 2$, both of which lie inside the interval of integration. We write

$$\int_0^3 \frac{dx}{x^2 - 3x + 2} = \int_0^1 \frac{dx}{(x-1)(x-2)} + \int_1^2 \frac{dx}{(x-1)(x-2)} + \int_2^3 \frac{dx}{(x-1)(x-2)}$$

$$= I_1 + I_2 + I_3.$$

We find that

(a) the integrand in I_1 has infinite discontinuity at $x = 1$,

(b) the integrand $f(x)$ in I_2 has infinite discontinuity at both the end points $x = 1$ and $x = 2$. We take any point, say $x = c$ inside the limits of integration, at which $f(x)$ is defined. We also find that $f(x) < 0$ when $1 < x < 2$. We write $g(x) = -f(x)$ so that $g(x) > 0$ when $1 < x < 2$. Therefore, we can write

$$I_2 = -\int_1^c \frac{dx}{(x-1)(2-x)} - \int_c^2 \frac{dx}{(x-1)(2-x)}$$

(c) the integrand in I_3 has infinite discontinuity at $x = 2$.

Hence, we can write

$$\int_0^3 \frac{dx}{x^2 - 3x + 2} = \lim_{\varepsilon_1 \to 0} \int_0^{1-\varepsilon_1} \frac{dx}{(x-1)(x-2)} - \lim_{\varepsilon_2 \to 0} \int_{1+\varepsilon_2}^{c} \frac{dx}{(x-1)(2-x)}$$

$$- \lim_{\varepsilon_3 \to 0} \int_c^{2-\varepsilon_3} \frac{dx}{(x-1)(2-x)} + \lim_{\varepsilon_4 \to 0} \int_{2+\varepsilon_4}^{3} \frac{dx}{(x-1)(x-2)}$$

$$= \lim_{\varepsilon_1 \to 0} \left[\ln\left(\frac{\varepsilon_1 + 1}{\varepsilon_1}\right) - \ln 2 \right] - \lim_{\varepsilon_2 \to 0} \left[\ln\left(\frac{c-1}{2-c}\right) - \ln\left(\frac{\varepsilon_2}{1-\varepsilon_2}\right) \right]$$

$$- \lim_{\varepsilon_3 \to 0} \left[\ln\left(\frac{1-\varepsilon_3}{\varepsilon_3}\right) - \ln\left(\frac{c-1}{2-c}\right) \right] + \lim_{\varepsilon_4 \to 0} \left[\ln\left(\frac{1}{2}\right) - \ln\left(\frac{\varepsilon_4}{\varepsilon_4 + 1}\right) \right]$$

Since the limits do not exist, the improper integral diverges.

Note that the improper integral I_1 diverges. We could have concluded that the improper integral diverges without discussing the convergence/divergence of I_2 and I_3.

Example 1.47 Discuss the convergence of the improper integral $\displaystyle\int_a^b \frac{dx}{(x-a)^p}$, $p > 0$.

Solution The integrand has infinite discontinuity at $x = a$. We write

$$\int_a^b \frac{dx}{(x-a)^p} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^p} = \frac{1}{1-p} \lim_{\varepsilon \to 0} \left[\frac{1}{(b-a)^{p-1}} - \frac{1}{\varepsilon^{p-1}} \right]$$

$$= \begin{cases} 1/[(1-p)(b-a)^{p-1}], & \text{if } p < 1 \\ \infty, & \text{if } p > 1. \end{cases}$$

For $p = 1$, we get

$$\int_a^b \frac{dx}{x-a} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{x-a} = \lim_{\varepsilon \to 0} \ln\left[\frac{b-a}{\varepsilon}\right] = \infty.$$

Therefore, the improper integral converges for $p < 1$ and diverges for $p \geq 1$.

Example 1.48 Show that the improper integral $\int_{-\pi/2}^{\pi/2} \tan x\, dx$ is divergent.

Solution The integrand has infinite discontinuity at $x = \pm\, \pi/2$. We write

$$\int_{-\pi/2}^{\pi/2} \tan x\, dx = \lim_{\varepsilon \to 0} \int_{-(\pi/2)+\varepsilon}^{c} \tan x\, dx + \lim_{\xi \to 0} \int_{c}^{(\pi/2)-\xi} \tan x\, dx$$

$$= \lim_{\varepsilon \to 0} [-\ln(\cos x)]_{-(\pi/2)+\varepsilon}^{c} + \lim_{\xi \to 0} [-\ln(\cos x)]_{c}^{(\pi/2)-\xi}$$

$$= \lim_{\varepsilon \to 0} \left\{\ln\left[\cos\left(-\frac{\pi}{2} + \varepsilon\right)\right] - \ln[\cos(c)]\right\}$$

$$- \lim_{\xi \to 0} \left\{\ln\left[\cos\left(\frac{\pi}{2} - \xi\right)\right] - \ln[\cos(c)]\right\}$$

Since the limits do not exist, the improper integral diverges.

Note that if we write

$$\int_{-\pi/2}^{\pi/2} \tan x\, dx = \lim_{\varepsilon \to 0} \int_{-(\pi/2)+\varepsilon}^{(\pi/2)-\varepsilon} \tan x\, dx$$

we get $\int_{-\pi/2}^{\pi/2} \tan x\, dx = 0$, which is not the correct solution.

Example 1.49 Discuss the convergence of the following improper integrals

(i) $\int_1^2 \frac{\sqrt{x}}{\ln x}\, dx$, (ii) $\int_0^{\pi/2} \frac{\sin x}{x\sqrt{x}}\, dx$.

Solution

(i) We have $f(x) = (\sqrt{x}/\ln x) \geq 0$, $1 < x \leq 2$. The point $x = 1$ is the only point of infinite discontinuity. Let $g(x) = 1/(x \ln x)$. Then, we have

$$\lim_{x \to 1^+} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(1+h)}{g(1+h)} = \lim_{h \to 0} \left[\frac{\sqrt{1+h}}{\ln(1+h)}\right] [(1+h)\ln(1+h)]$$

$$= \lim_{h \to 0} (1+h)^{3/2} = 1.$$

Therefore, both the integrals $\int_1^2 f(x)dx$ and $\int_1^2 g(x)dx$ converge or diverge together.

Now, $$\int_1^2 g(x)dx = \int_1^2 \frac{dx}{x\ln x} = \lim_{\varepsilon\to 0}\int_{1+\varepsilon}^2 \frac{dx}{x\ln x} = \lim_{\varepsilon\to 0}[\ln(\ln x)]_{1+\varepsilon}^2$$

$$= \lim_{\varepsilon\to 0}[\ln(\ln 2) - \ln(\ln(1+\varepsilon))] \to \infty.$$

Since $\int_1^2 g(x)dx$ is divergent, the given integral $\int_1^2 f(x)dx$ is also divergent by Comparison Test 4.

(ii) We have $f(x) = \dfrac{\sin x}{x\sqrt{x}} = \left(\dfrac{\sin x}{x}\right)\left(\dfrac{1}{\sqrt{x}}\right) \le \dfrac{1}{\sqrt{x}}$, since $\sin x/x$ is bounded and $(\sin x/x) \le 1$, $0 \le x \le \pi/2$. Let $g(x) = 1/\sqrt{x}$. We have $f(x) \le g(x)$, $0 < x < \pi/2$.
Now, $g(x)$ has a point of discontinuity at $x = 0$. Hence,

$$\int_0^{\pi/2} g(x)dx = \int_0^{\pi/2}\frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0}\int_\varepsilon^{\pi/2}\frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0}[\sqrt{2\pi} - 2\sqrt{\varepsilon}] = \sqrt{2\pi}.$$

Since $\int_0^{\pi/2} g(x)dx$ is convergent, the improper integral $\int_0^{\pi/2} f(x)dx$ is also convergent by Comparison Test 3 (*i*).

Example 1.50 Show that the improper integral $\displaystyle\int_0^{\pi/2}\frac{\cos^m x}{x^n}dx$ converges when $n < 1$.

Solution We have $f(x) = \dfrac{\cos^m x}{x^n} < \dfrac{1}{x^n}$, $0 < x < \pi/2$. The point $x = 0$ is the point of infinite discontinuity of $f(x)$. Let $g(x) = 1/x^n$. Then $f(x) < g(x)$.

Since the integral $\displaystyle\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2}\frac{dx}{x^n}$ is convergent for $n < 1$ (see Example 1.47), the given integral is also convergent for $n < 1$ by Comparison Test 3(i).

1.5.3 Absolute Convergence of Improper Integrals

In the previous sections, we had assumed that $f(x)$ is of the same sign throughout the interval of integration. Now, assume that $f(x)$ changes sign within the interval of integration. In this case, we consider absloute convergence of the improper integral.

Absolute convergence The improper integral $\int_a^b f(x)dx$ is said to be *absolutely convergent* if $\int_a^b |f(x)|\,dx$ is convergent.

Theorem 1.15 An absolutely convergent improper integral is convergent, that is if $\int_a^b |f(x)|\,dx$ converges, then $\int_a^b f(x)dx$ converges.

Since, $|f|$ is always positive within the interval of integration, all the comparison tests can be used to discuss the absolute convergence of the given improper integral.

Example 1.51 Show that the improper integral $\int_0^1 \frac{\sin(1/x)}{x^p}\,dx$ converges absolutely for $p < 1$.

Solution The integrand changes sign within the interval of integration. Hence, we consider the absolute convergence of the given integral. The function $f(x) = \sin(1/x)/x^p$ has a point of infinite discontinuity at $x = 0$. We have

$$|f(x)| = \left|\frac{\sin(1/x)}{x^p}\right| \le \frac{1}{x^p}.$$

Since $\int_0^1 \frac{1}{x^p}dx$ converges for $p < 1$, the given improper integral converges absolutely for $p < 1$.

Example 1.52 Show that the improper integral $\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$ converges.

Solution The integrand $f(x)$ changes sign within the interval of integration. Hence, we consider the absolute convergence of the given integral. We have

$$|I| = \left|\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx\right| \le \int_{-\infty}^{\infty} \left|\frac{\sin x}{1+x^2}\right| dx$$

$$= \lim_{a\to-\infty} \int_a^c \left|\frac{\sin x}{1+x^2}\right| dx + \lim_{b\to\infty} \int_c^b \left|\frac{\sin x}{1+x^2}\right| dx = I_1 + I_2.$$

Now,

$$I_1 = \lim_{a\to-\infty} \int_a^c \left|\frac{\sin x}{1+x^2}\right| dx \le \lim_{a\to-\infty} \int_a^c \frac{dx}{1+x^2} = \lim_{a\to-\infty} \left[\tan^{-1}c - \tan^{-1}a\right] = \tan^{-1}c + \frac{\pi}{2}.$$

$$I_2 = \lim_{b\to\infty} \int_c^b \left|\frac{\sin x}{1+x^2}\right| dx \le \lim_{b\to\infty} \int_c^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \left[\tan^{-1}b - \tan^{-1}c\right] = \frac{\pi}{2} - \tan^{-1}c\,.$$

Hence, $|I| \le I_1 + I_2 \le \pi$. Therefore, the given improper integral converges.

1.5.4 Beta and Gamma Functions

Beta and Gamma functions are improper integrals which are commonly encountered in many science and engineering applications. These functions are used in evaluating many definite integrals.

Gamma function

Consider the improper integral $\quad I(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx,\ \alpha > 0.$ (1.76)

We write the integral as $I(\alpha) = \int_0^c x^{\alpha-1}e^{-x}\,dx + \int_c^\infty x^{\alpha-1}e^{-x}dx = I_1 + I_2, 0 < c < \infty.$

The integral I_1 is an improper integral of the second kind as the integrand has a point of discontinuity at $x = 0$, whenever $0 < \alpha < 1$. For $\alpha \geq 1$, it is a proper integral. The integral I_2 is an improper integral of the first kind as its upper limit is infinite. We consider the two integrals separately.

Convergence at $x = 0$, $0 < \alpha < 1$, of the first integral I_1

In the integral I_1, let $f(x) = x^{\alpha-1}e^{-x}$ and $g(x) = x^{\alpha-1}$. Now, $\lim_{x\to 0^+} \dfrac{f(x)}{g(x)} = \lim_{x\to 0^+} \dfrac{x^{\alpha-1}e^{-x}}{x^{\alpha-1}} = 1$

Since $\int_0^c g(x)\,dx = \int_0^c \dfrac{dx}{x^{1-\alpha}}$ converges when $1 - \alpha < 1$, or $\alpha > 0$, the improper integral I_1 is convergent for all $\alpha > 0$.

Convergence at ∞, of the second integral I_2

Without loss of generality, let $c \geq 1$. Otherwise, the integral can be written as the sum of two integrals with the intervals $(c, 1)$, $(1, \infty)$. The first integral is a proper integral.

Let n be a positive integer such that $n > \alpha - 1$, $\alpha > 0$. Then,

$$\alpha - 1 < n, \; x^{\alpha-1} < x^n \quad \text{and} \quad x^{\alpha-1}e^{-x} < x^n e^{-x}, \; 1 < x < \infty.$$

Therefore,

$$\int_c^\infty x^{\alpha-1}e^{-x}\,dx < \int_c^\infty x^n e^{-x}\,dx = \lim_{b\to\infty} \int_c^b x^n e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[e^{-x}\{\text{polynomial of degree } n \text{ in } x, P_n(x)\}\right]_c^b$$

$$= \lim_{b\to\infty} \left[e^{-b}P_n(b) - e^{-c}P_n(c)\right] = -e^{-c}P_n(c)$$

since $\lim_{b\to\infty} [b^k/e^b] = 0$ for fixed k.

The limit exists and the integral I_2 converges for $\alpha > 0$.

Hence the given improper integral (Eq. (1.76)) converges when $\alpha > 0$. This improper integral is called the *Gamma function* and is denoted by $\Gamma(\alpha)$. Therefore,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx, \; \alpha > 0. \tag{1.77}$$

Some identities of Gamma functions

1. $\Gamma(1) = \int_0^\infty e^{-x}dx = 1.$ (1.78)

2. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$ (1.79)

   Integrating Eq. (1.76) by parts, we get

$$\Gamma(\alpha + 1) = \int_0^\infty x^\alpha e^{-x}dx = -\left[x^\alpha e^{-x}\right]_0^\infty + \alpha\int_0^\infty x^{\alpha-1}e^{-x}dx = \alpha\Gamma(\alpha).$$

3. $\Gamma(m + 1) = m!$, for any positive integer m. (1.80)

We have $\Gamma(m+1) = m\Gamma(m) = m(m-1)\Gamma(m-1) = \ldots = m(m-1) \ldots 1\,\Gamma(1) = m!$ (1.81)

4. $\Gamma(1/2) = \sqrt{\pi}$.

We have

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x} dx = 2\int_0^\infty e^{-u^2} du. \text{ (set } x = u^2\text{)}.$$

We write

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \left[2\int_0^\infty e^{-u^2} du\right]\left[2\int_0^\infty e^{-v^2} dv\right] = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} du\, dv.$$

Changing to polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, we obtain $du dv = r\, dr\, d\theta$ and

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_{\theta=0}^{\pi/2} \int_{r=0}^\infty re^{-r^2} dr\, d\theta = 2\pi\int_0^\infty re^{-r^2} dr = -\pi[e^{-r^2}]_0^\infty = \pi.$$

Hence, $$\Gamma(1/2) = \sqrt{\pi}.$$

(In Chapter 2, we shall discuss evaluation of double integrals and change of variables.) (1.82)

5. $\Gamma(-1/2) = -2\sqrt{\pi}$.

We have $\Gamma(\alpha) = [\Gamma(\alpha+1)]/\alpha$. Substituting $\alpha = -1/2$, we get

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(1/2)}{(-1/2)} = -2\sqrt{\pi}.$$

Beta function

Consider the improper integral

$$I = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad 0 < m < 1, 0 < n < 1. \tag{1.83}$$

Note that I is a proper integral for $m \geq 1$ and $n \geq 1$. The improper integral has points of infinite discontinuity at (i) $x = 0$, when $m < 1$ and (ii) $x = 1$, when $n < 1$. When $m < 1$ and $n < 1$, we take any number, say c between 0 and 1 and write the improper integral as

$$I = \int_0^c x^{m-1}(1-x)^{n-1} dx + \int_c^1 x^{m-1}(1-x)^{n-1} dx = I_1 + I_2$$

where $$I_1 = \int_0^c x^{m-1}(1-x)^{n-1} dx \quad \text{and} \quad I_2 = \int_c^1 x^{m-1}(1-x)^{n-1} dx.$$

I_1 is an improper integral, since $x = 0$ is a point of infinite discontinuity, while I_2 is an improper integral, since $x = 1$ is a point of infinite discontinuity. We consider these two integrals separately.

Convergence at $x = 0$, $0 < m < 1$, of the integral I_1

In the integral I_1, let $f(x) = x^{m-1}(1-x)^{n-1}$ and $g(x) = x^{m-1}$.

Now,
$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{x^{m-1}(1-x)^{n-1}}{x^{m-1}} = 1$$

and
$$\int_0^c g(x)dx = \int_0^c \frac{dx}{x^{1-m}} \text{ is convergent only when } 1-m<1, \text{ or } m>0.$$

Therefore, the improper integral I_1 converges when $m > 0$.

Convergence at $x = 1$, $0 < n < 1$, of the integral I_2

In the integral I_2, let $f(x) = x^{m-1}(1-x)^{n-1}$ and $g(x) = (1-x)^{n-1}$.

Now,
$$\lim_{x\to 1^-} \frac{f(x)}{g(x)} = \lim_{x\to 1^-} \frac{x^{m-1}(1-x)^{n-1}}{(1-x)^{n-1}} = 1$$

and
$$\int_c^1 g(x)dx = \int_c^1 \frac{dx}{(1-x)^{1-n}} \text{ converges when } 1-n<1, \text{ or } n>0.$$

Therefore, the improper integral I_2 converges when $n > 0$. Combining the two results, we deduce that the given improper integral (Eq. (1.83)) converges when $m > 0$ and $n > 0$. This improper integral is called the *Beta function* and is denoted by $\beta(m, n)$. Therefore,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad m>0, n>0. \tag{1.84}$$

Some identities of Beta functions

1. $\beta(m, n) = \beta(n, m)$ (1.85)
   (substitute $x = 1 - t$ in Eq. (1.84) and simplify).

2. $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\cos^{2n-1}(\theta)d\theta = 2\int_0^{\pi/2} \sin^{2n-1}(\theta)\cos^{2m-1}(\theta)d\theta. \tag{1.86}$$
   (substitute $x = \sin^2\theta$ in Eq. (1.84) and simplify).

3. $$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx \tag{1.87}$$
   (substitute $x = t/(1 + t)$ in Eq. (1.84) and simplify).

4. $$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{1.88}$$

   We can prove this result using double integrals and change of variables. We have

$$\Gamma(m) = \int_0^\infty x^{m-1}e^{-x}\,dx = 2\int_0^\infty u^{2m-1}e^{-u^2}\,du, \ (\text{set } x = u^2)$$

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx = 2\int_0^\infty v^{2n-1}e^{-v^2}\,dv, \ (\text{set } x = v^2)$$

$$\Gamma(m)\,\Gamma(n) = 4\int_0^\infty\int_0^\infty u^{2m-1}v^{2n-1}e^{-(u^2+v^2)}\,du\,dv.$$

Changing to polar coordinates, $u = r\cos\theta$, $v = r\sin\theta$, we get

$$\Gamma(m)\Gamma(n) = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty}\cos^{2m-1}(\theta)\sin^{2n-1}(\theta)\,r^{2m+2n-1}e^{-r^2}\,dr\,d\theta$$

$$= 4\left[\int_0^\infty r^{2m+2n-1}e^{-r^2}\,dr\right]\left[\int_0^{\pi/2}\cos^{2m-1}(\theta)\sin^{2n-1}(\theta)\,d\theta\right]$$

$$= 2\beta(m,n)\int_0^\infty r^{2m+2n-1}e^{-r^2}\,dr, \quad \text{(using Eq. (1.86))}$$

We also have

$$\Gamma(m+n) = \int_0^\infty x^{m+n-1}e^{-x}\,dx = 2\int_0^\infty r^{2m+2n-1}e^{-r^2}\,dr, \ (\text{set } x = r^2).$$

Combining the two results, we obtain

$$\Gamma(m)\Gamma(n) = \beta(m,n)\Gamma(m+n), \ \text{ or } \ \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

5. $\beta(m, n) = \beta(m + 1, n) + \beta(m, n + 1)$.

We have

$$\beta(m+1,n) = 2\int_0^{\pi/2}\sin^{2m+1}(\theta)\cos^{2n-1}(\theta)\,d\theta = 2\int_0^{\pi/2}\sin^{2m-1}(\theta)\sin^2\theta\cos^{2n-1}(\theta)\,d\theta$$

$$= 2\int_0^{\pi/2}\sin^{2m-1}(\theta)\cos^{2n-1}(\theta)(1-\cos^2\theta)\,d\theta$$

$$= 2\int_0^{\pi/2}\sin^{2m-1}(\theta)\cos^{2n-1}(\theta)\,d\theta - 2\int_0^{\pi/2}\sin^{2m-1}(\theta)\cos^{2n+1}(\theta)\,d\theta$$

$$= \beta(m,n) - \beta(m,n+1)$$

Therefore, $\beta(m, n) = \beta(m + 1, n) + \beta(m, n + 1)$.

Example 1.53 Given that $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x} = \frac{\pi}{\sin p\pi}$, show that $\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$.

Solution Let $\dfrac{x}{1+x} = y$. Solving for x, we get $x = \dfrac{y}{1-y}$ and $dx = \dfrac{1}{(1-y)^2}dy$.

Then, $I = \displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \int_0^1 y^{p-1}(1-y)^{-p} = \beta(p, 1-p) = \frac{\Gamma(p)\Gamma(1-p)}{\Gamma(1)} = \Gamma(p)\Gamma(1-p)$

Hence, the result.

Example 1.54 Evaluate the following improper integrals

(i) $\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx$, (ii) $\int_0^\infty e^{-x^3}\, dx$

in terms of Gamma functions.

Solution

(i) Substitute $x = \sqrt{t}$. We get $dx = dt/(2\sqrt{t})$ and

$$I = \int_0^\infty \sqrt{x}\, e^{-x^2}\, dx = \frac{1}{2}\int_0^\infty t^{-1/4} e^{-t}\, dt = \frac{1}{2}\int_0^\infty t^{(3/4)-1} e^{-t}\, dt = \frac{1}{2}\Gamma\left(\frac{3}{4}\right).$$

(ii) Substitute $x = t^{1/3}$. We get $dx = \frac{1}{3}t^{-2/3}\, dt$ and

$$I = \int_0^\infty e^{-x^3}\, dx = \frac{1}{3}\int_0^\infty t^{-2/3} e^{-t}\, dt = \frac{1}{3}\int_0^\infty t^{(1/3)-1} e^{-t}\, dt = \frac{1}{3}\Gamma\left(\frac{1}{3}\right).$$

Example 1.55 Using Beta and Gamma functions, evaluate the integral

$$I = \int_{-1}^{1} (1 - x^2)^n\, dx, \text{ where } n \text{ is a positive integer.}$$

Solution We have $$I = \int_{-1}^{1} (1 + x)^n (1 - x)^n\, dx.$$

Let $1 + x = 2t$. Then, $dx = 2dt$ and $1 - x = 2(1 - t)$. We obtain

$$I = 2^{2n+1}\int_0^1 t^n (1 - t)^n\, dt = 2^{2n+1}\beta(n + 1, n + 1)$$

$$= 2^{2n+1}\frac{\Gamma(n + 1)\,\Gamma(n + 1)}{\Gamma(2n + 2)} = \frac{2^{2n+1}(n!)^2}{(2n + 1)!}.$$

Example 1.56 Express $\int_0^1 x^m (1 - x^p)^n\, dx$ in terms of Beta function and hence evaluate the integral $\int_0^1 x^{3/2}(1 - \sqrt{x})^{1/2}\, dx$.

Solution Let $x^p = y$. Then $px^{p-1}dx = dy$. We obtain

$$I = \int_0^1 x^m (1 - x^p)^n\, dx = \frac{1}{p}\int_0^1 y^{(m-p+1)/p}(1 - y)^n\, dy$$

$$= \frac{1}{p}\int_0^1 y^{[(m+1)/p-1]}(1 - y)^n\, dy = \frac{1}{p}\beta\left(\frac{m + 1}{p}, n + 1\right)$$

Now, comparing the integral $\int_0^1 x^{3/2}(1-\sqrt{x})^{1/2}\,dx$ with the given integral, we find that $m = 3/2$, $p = 1/2$ and $n = 1/2$. Therefore,

$$\int_0^1 x^{3/2}(1-\sqrt{x})^{1/2}\,dx = 2\beta\left(5, \frac{3}{2}\right) = \frac{2\Gamma(5)\Gamma(3/2)}{\Gamma(13/2)}$$

Now, $$\Gamma(5) = 4! = 24,\ \Gamma\left(\frac{13}{2}\right) = \frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{10395}{32}\Gamma\left(\frac{3}{2}\right).$$

Hence, $$I = \frac{2(24)(32)\Gamma(3/2)}{10395\,\Gamma(3/2)} = \frac{1536}{10395} = \frac{512}{3465}.$$

Example 1.57 Using Beta and Gamma functions, show that for any positive integer m

(i) $$\int_0^{\pi/2} \sin^{2m-1}(\theta)d\theta = \frac{(2m-2)(2m-4)\ldots 2}{(2m-1)(2m-3)\ldots 3},$$

(ii) $$\int_0^{\pi/2} \sin^{2m}(\theta)d\theta = \frac{(2m-1)(2m-3)\ldots 1}{(2m)(2m-2)\ldots 2}\,\frac{\pi}{2}.$$

Solution From Eq. (1.86), we obtain

$$\beta\left(m, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m-1}(\theta)d\theta \quad\text{and}\quad \beta\left(m+\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m}(\theta)d\theta.$$

(i) $$I = \int_0^{\pi/2} \sin^{(2m-1)}(\theta)\,d\theta = \frac{1}{2}\beta\left(m, \frac{1}{2}\right) = \frac{\Gamma(m)\,\Gamma(1/2)}{2\Gamma(m+1/2)}.$$

We have $\Gamma(m) = (m-1)!$, and

$$\Gamma\left(m+\frac{1}{2}\right) = \left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2^m}\,[(2m-1)\,(2m-3)\ldots 3\cdot 1]\,\Gamma\left(\frac{1}{2}\right).$$

Therefore,

$$I = \frac{(m-1)!\,2^m\,\Gamma(1/2)}{2(2m-1)\,(2m-3)\ldots 3\cdot 1\cdot\Gamma(1/2)} = \frac{2^{m-1}[(m-1)(m-2)\ldots 2\cdot 1]}{(2m-1)(2m-3)\ldots 3\cdot 1}$$

$$= \frac{(2m-2)\,(2m-4)\ldots 4\cdot 2}{(2m-1)\,(2m-3)\ldots 3\cdot 1}.$$

(ii) $$I = \int_0^{\pi/2} \sin^{2m}(\theta)d\theta = \frac{1}{2}\,\beta\left(m+\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma(m+1/2)\,\Gamma(1/2)}{2\Gamma(m+1)}$$

$$= \frac{1}{2(m!)}\left[\frac{(2m-1)(2m-3)\ldots 3\cdot 1}{2^m}\right](\sqrt{\pi})^2 = \frac{(2m-1)(2m-3)\ldots 3\cdot 1}{2^{m+1}[m(m-1)\ldots 2\cdot 1]}(\pi)$$

$$= \frac{(2m-1)(2m-3)\ldots 3\cdot 1}{(2m)(2m-2)\ldots 4\cdot 2}\,\frac{\pi}{2}.$$

Example 1.58 Evaluate $\int_0^\infty 2^{-9x^2}\,dx$ using the Gamma function.

Solution We write

$$I = \int_0^\infty 2^{-9x^2}\,dx = \int_0^\infty e^{-9x^2 \ln 2}\,dx$$

Substitute $9x^2 \ln 2 = y$. Then, $x = \dfrac{\sqrt{y}}{3\sqrt{\ln 2}}$ and $dx = \dfrac{y^{-1/2}dy}{6\sqrt{\ln 2}}$.

Therefore,
$$I = \frac{1}{6\sqrt{\ln 2}}\int_0^\infty y^{-1/2}e^{-y}\,dy = \frac{1}{6\sqrt{\ln 2}}\int_0^\infty y^{(1/2)-1}e^{-y}\,dy$$

$$= \frac{\Gamma(1/2)}{6\sqrt{\ln 2}} = \frac{1}{6}\sqrt{\frac{\pi}{\ln 2}}.$$

Example 1.59 Show that

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right)\Gamma(n) \tag{1.89}$$

and
$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}. \tag{1.90}$$

Solution From Eq. (1.88), we have

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\cos^{2n-1}(\theta)d\theta. \tag{1.91}$$

Setting $m = n$, we get

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = \beta(n,n) = 2\int_0^{\pi/2} \sin^{2n-1}(\theta)\cos^{2n-1}(\theta)d\theta$$

$$= \frac{1}{2^{2n-2}}\int_0^{\pi/2} \sin^{2n-1}(2\theta)d\theta.$$

Substituting, $2\theta = \dfrac{\pi}{2} - \phi$, we get $d\theta = -\dfrac{1}{2}d\phi$. Hence, we obtain

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = \frac{-1}{2^{2n-1}} \int_{\pi/2}^{-\pi/2} \cos^{2n-1}(\phi) d\phi$$

$$= \frac{1}{2^{2n-1}} \int_{-\pi/2}^{\pi/2} \cos^{2n-1}\phi \, d\phi = \frac{2}{2^{2n-1}} \int_0^{\pi/2} \cos^{2n-1}(\theta) d\theta \tag{1.92}$$

since $\cos\theta$ is an even function.

Setting $m = 1/2$ in Eq. (1.91), we obtain

$$\frac{\Gamma(n)\,\Gamma(1/2)}{\Gamma(n + 1/2)} = 2 \int_0^{\pi/2} \cos^{2n-1}(\theta) d\theta. \tag{1.93}$$

Comparing Eqs. (1.92) and (1.93), we have

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = \frac{1}{2^{2n-1}} \left[\frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n + 1/2)}\right]$$

or

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma(n)\,\Gamma(n + 1/2), \left(\text{since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

which is the required result.

Setting $n = 1/4$ in Eq. (1. 89), we obtain

$$\Gamma\left(\frac{1}{2}\right) = \frac{2^{-1/2}}{\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)$$

or

$$\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}.$$

Example 1.60 Show that $\int_0^{\pi/2} \sqrt{\tan x} = \frac{\pi}{\sqrt{2}}$.

Solution We have

$$I = \int_0^{\pi/2} \sqrt{\tan x}\, dx = \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x \, dx = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma(1/4)\Gamma(3/4)}{\Gamma(1)} = \frac{\pi\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}} \quad \text{(using Eq. (1.90).}$$

1.5.5 Improper Integrals Involving a Parameter

Often, we come across integrals of the form

$$\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)\, dx \tag{1.94}$$

where α is a parameter and the integrand f is such that the integral cannot be evaluated by standard methods. We can evaluate some of these integrals by differentiating the integral with respect to the

parameter, that is first obtain $\phi'(\alpha)$, evaluate the integral (that is integrate with respect to x) and then integrate $\phi'(\alpha)$ with respect to α. Note that f is a function of two variables x and α. When we differentiate f with respect to α, we treat x as a constant and denote the derivative as $\partial f/\partial\alpha$ (partial derivative of f with respect to α. Chapter 2 discusses partial derivatives in detail). We assume that f, $\partial f/\partial\alpha$, $a(\alpha)$ and $b(\alpha)$ are continuous functions of α.

We now present the formula which gives the derivative of $\phi(\alpha)$.

Theorem 1.16 (Leibniz formula) If $a(\alpha)$, $b(\alpha)$, $f(x, \alpha)$ and $\partial f/\partial\alpha$ are continuous functions of α, then

$$\frac{d\phi}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial\alpha}(x, \alpha)dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha}. \tag{1.95}$$

Proof Let $\Delta\alpha$ be an increment in α and Δa, Δb be the corresponding increments in a and b. We have

$$\Delta\phi = \phi(\alpha + \Delta\alpha) - \phi(\alpha) = \int_{a+\Delta a}^{b+\Delta b} f(x, \alpha + \Delta\alpha)dx - \int_a^b f(x, \alpha)dx$$

$$= \int_{a+\Delta a}^{a} f(x, \alpha + \Delta\alpha)dx + \int_a^b f(x, \alpha + \Delta\alpha)dx + \int_b^{b+\Delta b} f(x, \alpha + \Delta\alpha)dx - \int_a^b f(x, \alpha)dx$$

or

$$\frac{\Delta\phi}{\Delta\alpha} = \int_{a+\Delta a}^{a} \frac{1}{\Delta\alpha} f(x, \alpha + \Delta\alpha)dx + \int_a^b \frac{1}{\Delta\alpha}[f(x, \alpha + \Delta\alpha) - f(x, \alpha)]dx$$

$$+ \int_b^{b+\Delta b} \frac{1}{\Delta\alpha} f(x, \alpha + \Delta\alpha)dx. \tag{1.96}$$

Using the mean value theorem of integrals

$$\int_{x_0}^{x_1} f(x)dx = (x_1 - x_0)f(\xi), x_0 < \xi < x_1$$

we get

$$\int_{a+\Delta a}^{a} f(x, \alpha + \Delta\alpha)dx = -\Delta a f(\xi_1, \alpha + \Delta\alpha), \ a < \xi_1 < a + \Delta a \tag{1.97}$$

and

$$\int_b^{b+\Delta b} f(x, \alpha + \Delta\alpha)dx = \Delta b f(\xi_2, \alpha + \Delta\alpha), \ b < \xi_2 < b + \Delta b. \tag{1.98}$$

Using the Lagrange mean value theorem, we get

$$f(x, \alpha + \Delta\alpha) - f(x, \alpha) = \Delta\alpha \frac{\partial f}{\partial\alpha}(x, \xi_3), \alpha < \xi_3 < \alpha + \Delta\alpha. \tag{1.99}$$

We note that

$$\lim_{\Delta\alpha\to 0} \xi_1 = a, \ \lim_{\Delta\alpha\to 0} \xi_2 = b \text{ and } \lim_{\Delta\alpha\to 0} \xi_3 = \alpha. \tag{1.100}$$

Taking limits as $\Delta\alpha \to 0$ on both sides of Eq. (1.96) and using the results in Eqs. (1.97) to (1.100), we obtain

$$\frac{d\phi}{d\alpha} = \int_a^b \frac{\partial f}{\partial\alpha}(x, \alpha)dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha}.$$

Remark 8

(a) If the limits $a(\alpha)$ and $b(\alpha)$ are constants, then we obtain from Eq. (1.95)

$$\frac{d\phi}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}(x, \alpha)\, dx. \tag{1.101}$$

(b) If the integrand f is independent of α, then we obtain from Eq. (1.95)

$$\frac{d\phi}{d\alpha} = f(b)\frac{db}{d\alpha} - f(a)\frac{da}{d\alpha}. \tag{1.102}$$

(c) Leibniz formula is often used to evaluate certain types of improper integrals.

Example 1.61 Evaluate the integral $\displaystyle\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\, dx$, $\alpha > 0$ and deduce that

(i) $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$, (ii) $\displaystyle\int_0^\infty \frac{\sin ax}{x}\, dx = \frac{\pi}{2}$, $a > 0$.

Solution Let $\displaystyle\phi(\alpha) = \int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\, dx.$ (1.103)

The limits of integration are independent of the parameter α. We obtain

$$\frac{d\phi}{d\alpha} = \int_0^\infty \frac{\partial}{\partial \alpha}\left[\frac{e^{-\alpha x}\sin x}{x}\right] dx = -\int_0^\infty \frac{xe^{-\alpha x}\sin x}{x}\, dx = -\int_0^\infty e^{-\alpha x}\sin x\, dx.$$

Using the result $\displaystyle\int e^{-\alpha x}\sin x\, dx = -\frac{e^{-\alpha x}}{1+\alpha^2}(\alpha\sin x + \cos x)$, we obtain

$$\frac{d\phi}{d\alpha} = \left[\frac{e^{-\alpha x}}{1+\alpha^2}(\alpha\sin x + \cos x)\right]_0^\infty = -\frac{1}{1+\alpha^2}.$$

Integrating with respect to α, we get

$\phi(\alpha) = -\tan^{-1}\alpha + c$, where c is the constant of integration.

From Eq. (1.103), we get the condition $\phi(\infty) = 0$. Hence,

$$\phi(\infty) = 0 = -\tan^{-1}\infty + c, \quad \text{or} \quad c = \pi/2.$$

Therefore,

$$\phi(\alpha) = \int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\, dx = \frac{\pi}{2} - \tan^{-1}\alpha.$$

(i) Setting $\alpha = 0$, we obtain $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$ (1.104)

(ii) Substituting $x = ay$ on the left hand side of Eq. (1.104), we obtain

$$\int_0^\infty \frac{\sin x}{x}\, dx = \int_0^\infty \frac{\sin ay}{y}\, dy = \frac{\pi}{2}.$$

Example 1.62 Using the result $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$, evaluate the integral $\int_0^{\infty} e^{-x^2}\cos(2\alpha x)\,dx$.

Solution Let $\phi(\alpha) = \int_0^{\infty} e^{-x^2}\cos(2\alpha x)\,dx$. (1.105)

The limits of integration are independent of the parameter α. Hence,

$$\frac{d\phi}{d\alpha} = \int_0^{\infty} \frac{\partial}{\partial \alpha}[e^{-x^2}\cos(2\alpha x)]\,dx = \int_0^{\infty} (-2x)e^{-x^2}\sin(2\alpha x)\,dx$$

$$= [e^{-x^2}\sin(2\,\alpha x)]_0^{\infty} - 2\alpha\int_0^{\infty} e^{-x^2}\cos(2\alpha x)\,dx = -2\alpha\phi.$$

Integrating the differential equation $\frac{d\phi}{d\alpha} + 2\alpha\phi = 0$, we obtain $\phi(\alpha) = c\,e^{-\alpha^2}$.

From Eq. (1.105), we get the condition $\phi(0) = \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

Using this condition, we obtain $\phi(0) = \frac{\sqrt{\pi}}{2} = c$.

Therefore, $\phi(\alpha) = \int_0^{\infty} e^{-x^2}\cos(2\alpha x)\,dx = \frac{\sqrt{\pi}}{2}\,e^{-\alpha^2}$.

Example 1.63 Evaluate the integral $\int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$, $a > 0$ and $a \neq 1$.

Solution Let $\phi(a) = \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$. (1.106)

We have

$$\frac{d\phi}{da} = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{\tan^{-1}(ax)}{x(1+x^2)}\right]dx = \int_0^{\infty} \frac{dx}{(1+x^2)(1+a^2x^2)}$$

$$= \frac{1}{a^2-1}\int_0^{\infty}\left[\frac{a^2}{a^2x^2+1} - \frac{1}{1+x^2}\right]dx$$

$$= \frac{1}{a^2-1}[\{a\tan^{-1}(ax)\}_0^{\infty} - \{\tan^{-1}(x)\}_0^{\infty}] = \frac{\pi}{2}\left[\frac{a-1}{a^2-1}\right] = \frac{\pi}{2(a+1)}.$$

Integrating with respect to a, we obtain

$$\phi(a) = \frac{\pi}{2}\ln(a+1) + c.$$

From Eq. (1.106), we get the condition $\phi(0) = 0$. Using this condition, we obtain $\phi(0) = 0 = c$.

Therefore, $$\phi(a) = \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx = \frac{\pi}{2}\ln(a+1).$$

1.5.6 Error Functions

Error functions arise in the theory of probability and solution of certain types of partial differential equations (see section 8.7).

Let us first consider the following function that arises in defining the normal probability distribution (the case when mean = $\mu = 0$ and variance = $\sigma^2 = 1$)

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}. \tag{1.107}$$

This function is also called the *Gaussian function*. The bell shaped normal curve defined by Eq. (1.107) is given in Fig. 1.20. The area under the curve and above the x-axis is given by

$$I = \int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}dx. \tag{1.108}$$

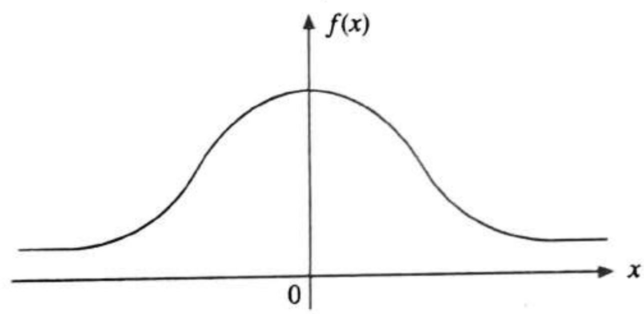


Fig. 1.20 Normal curve.

Setting $u = x/\sqrt{2}$, we get

$$I = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\,du.$$

It was shown in equation (1.81), that

$$\int_0^{\infty} e^{-u^2}\,du = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Hence, $$I = \int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\,du = \frac{1}{\sqrt{\pi}}(2)\left(\frac{\sqrt{\pi}}{2}\right) = 1,$$

that is, the total area under the normal curve is 1. Since the area is symmetric about the y-axis, we get

$$\int_{-\infty}^{0} f(x)\,dx = \int_0^{\infty} f(x)dx = \frac{1}{2}.$$

The area under the curve, from $-\infty$ to any point z, is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx. \tag{1.109}$$

Hence, by definition $\phi(0) = 1/2$. The function $\phi(z)$ is called the distribution function of the normal distribution with mean 0 and variance 1. Setting $x = -y$ in Eq. (1.109), we get $dx = -dy$ and

$$\phi(z) = -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{-z} e^{-y^2/2} dy = \frac{1}{\sqrt{2\pi}} \int_{-z}^{\infty} e^{-y^2/2} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-y^2/2} dy$$

$$= 1 - \phi(-z)$$

or $$\phi(-z) = 1 - \phi(z). \tag{1.110}$$

Values of the distribution function are tabulated for various values of z. Further, the area under the curve from $x = 0$ to $x = z$ is given by .

$$I(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{z} e^{-x^2/2} dx - \int_{-\infty}^{0} e^{-x^2/2} dx \right] = \phi(z) - \frac{1}{2} \tag{1.111}$$

or $$\phi(z) = \frac{1}{2} + I(z).$$

Error function *erf (x)*

The *error function* is also called the error integral function. It is defined by

$$erf\,(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt. \tag{1.112}$$

Let $t^2 = u$. Then, $dt = \frac{1}{2t} du = \frac{1}{2\sqrt{u}} du$, and

$$erf\,(x) = \frac{1}{\sqrt{\pi}} \int_{0}^{x^2} u^{-1/2} e^{-u} du. \tag{1.113}$$

This is another form of the error function. Using this definition, we obtain

$$erf\,(\infty) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} u^{-1/2} e^{-u} du = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} (\sqrt{\pi}) = 1. \tag{1.114}$$

Let $t^2 = u^2/2$ in Eq. (1.112). Then,

$$2t\, dt = u\, du, \quad dt = \frac{u}{2t} du = \frac{du}{\sqrt{2}}, \text{ and}$$

$$erf\,(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{2}x} e^{-u^2/2} \frac{du}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{\sqrt{2}x} e^{-u^2/2} du. \tag{1.115}$$

Using Eq. (1.111), we can write

$$I(\sqrt{2}\,x) = \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{2}x} e^{-x^2/2} dx = \phi(\sqrt{2}\,x) - \frac{1}{2}.$$

Therefore,
$$erf\,(x) = 2I(\sqrt{2}\,x) = 2\phi\,(\sqrt{2}\,x) - 1. \tag{1.116}$$

Hence, the error function can be evaluated using this relation.

Complementary error function *erfc* (*x*)

Using the definition of the error function given in Eqs. (1.113) and (1.114), we write

$$erf\,(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} u^{-1/2} e^{-u} du = \frac{1}{\sqrt{\pi}} \int_0^{\infty} u^{-1/2} e^{-u} du - \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} u^{-1/2} e^{-u} du$$

$$= 1 - \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} u^{-1/2} e^{-u} du = 1 - erfc\,(x) \tag{1.117}$$

where we define

$$erfc\,(x) = \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} u^{-1/2} e^{-u} du. \tag{1.118}$$

This function *erfc* (*x*) is called the complementary error function.

Using Eqs. (1.117), (1.112) and (1.114), we can write

$$erfc(x) = 1 - erf\,(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt - \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt. \tag{1.119}$$

Equations (1.112) and (1.119) are the commonly used definitions of error function and complementary error function respectively. The graphs of $erf\,(x)$ and $erfc\,(x)$ are given in Fig. 1.21.

Some properties of error functions

1. $erf\,(-x) = -\,erf\,(x)$. (1.120)

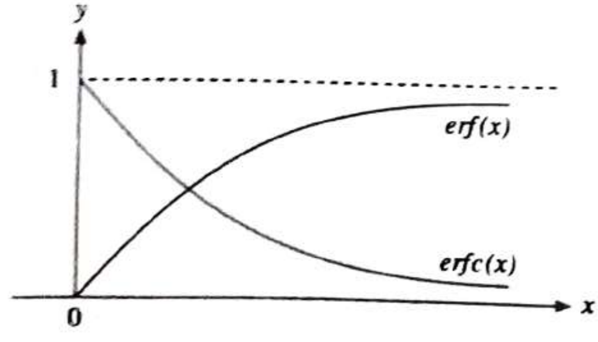


Fig. 1.21. Error function and complementary error function.

Using the definition given in Eq. (1.112), we get

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2} (-du) \quad (\text{setting } t = -u)$$

$$= -erf(x).$$

2. $$erfc(-x) = 1 + erf(x) = 2 - erfc(x). \tag{1.121}$$

Using equation (1.117), we get

$$erfc(-x) = 1 - erf(-x) = 1 + erf(x)$$

$$= 1 + [1 - erfc(x)] = 2 - erfc(x).$$

3. *Derivative of error function* We have

$$\frac{d}{dx}[erf(\alpha x)] = \frac{2\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}. \tag{1.122}$$

From the definition, we have

$$erf(\alpha x) = \frac{2}{\sqrt{\pi}} \int_0^{\alpha x} e^{-t^2} dt. \tag{1.123}$$

Consider x as a parameter. Comparing Eq. (1.123) with Eq. (1.94)

$$\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha) d\alpha, \quad \text{where } \alpha \text{ is a parameter} \tag{1.124}$$

we get $f(t, x) = \frac{2}{\sqrt{\pi}} e^{-t^2}$, $b(x) = \alpha x$, $a(x) = 0$, $\phi(x) = erf(\alpha x)$.

Using Eq. (1.95)

$$\frac{d\phi}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}(x, \alpha) dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha} \tag{1.125}$$

we obtain

$$\frac{d}{dx}[erf(\alpha x)] = \frac{2}{\sqrt{\pi}} \int_0^{\alpha x} \frac{\partial}{\partial x}(e^{-t^2}) dt + f(\alpha x, x)\frac{d}{dx}(\alpha x) - 0$$

$$= \frac{2\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}$$

4. *Integral of error function* We have

$$\int_0^t erf(\alpha x) dx = t\, erf(\alpha t) + \frac{1}{\alpha\sqrt{\pi}}[e^{-\alpha^2 t^2} - 1] \tag{1.126}$$

Integrating the left hand side by parts, we obtain

$$\int_0^t 1 \cdot erf(\alpha x)\, dx = \left[x\, erf(\alpha x)\right]_0^t - \int_0^t x \frac{d}{dx}[erf(\alpha x)]dx$$

$$= t\, erf(\alpha t) - \frac{2\alpha}{\sqrt{\pi}} \int_0^t x e^{-\alpha^2 x^2} dx$$

using Eq. (1.122). Let $\alpha^2 x^2 = u$. Then, $2\alpha^2 x\, dx = du$ or $x\, dx = du/(2\alpha^2)$. Hence,

$$\int_0^t erf(\alpha x)\, dx = t\, erf(\alpha t) - \left(\frac{2\alpha}{\sqrt{\pi}}\right)\left(\frac{1}{2\alpha^2}\right) \int_0^{\alpha^2 t^2} e^{-u} du$$

$$= t\, erf(\alpha t) + \frac{1}{\alpha\sqrt{\pi}} \left[e^{-u}\right]_0^{\alpha^2 t^2}$$

$$= t\, erf(\alpha t) + \frac{1}{\alpha\sqrt{\pi}} [e^{-\alpha^2 t^2} - 1].$$

Exercise 1.4

In problems **1** to **25**, discuss the convergence or divergence of the given improper integral. Find its value if it exists.

1. $\int_0^\infty \frac{dx}{4+x}$, **2.** $\int_2^\infty \frac{\ln x}{x} dx$, **3.** $\int_3^\infty \frac{dx}{x^2+2x}$,

4. $\int_0^\infty \frac{x\, dx}{x^4+1}$, **5.** $\int_0^\infty x^2 e^{-ax} dx,\ a > 0$, **6.** $\int_0^\infty e^{-ax} \sin bx\, dx,\ a > 0$,

7. $\int_1^\infty x e^{-x^2} dx$, **8.** $\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$, **9.** $\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}}$,

10. $\int_0^\infty e^{-x} dx$ **11.** $\int_0^\infty \frac{x^{p-1}}{1+x} dx,\ 0 < p < 1$, **12.** $\int_1^\infty \frac{x+4}{x^{3/2}} dx$,

13. $\int_0^\infty \frac{dx}{x^3+1}$, **14.** $\int_1^3 \frac{dx}{x \ln x}$, **15.** $\int_0^4 \frac{dx}{x^2-2x-8}$,

16. $\int_0^{\pi/2} \frac{dx}{\cos x}$, **17.** $\int_0^2 \ln x\, dx$, **18.** $\int_0^1 \frac{x}{\sqrt{1-x^2}} dx$,

19. $\int_{-1}^1 \frac{dx}{x^4}$, **20.** $\int_0^1 \frac{dx}{\sqrt{x}+x^3}$, **21.** $\int_1^3 \frac{\sqrt{x}}{\ln x} dx$

22. $\int_0^{\pi/2} \frac{\sin^n x}{x^m} dx$ **23.** $\int_2^\infty \frac{\sin x}{x(\ln x)^2} dx$ **24.** $\int_0^\pi \frac{\cos x}{\sqrt{x}} dx$.

25. $\displaystyle\int_0^{\infty} \frac{x^p}{1+x^q}\,dx$, (i) $q \geq 0$, (ii) $q < 0$

In problems **27** to **40**, evaluate the integrals using the Beta and Gamma functions.

26. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}}$

27. $\displaystyle\int_0^{\pi/2} \sin^2\theta \cos^4\theta\, d\theta$.

28. $\displaystyle\int_0^{\pi/2} \sin^3\theta \cos^5\theta\, d\theta$.

29. $\displaystyle\int_0^{\pi/2} \cos^m\theta\, d\theta$, m integer.

30. $\displaystyle\int_0^{a} x\sqrt{a^3 - x^3}\, dx$.

31. $\displaystyle\int_0^{1} \frac{dx}{\sqrt[3]{1-x^3}}$.

32. $\displaystyle\int_0^{1} x^n (\ln x)^m\, dx$.

33. $\displaystyle\int_0^{a} \frac{x^{3/2}}{\sqrt{a^2 - x^2}}\, dx$.

34. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{-\ln x}}$.

35. $\displaystyle\int_0^{1} x^k (1-x)^{n-k}\, dx$, $k > 0$.

36. $\displaystyle\int_0^{\infty} \frac{dx}{1+x^4}$.

37. $\displaystyle\int_0^{\infty} \frac{x^a}{a^x}\, dx$, $a > 1$.

38. $\displaystyle\int_0^{\infty} t^k e^{-st}\, dt$, $s > 0, k > 0$.

39. $\displaystyle\int_0^{\infty} t^4 e^{-2t^2}\, dt$.

40. $\displaystyle\int_0^{\infty} x^{1/3} e^{-x^2}\, dx$.

Establish the following results.

41. $\displaystyle\int_0^{p} x^m (p^q - x^q)^n\, dx = \left(\frac{p^{m+nq+1}}{q}\right) \beta\left(n+1, \frac{m+1}{q}\right)$, m, n, p, q are positive constants.

42. $\displaystyle\int_a^{b} (x-a)^{m-1} (b-x)^{n-1}\, dx = (b-a)^{m+n-1}\, \beta(m, n)$, m, n, a, b are positive constants.

43. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{mx}}{ae^{nx}+b}\, dx = \frac{\pi}{n}\left(\frac{b}{a}\right)^{m/n} \left[\frac{1}{b\sin(m\pi/n)}\right]$, a, b, m, n are positive constants.

44. $\displaystyle\int_{-1}^{1} (1-x^2)^n\, dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$, n is a positive integer.

45. $\displaystyle\int_0^{m} x^n \left(1 - \frac{x}{m}\right)^{m-1} dx = m^{n+1}\, \beta(m, n+1)$, m, n are positive constants.

46. $\displaystyle\int x^m e^{-\alpha x^n}\, dx = \frac{1}{n\alpha^{(m+1)/n}}\, \Gamma\left(\frac{m+1}{n}\right)$, m, n, α are positive constants.

47. $\displaystyle\int_0^{\infty} e^{-mx} (1-e^{-x})^n\, dx = \beta(m, n+1)$, m, n, are positive constants.

48. $\displaystyle\int_0^{1} x^m (\ln x)^n\, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$, $m > 1$ and n is a positive integer.

49. $\displaystyle\int_0^{\infty} \frac{\sin x}{x^p}\, dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}$, $0 < p < 1$, $\left(\text{given that } \displaystyle\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}\right)$.

50. For large n, $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$ (Stirling's formula).

51. $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n (a+b)^m}\,\beta(m,n)$

Using the concept of differentiation of integrals (assuming that the differentiation is valid) evaluate the following integrals:

52. $\displaystyle\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx,\ a>0, b>0.$

53. $\displaystyle\int_0^1 \frac{x^a - x^b}{\log x}\,dx,\ a>b>-1.$

54. $\displaystyle\int_0^1 x^n (\log x)^k\,dx,$ n any integer > -1.

55. $\displaystyle\int_0^{\pi/2\alpha} \alpha \sin \alpha x\,dx.$

56. $\displaystyle\int_{-\infty}^\infty x^2 e^{-x^2}\,dx,$ where $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,dx\sqrt{\pi},$

57. $\displaystyle\int_0^{\alpha^3} \cot^{-1}(x/\alpha^3)\,dx.$

58. $\displaystyle\int_0^\pi \frac{\cos x}{(a+b\cos x)^3}\,dx,$ given that $\displaystyle\int_0^\pi \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}},\ a>b>0,$

59. $\displaystyle\int_0^\infty e^{-x^2-(a^2/x^2)}\,dx,\ a>0,$ given that $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$

60. $\displaystyle\int_0^{\pi/2} \log(1-\alpha^2\sin^2 x)\,dx,\ |x|<1$

61. $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^{n+1}},$ n any positive integer.

62. Show that $\displaystyle\frac{d}{dx}[erfc(\alpha x)] = -\frac{2\alpha}{\sqrt{\pi}}\,e^{-\alpha^2 x^2}.$

63. Show that $\displaystyle erf(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3(1!)} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \dots\right]$

64. Show that $\displaystyle\int_0^t erfc(\alpha x)\,dx = t\,erfc(\alpha t) - \frac{1}{\alpha\sqrt{\pi}}[e^{-\alpha^2 t^2} - 1].$

65. Show that $\displaystyle\int_0^\infty e^{-t^2-2\alpha t}\,dt = \frac{\sqrt{\pi}}{2}\,e^{\alpha^2}[1-erf(\alpha)].$

66. The relation $\displaystyle\int_0^\infty e^{-x^2}\cos(2bx)\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$ is given (see Example 13.40). Deduce the result for $\displaystyle\int_0^\infty e^{-\alpha^2x^2}\cos(px)\,dx.$ Integrate this result with respect p, taken as a parameter, from $p=0$ to $p=s$ and show that

$$\int_0^\infty e^{-\alpha^2 x^2}\left(\frac{\sin(sx)}{x}\right)dx = \frac{\pi}{2}\,erf\left(\frac{s}{2\alpha}\right).$$

• Substitution

(1) δ-ε approach → Verify if the given limit exists

(2) Diff path → Prove lim doesnt exist

(3) Polar coordinates → To find limit [If lim dependent on ⇒ doesnt exist

(4) δ-ε in Polar → Verify

Chapter 2

Functions of Several Real Variables

• $|xy| \leq \frac{x^2+y^2}{2}$ ∵ $(x-y)^2 \geq$

• $|x|+|y| \leq 2\sqrt{x^2+y^2}$

2.1 Introduction

In Chapter 1 we studied the calculus of functions of a single real variable defined by $y = f(x)$. In this chapter we shall extend the concepts of functions of one variable to functions of two or more variables.

If to each point (x, y) of a certain part of the x-y plane, $x \in \mathbb{R}$, $y \in \mathbb{R}$ or $(x, y) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, there corresponds a real value z according to some rule $f(x, y)$, then $f(x, y)$ is called a *real valued function of two variables* x and y and is written as

$$z = f(x, y),\ x \in \mathbb{R},\ y \in \mathbb{R}, \text{ or } (x, y) \in \mathbb{R}^2,\ z \in \mathbb{R}. \tag{2.1}$$

We call x, y as the independent variables and z as the dependent variable.

In general, we define a real valued function of n variables as

$$z = f(x_1, x_2, \ldots, x_n),\quad (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,\ z \in \mathbb{R} \tag{2.2}$$

where $x_1, x_2, \ldots, x_n$ are the n independent variables and z is the dependent variable. The point $(x_1, x_2, \ldots, x_n)$ is called an *n-tuple* and lies in an n-dimensional space. In this case, the function f maps $\mathbb{R}^n$ into $\mathbb{R}$.

The function as defined by Eq. (2.2) is called an *explicit* function, whereas a function defined by $\phi(z, x_1, x_2, \ldots, x_n) = 0$ is called an *implicit* function.

We shall discuss the calculus of the functions of two variables in detail and then generalize to the case of several variables.

2.2 Functions of Two Variables

Consider the function of two variables

$$z = f(x, y). \tag{2.3}$$

The set of points (x, y) in the x-y plane for which $f(x, y)$ is defined is called the *domain* of definition of the function and is denoted by D. This domain may be the entire x-y plane or a part of the x-y plane. The collection of the corresponding values of z is called the *range* of the function. The following are some examples

$z = \sqrt{1 - x^2 - y^2}$: z is real. Therefore, we have $1 - x^2 - y^2 \geq 0$, or $x^2 + y^2 \leq 1$, that is, the domain is the region $x^2 + y^2 \leq 1$. The range is the set of all real, positive numbers.

$z = 1/(x^2 - y^2)$: The domain is the set of all points (x, y) such that $x^2 - y^2 \neq 0$, that is $y \neq \pm x$. The range is $\mathbb{R}$.

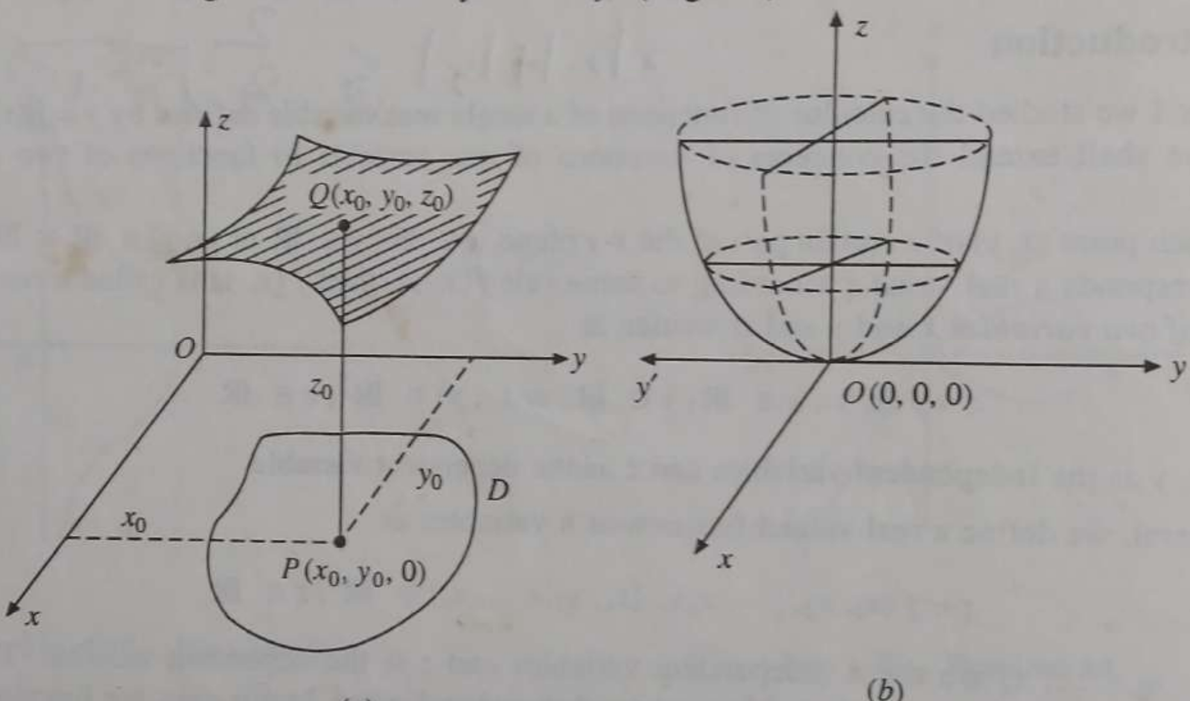
$z = \log(x + y)$: The domain is the set of all points (x, y) such that $x + y > 0$. The range is $\mathbb{R}$.

The domain of a function and its *natural domain* can be different. For example, we have

$$f(x, y) = \text{area of a triangle} = x\,y/2$$

where x and y are respectively the base and the altitude of the triangle. The domain is $x > 0$, $y > 0$, whereas the natural domain of the function is the entire x-y plane.

Consider the rectangular coordinate system $Oxyz$ (Fig. 2.1).



(*a*) (*b*)

Fig. 2.1. Function of two variables.

At each point $P(x_0, y_0, 0)$ in the x-y plane, construct a perpendicular to the x-y plane. Take a point Q on it such that $PQ = z_0 = f(x_0, y_0)$. This gives a point $Q(x_0, y_0, z_0)$, or $Q(x_0, y_0, f(x_0, y_0))$ in space. The locus of all such points (x, y, z) satisfying $z = f(x, y)$ is called a surface. For example, the graph of the function $z = x^2 + y^2$, $(x, y) \in \mathbb{R}^2$ is the paraboloid of revolution as given in Fig. 2.1b. Each perpendicular to the x-y plane intersects the surface $z = f(x, y)$ at exactly one point if $(x, y) \in D$ and at no point if $(x, y) \notin D$.

The graph of $z = f(x, y) = c$, where c is a real constant is called a *level curve*. For example, for the paraboloid of revolution $z = x^2 + y^2$, the level curves are the circles $x^2 + y^2 = c$, $c > 0$.

We define the following:

Distance between two points Let $P(x_0, y_0)$ and $Q(x_1, y_1)$ be any two points in $\mathbb{R}^2$. Then

$$d(P, Q) = |PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \tag{2.4}$$

is called the distance between the points P and Q.

{NEIGHBOURHOOD of a POINT}

Neighborhood of a point Let $P(x_0, y_0)$ be a point in $\mathbb{R}^2$. Then the δ-neighborhood of the point $P(x_0, y_0)$ is the set of all points (x, y) which lie inside a circle of radius δ with centre at the point (x_0, y_0), (Fig. 2.2). We usually denote this neighborhood by $N_\delta(P)$ or by $N(P, \delta)$.

δ

$N_\delta(P)$

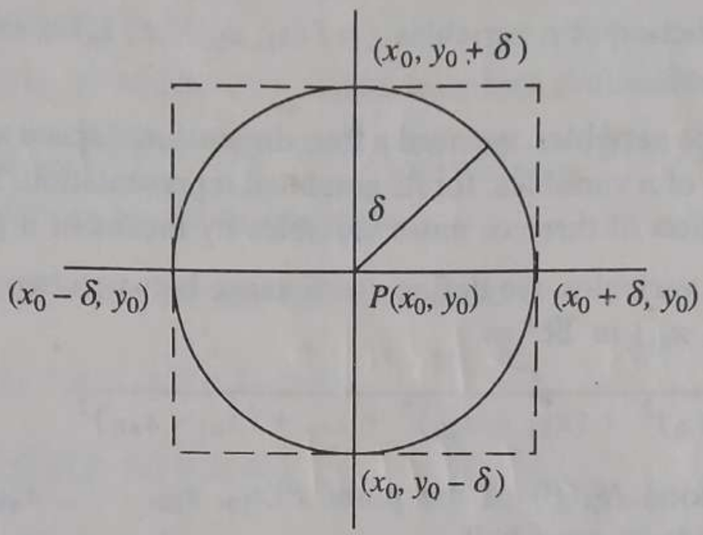


Fig. 2.2. Neighborhood of a point $P(x_0, y_0)$.

Therefore,

$$N_\delta(P) = \left\{(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}. \tag{2.5}$$

Since $|x - x_0| \le \sqrt{(x - x_0)^2 + (y - y_0)^2}$ and $|y - y_0| \le \sqrt{(x - x_0)^2 + (y - y_0)^2}$,

the neighborhood of the point $P(x_0, y_0)$ can also be defined as

$$N_\delta(P) = \{(x, y): |x - x_0| < \delta \text{ and } |y - y_0| < \delta\}. \tag{2.6}$$

that is, the set of all points which lie inside a square of side 2δ with centre at (x_0, y_0) and sides parallel to the coordinate axes (Fig. 2.2).

* If the point $P(x_0, y_0)$ is not included in the set, then it is called the *deleted δ-neighborhood* of the point, that is, the set of points which satisfy

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \tag{2.7}$$

is called the deleted neighborhood of $P(x_0, y_0)$.

Open domain A domain D is open, if for every point P in D, there exists a $\delta > 0$ such that all points in the δ-neighborhood of P are in D.

Connected domain A domain D is connected, if any two points $P, Q \in D$ can be joined by finitely many number of line segments all of which lie entirely in D.

Bounded domain A domain D is bounded, if there exists a real finite positive number M (no matter how large) such that D can be enclosed within a circle with radius M and centre at the origin. That is, the distance of any point P in D from the origin is less than M, $|OP| < M$.

Closed region A closed region is a bounded domain together with its boundary.

Bounded function A function $f(x, y)$ defined in some domain D in $\mathbb{R}^2$ is bounded, if there exists a real finite positive number M such that $|f(x, y)| \leq M$ for all $(x, y) \in D$.

Remark 1

(a) The domain of a function of n variables $z = f(x_1, x_2, \ldots, x_n)$ is the set of all n-tuples in $\mathbb{R}^n$ for which f is defined.

(b) For functions of three variables, we need a four-dimensional space and an $(n + 1)$ - dimensional space for a function of n variables, for its graphical representation. Therefore, it is not possible to represent a function of three or more variables by means of a graph in space.

(c) For a function of n variables, we define the distance between two points $P(x_{10}, x_{20}, \ldots, x_{n0})$ and $Q(x_{11}, x_{21}, \ldots, x_{n1})$ in $\mathbb{R}^n$ as

$$|PQ| = \sqrt{(x_{11} - x_{10})^2 + (x_{21} - x_{20})^2 + \ldots + (x_{n1} - x_{n0})^2}$$

and the neighborhood $N_\delta(P)$ of the point $P(x_{10}, x_{20}, \ldots, x_{n0})$ is the set of all points $(x_1, x_2, \ldots, x_n)$ inside an open ball

$$\sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + \ldots + (x_n - x_{n0})^2} < \delta.$$

2.2.1 Limits

Let $z = f(x, y)$ be a function of two variables defined in a domain D. Let $P(x_0, y_0)$ be a point in D. If for a given real number $\varepsilon > 0$, however small, we can find a real number $\delta > 0$ such that for every point (x, y) in the δ-neighborhood of $P(x_0, y_0)$

$$|f(x, y) - L| < \varepsilon, \quad \text{whenever} \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \tag{2.8}$$

then the real, finite number L is called the limit of the function $f(x, y)$ as $(x, y) \to (x_0, y_0)$. Symbolically, we write it as

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L.$$

Note that for the limit to exist, the function $f(x, y)$ may or may not be defined at (x_0, y_0). If $f(x, y)$ is not defined at $P(x_0, y_0)$, then we write

$$|f(x, y) - L| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

This definition is called the δ-ε approach to study the existence of limits.

Remark 2

(a) $\lim_{(x, y) \to (x_0, y_0)} f(x, y)$, if it exists is unique. (The proof is similar to the case of functions of one variable).

(b) Let $x = r \cos \theta$, $y = r \sin \theta$ so that $x^2 + y^2 = r^2$ and $\theta = \tan^{-1}(y/x)$. Then, we can define the limit given in Eq. (2.8) as

$$\lim_{r \to 0} |f(r \cos \theta, r \sin \theta) - L| < \varepsilon, \quad \text{whenever} \quad r < \delta, \text{ independent of } \theta.$$

Final exp must be independent of θ

depending upon θ makes it path dependent

(c) Since $(x, y) \to (x_0, y_0)$ in the two-dimensional plane, there are infinite number of paths joining (x, y) to (x_0, y_0). Since the limit is unique, the limit is same along all the paths, that is the limit is independent of the path. Thus, the limit of a function cannot be obtained by approaching the point P along a particular path and finding the limit of $f(x, y)$. If the limit is dependent on a path, then the limit does not exist.

Let $u = f(x, y)$ and $v = g(x, y)$ be two real valued functions defined in a domain D. Let

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L_1 \quad \text{and} \quad \lim_{(x, y) \to (x_0, y_0)} g(x, y) = L_2.$$

Then, the following results can be easily established.

(i) $\lim_{(x, y) \to (x_0, y_0)} [k f(x, y)] = k L_1$ for any real constant k.

(ii) $\lim_{(x, y) \to (x_0, y_0)} [f(x, y) \pm g(x, y)] = L_1 \pm L_2$.

(iii) $\lim_{(x, y) \to (x_0, y_0)} [f(x, y)\, g(x, y)] = L_1 L_2$.

(iv) $\lim_{(x, y) \to (x_0, y_0)} [f(x, y)/g(x, y)] = L_1 / L_2,\ L_2 \neq 0$.

Remark 3

Let $z = f(x_1, x_2, \ldots, x_n)$ be a function of n variables defined in some domain D in $\mathbb{R}^n$. Then, for any fixed point $P_0(x_{10}, x_{20}, \ldots, x_{n0})$ in D

$$\lim_{P \to P_0} f(x_1, x_2, \ldots, x_n) = L.$$

if $| f(x_1, x_2, \ldots, x_n) - L | < \varepsilon$, whenever $\sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + \ldots + (x_n - x_{n0})^2} < \delta$ where $P(x_1, x_2, \ldots, x_n)$ is a point in the neighborhood or the deleted neighborhood of P_0.

Example 2.1 Using the δ-ε approach, show that

(i) $\lim_{(x, y) \to (2, 1)} (3x + 4y) = 10$, (ii) $\lim_{(x, y) \to (1, 1)} (x^2 + 2y) = 3$.

Solution

(i) Here $f(x, y) = 3x + 4y$ is defined at $(2, 1)$. We have

$| f(x, y) - 10 | = | 3x + 4y - 10 | = | 3(x - 2) + 4(y - 1) | \leq 3 | x - 2 | + 4 | y - 1 |$.

If we take $| x - 2 | < \delta$ and $| y - 1 | < \delta$, we get $| f(x, y) - 10 | < 7\delta < \varepsilon$, which is satisfied when $\delta < \varepsilon / 7$.

Hence, $\lim_{(x, y) \to (2, 1)} f(x, y) = 10$.

Note that the value of δ is not unique.

(ii) Here $f(x, y) = x^2 + 2y$ is defined at $(1, 1)$. We have

$$\begin{aligned} | f(x, y) - 3 | &= | x^2 + 2y - 3 | = | (x - 1 + 1)^2 + 2(y - 1 + 1) - 3 | \\ &= | (x - 1)^2 + 2(x - 1) + 2(y - 1) | \leq | x - 1 |^2 + 2 | x - 1 | + 2 | y - 1 | \end{aligned}$$

If we take $| x - 1 | < \delta$ and $| y - 1 | < \delta$, we get $| f(x, y) - 3 | < \delta^2 + 4\delta < \varepsilon$ which is satisfied when

$$(\delta + 2)^2 < \varepsilon + 4 \text{ or } \delta < \sqrt{\varepsilon + 4} - 2.$$

Hence, $\lim_{(x, y) \to (1, 1)} f(x, y) = 3.$

We can also write $\quad |f(x, y) - 3| < \delta^2 + 4\delta < 5\delta < \varepsilon$

which is satisfied when $\delta < \varepsilon/5$.

Example 2.2 Using δ-ε approach, show that

(i) $\displaystyle \lim_{(x, y) \to (0, 0)} \left(\frac{xy}{\sqrt{x^2 + y^2}} \right) = 0$, (ii) $\displaystyle \lim_{(x, y, z) \to (0, 0, 0)} \left(\frac{xy + xz + yz}{\sqrt{x^2 + y^2 + z^2}} \right) = 0.$

Solution

(i) Here $f(x, y) = xy/(\sqrt{x^2 + y^2})$ is not defined at $(0, 0)$. We have

$$\left| \frac{xy}{\sqrt{x^2 + y^2}} - 0 \right| = \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| \le \frac{1}{2} \frac{(x^2 + y^2)}{\sqrt{x^2 + y^2}} = \frac{1}{2} \sqrt{x^2 + y^2} < \varepsilon, (x, y) \ne (0, 0)$$

since $|xy| \le (x^2 + y^2)/2$. If we choose $\delta < 2\varepsilon$, then we get

$$\left| \frac{xy}{\sqrt{x^2 + y^2}} - 0 \right| < \varepsilon, \text{ whenever } 0 < \sqrt{x^2 + y^2} < \delta.$$

Hence, $\displaystyle \lim_{(x, y) \to (0, 0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0.$

Alternative Writing $x = r \cos \theta$, $y = r \sin \theta$, we obtain

$$\lim_{(x, y) \to (0, 0)} \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = \lim_{r \to 0} \left| \frac{r^2 \sin \theta \cos \theta}{r} \right| = 0$$

which is independent of θ.

(ii) Here $f(x, y, z) = (xy + xz + yz)/\sqrt{x^2 + y^2 + z^2}$ is not defined at $(0, 0, 0)$.

Since $|xy| \le (x^2 + y^2)/2$, $|xz| \le (x^2 + z^2)/2$, $|yz| < (y^2 + z^2)/2$, we get

$$\left| \frac{xy + xz + yz}{\sqrt{x^2 + y^2 + z^2}} - 0 \right| \le \frac{1}{2} \left[\frac{x^2 + y^2 + x^2 + z^2 + y^2 + z^2}{\sqrt{x^2 + y^2 + z^2}} \right] = \left| \sqrt{x^2 + y^2 + z^2} \right| < \varepsilon.$$

If we choose $\delta < \varepsilon$, we obtain

$$\left| \frac{xy + xz + yz}{\sqrt{x^2 + y^2 + z^2}} - 0 \right| < \varepsilon \text{ whenever } 0 < \sqrt{x^2 + y^2 + z^2} < \delta.$$

Hence, $\displaystyle \lim_{(x, y, z) \to (0, 0, 0)} \left[\frac{xy + xz + yz}{\sqrt{x^2 + y^2 + z^2}} \right] = 0.$

Example 2.3 Show that the following limits

(i) $\lim\limits_{(x,\,y) \to (0,\,0)} \dfrac{xy}{x^2 + y^2}$, (ii) $\lim\limits_{(x,\,y) \to (0,\,0)} \dfrac{x + \sqrt{y}}{x^2 + y}$, How do I explain variance in Limit using θ.

(iii) $\lim\limits_{(x,\,y) \to (0,\,0)} \dfrac{x^3 y}{x^6 + y^2}$. (iv) $\lim\limits_{(x,\,y) \to (0,\,1)} \tan^{-1}\left(\dfrac{y}{x}\right)$.

do not exist.

Solution The limit does not exist if it is not finite, or if it depends on a particular path.

(i) Consider the path $y = mx$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore

$$\lim_{(x,\,y) \to (0,\,0)} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{mx^2}{(1 + m^2)x^2} = \frac{m}{1 + m^2}$$

which depends on m. For different values of m, we obtain different limits. Hence, the limit does not exist.

Alternative Setting $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\lim_{(x,\,y) \to (0,\,0)} \frac{xy}{x^2 + y^2} = \lim_{r \to 0} \frac{r^2 \sin\theta\cos\theta}{r^2} = \sin\theta\cos\theta$$

which depends on θ. Hence, the limit is dependent on different radial paths θ = constant. Hence, the limit does not exist.

(ii) Choose the path $y = mx^2$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore,

$$\lim_{(x,\,y) \to (0,\,0)} \frac{x + \sqrt{y}}{x^2 + y} = \lim_{x \to 0} \frac{1 + \sqrt{m}}{(1 + m)x} = \infty.$$

Since the limit is not finite, the limit does not exist.

(iii) Choose the path $y = mx^3$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore

$$\lim_{(x,\,y) \to (0,\,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{x \to 0} \frac{mx^6}{(1 + m^2)x^6} = \frac{m}{1 + m^2}$$

which depends on m. For different values of m, we obtain different limits. Hence, the limit does not exist.

(iv) We have

$$\lim_{(x,\,y) \to (0,\,1)} \tan^{-1}\frac{y}{x} = \tan^{-1}(\pm\infty) = \pm\frac{\pi}{2}$$

depending on whether the point $(0, 1)$ is approached from left or from right along the line $y = 1$. If we approach from left, we obtain the limit as $-\pi/2$ and if we approach from right, we obtain the limit as $\pi/2$. Since the limit is not unique, the limit does not exist as $(x, y) \to (0, 1)$.

2.2.2 Continuity

A function $z = f(x, y)$ is said to be *continuous* at a point (x_0, y_0), if

(i) $f(x, y)$ is defined at the point (x_0, y_0), (ii) $\lim\limits_{(x,\,y) \to (x_0,\,y_0)} f(x, y)$ exists, and

(iii) $\lim\limits_{(x,\,y) \to (x_0,\,y_0)} f(x, y) = f(x_0, y_0)$.

If any one of the above conditions is not satisfied, then the function is said to be discontinuous at the point (x_0, y_0).

Therefore, a function $f(x, y)$ is continuous at (x_0, y_0) if

$$|f(x, y) - f(x_0, y_0)| < \varepsilon, \quad \text{whenever} \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta. \tag{2.9}$$

If $f(x_0, y_0)$ is defined and $\lim\limits_{(x, y) \to (x_0, y_0)} f(x, y) = L$ exists, but $f(x_0, y_0) \neq L$, then the point (x_0, y_0) is called a point of *removable discontinuity*. We can redefine the function at the point (x_0, y_0) as $f(x_0, y_0) = L$ so that the new function becomes continuous at the point (x_0, y_0).

If the function $f(x, y)$ is continuous at every point in a domain D, then it is said to be continuous in D.

In the definition of continuity, $\lim\limits_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0)$ holds for all paths going to the point (x_0, y_0). Hence, if the continuity of a function is to be proved, we cannot choose a path and find the limit. However, to show that a function is discontinuous, it is sufficient to choose a path and show that the limit does not exist.

A continuous function has the following properties:

P1 A continuous function in a closed and bounded domain D attains atleast once its maximum value M and its minimum value m at some point inside or on the boundary of D.

P2 For any number μ that satisfies $m < \mu < M$, there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = \mu$.

P3 A continuous function, in a closed and bounded domain D, that attains both positive and negative values will have the value zero at some point in D.

P4 If $z = f(x, y)$ is continuous at some point $P(x_0, y_0)$ and $w = g(z)$ is a composite function defined at $z_0 = f(x_0, y_0)$, then the composite function $g(f(z))$ is also continuous at P. For example, the functions e^{x-y}, $\log(x^2 + y^2)$, $\sin(x + y)$ etc. are continuous functions.

Example 2.4 Show that the following functions are continuous at the point (0, 0).

(i) $f(x, y) = \begin{cases} \dfrac{2x^4 + 3y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0), \end{cases}$ (ii) $f(x, y) = \begin{cases} \dfrac{2x(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0), \end{cases}$

(iii) $f(x, y) = \begin{cases} \dfrac{\sin^{-1}(x + 2y)}{\tan^{-1}(2x + 4y)}, & (x, y) \neq (0, 0) \\ 1/2, & (x, y) = (0, 0). \end{cases}$

Solution

4th method - Polar coordinates in δ-ε

(i) Let $x = r\cos\theta$, $y = r\sin\theta$. Then, $r = \sqrt{x^2 + y^2} \neq 0$. We have

$$|f(x, y) - f(0, 0)| = \left|\frac{2x^4 + 3y^4}{x^2 + y^2}\right| = \left|\frac{r^4(2\cos^4\theta + 3\sin^4\theta)}{r^2(\cos^2\theta + \sin^2\theta)}\right|$$

$$< r^2[2|\cos^4\theta| + 3|\sin^4\theta|] < 5r^2 < \varepsilon$$

or $$r = \sqrt{x^2 + y^2} < \sqrt{\varepsilon/5}.$$

If we choose $\delta < \sqrt{\varepsilon/5}$, we find that $|f(x, y) - f(0, 0)| < \varepsilon$, whenever $0 < \sqrt{x^2 + y^2} < \delta$. Therefore, $\lim_{(x, y)\to(0, 0)} f(x, y) = f(0, 0) = 0$. Hence, $f(x, y)$ is continuous at $(0, 0)$.

(ii) Let $x = r\cos\theta$, $y = r\sin\theta$. Then, $r = \sqrt{x^2 + y^2} \neq 0$. We have

$$|f(x, y) - f(0, 0)| = \left|\frac{2x(x^2 - y^2)}{x^2 + y^2}\right| = \left|\frac{2r^3(\cos^2\theta - \sin^2\theta)\cos\theta}{r^2(\cos^2\theta + \sin^2\theta)}\right|$$

$$= |2r\cos 2\theta\cos\theta| \leq 2r < \varepsilon$$

or $$r = \sqrt{x^2 + y^2} < \varepsilon/2.$$

If we choose $\delta < \varepsilon/2$, we find that

$$|f(x, y) - f(0, 0)| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim_{(x, y)\to(0, 0)} f(x, y) = f(0, 0) = 0$. Hence, $f(x, y)$ is continuous at $(0, 0)$.

(iii) Let $x + 2y = t$. Therefore, $t \to 0$ as $(x, y) \to (0, 0)$.

We can now write

$$\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{t\to 0} \frac{\sin^{-1} t}{\tan^{-1} 2t} = \lim_{t\to 0}\left[\frac{(\sin^{-1} t)/t}{(\tan^{-1}(2t))/(2t)}\right]\left[\frac{t}{2t}\right] = \frac{1}{2}.$$

Since $\lim_{(x, y)\to(0, 0)} f(x, y) = f(0, 0) = \frac{1}{2}$, the given function is continuous at $(x, y) = (0, 0)$.

Example 2.5 Show that the following functions are discontinuous at the given points

(i) $f(x, y) = \begin{cases} \dfrac{x - y}{x + y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ at the point $(0, 0)$.

(ii) $f(x, y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2 + y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ at the point $(0, 0)$.

(iii) $f(x, y) = \begin{cases} \dfrac{x^2 + xy + x + y}{x + y}, & (x, y) \neq (2, 2) \\ 4, & (x, y) = (2, 2) \end{cases}$ at the point $(2, 2)$.

Solution

(i) Choose the path $y = mx$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore,

$$\lim_{(x, y)\to(0, 0)} \frac{x - y}{x + y} = \lim_{x\to 0} \frac{(1 - m)x}{(1 + m)x} = \frac{1 - m}{1 + m}$$

which depends on m. Since, the limit does not exist, the function is not continuous at $(0, 0)$.

(ii) Choose the path $y = m^2x^2$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore,

$$\lim_{(x, y)\to(0, 0)} \frac{x^2 - x\sqrt{y}}{x^2 + y} = \lim_{x \to 0} \frac{(1-m)x^2}{(1+m^2)x^2} = \frac{1-m}{1+m^2}$$

which depends on m. Since the limit does not exist, the function is not continuous at $(0, 0)$.

(iii) $$\lim_{(x, y)\to(2, 2)} f(x, y) = \lim_{(x, y)\to(2, 2)} \frac{(x+y)(x+1)}{(x+y)} = \lim_{(x, y)\to(2, 2)} (x+1) = 3.$$

Since $\lim_{(x, y)\to(2, 2)} f(x, y) \neq f(2, 2)$, the function is not continuous at $(2, 2)$.

Note that the point $(2, 2)$ is a point of removable discontinuity.

Example 2.6 Let $f(x, y) = \begin{cases} \dfrac{x^4y - 3x^2y^3 + y^5}{(x^2+y^2)^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

Find a $\delta > 0$ such that $|f(x, y) - f(0, 0)| < 0.01$, whenever $\sqrt{x^2+y^2} < \delta$.

Solution We have

$$|f(x, y) - f(0, 0)| = \left| \frac{x^4y - 3x^2y^3 + y^5}{(x^2+y^2)^2} \right|.$$

Substituting $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$|f(x, y) - f(0, 0)| = \left| \frac{r^5(\cos^4\theta\sin\theta - 3\cos^2\theta\sin^3\theta + \sin^5\theta)}{r^4(\cos^2\theta + \sin^2\theta)^2} \right|$$

$$= |r(\cos^4\theta\sin\theta - 3\cos^2\theta\sin^3\theta + \sin^5\theta)|$$

$$\leq r(1+3+1) = 5r = 5\sqrt{x^2+y^2} < 0.01.$$

Therefore, $\sqrt{x^2+y^2} \leq 0.01/5 = 0.002$. Hence, $\delta < 0.002$.

Exercise 2.1

Using the δ-ε approach, establish the following limits.

1. $\lim_{(x, y)\to(1, 1)} (x^2 + y^2 - 1) = 1.$

2. $\lim_{(x, y)\to(2, 1)} (x^2 + 2x - y^2) = 7.$

3. $\lim_{(x, y)\to(0, 0)} \dfrac{x+y}{x^2+y^2+1} = 0.$

4. $\lim_{(x, y)\to(0, 0)} \dfrac{x^3+y^3}{x^2+y^2} = 0.$

5. $\lim_{(x, y)\to(0, 0)} \left[y + x\cos\left(\dfrac{1}{y}\right) \right] = 0.$

6. $\lim_{(x, y)\to(0, 0)} (x^2+y^2)\sin\dfrac{1}{xy} = 0.$

Determine the following limits if they exist.

7. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}}$.

8. $\lim\limits_{(x,y)\to(1,-1)} \dfrac{x^3-y^3}{x-y}$.

9. $\lim\limits_{(x,y)\to(\alpha,0)} \left(1+\dfrac{x}{y}\right)^y$.

10. $\lim\limits_{(x,y)\to(0,0)} \cot^{-1}\left(\dfrac{1}{\sqrt{x^2+y^2}}\right)$.

11. $\lim\limits_{(x,y)\to(0,1)} \dfrac{(y-1)\tan^2 x}{x^2(y^2-1)}$.

12. $\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)\sin y}{y\ln x}$.

13. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-x-y}{x^2+y^2}$.

14. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x^2+y^2}$.

15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^3+y^3}$.

16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4y^2}{(x^4+y^2)^2}$.

17. $\lim\limits_{(x,y,z)\to(0,0,0)} \log\left(\dfrac{z}{xy}\right)$.

18. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy+z}{x+y+z^2}$.

19. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy^2z^2}{x^4+y^4+z^8}$.

20. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x(x+y+z)}{x^2+y^2+z^2}$.

Discuss the continuity of the following functions at the given points.

21. $f(x,y)=\begin{cases}\dfrac{(x-y)^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at (0, 0).

22. $f(x,y)=\begin{cases}\dfrac{1}{1+e^{1/x}}+y^2, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at (0, 0).

23. $f(x,y)=\begin{cases}\dfrac{e^{xy}}{x^2+1}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at (0, 0).

24. $f(x,y)=\begin{cases}\dfrac{x^2+y^2}{xy}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at (0, 0).

25. $f(x,y)=\begin{cases}\dfrac{x^2+y^2}{\tan xy}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at (0, 0).

26. $f(x,y)=\begin{cases}\dfrac{x^2-2xy+y^2}{x-y}, & (x,y)\neq(1,-1)\\ 0, & (x,y)=(1,-1)\end{cases}$

at (1, −1).

27. $f(x,y)=\begin{cases}\dfrac{xy(x-y)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at (0, 0).

28. $f(x,y)=\begin{cases}\dfrac{x^4y^4}{(x^2+y^4)^3}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at (0, 0).

29. $f(x,y)=\begin{cases}\dfrac{\sin\sqrt{|xy|}-\sqrt{|xy|}}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at (0, 0).

30. $f(x,y)=\begin{cases}\dfrac{2x^2+y^2}{3+\sin x}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at (0, 0).

31. $f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^3+y^3}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at $(0, 0)$.

32. $f(x,y)=\begin{cases}\dfrac{x^5-y^5}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at $(0, 0)$.

33. $f(x,y)=\begin{cases}\dfrac{x^2y}{1+x}, & x\neq-1\\ y, & (x,y)=(-1,\alpha)\end{cases}$

at $(-1, \alpha)$.

34. $f(x,y,z)=\begin{cases}\dfrac{xyz}{x^2+y^2+z^2}, & (x,y,z)\neq(0,0,0)\\ 0, & (x,y,z)=(0,0,0)\end{cases}$

at $(0, 0, 0)$.

35. $f(x,y,z)=\begin{cases}\dfrac{2xy}{x^2-3z^2}, & (x,y,z)\neq(0,0,0)\\ 0, & (x,y,z)=(0,0,0)\end{cases}$

at $(0, 0, 0)$.

2.3 Partial Derivatives

The derivative of a function of several variables with respect to one of the independent variables keeping all the other independent variables as constant is called the *partial derivative* of the function with respect to that variable.

Consider the function of two variables $z=f(x,y)$ defined in some domain D of the x-y plane. Let y be held constant, say $y=y_0$. Then, the function $f(x, y_0)$ depends on x alone and is defined in an interval about x, that is $f(x,y_0)$ is a function of one variable x. Let the points (x, y_0) and $(x+\Delta x, y_0)$ be in D, where Δx is an increment in the independent variable x. Then

$$\Delta_x z = f(x+\Delta x, y_0) - f(x, y_0) \tag{2.10}$$

is called the *partial increment* in z with respect to x and is a function of x and Δx.

Similarly, if x is held constant, say $x=x_0$, then the function $f(x_0, y)$ depends only on y and is defined in some interval about y, that is $f(x_0, y)$ is a function of one variable y. Let the points (x_0, y) and $(x_0, y+\Delta y)$ be in D, where Δy is an increment in the independent variable y. Then

$$\Delta_y z = f(x_0, y+\Delta y) - f(x_0, y) \tag{2.11}$$

is called the partial increment in z with respect to y and is a function of y and Δy.

When both x and y are given increments Δx and Δy respectively, then the increment Δz in z is given by

$$\boxed{\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y).} \tag{2.12}$$

This increment is called the *total increment* in z and is a function of x, y, Δx and Δy.

In general, $\Delta z \neq \Delta_x z + \Delta_y z$. For example, consider the function $z=f(x,y)=xy$ and a point (x_0, y_0). We have

$$\Delta_x z = (x_0+\Delta x)y_0 - x_0y_0 = y_0\Delta x$$

& partial der are to be found at (x, y); then Ist principle is to be used

$$\Delta_y z = x_0(y_0 + \Delta y) - x_0 y_0 = x_0 \Delta y$$

$$\Delta z = (x_0 + \Delta x)(y_0 + \Delta y) - x_0 y_0 = x_0 \Delta y + y_0 \Delta y + \Delta x \Delta y \neq \Delta_x z + \Delta_y z.$$

Now, consider the limit

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}. = f_x(x_0, y_0) \tag{2.13}$$

If this limit exists, then this limit is called the first order partial derivative of z or $f(x, y)$ with respect to x at the point (x_0, y_0) and is denoted by $z_x(x_0, y_0)$ or $f_x(x_0, y_0)$ or $(\partial f/\partial x)(x_0, y_0)$ or $(\partial z/\partial x)(x_0, y_0)$.

Similarly, if the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = f_y(x_0, y_0) \tag{2.14}$$

exists, then this limit is called the first order partial derivative of z or $f(x, y)$ with respect to y at the point (x_0, y_0) and is denoted by $z_y(x_0, y_0)$ or $f_y(x_0, y_0)$ or $(\partial z/\partial y)(x_0, y_0)$ or $(\partial f/\partial y)(x_0, y_0)$.

Remark 4

Let $z = f(x_1, x_2, \ldots, x_n)$ be a function of n variables defined in some domain D in $\mathbb{R}^n$. Let $P_0(x_1, x_2, \ldots, x_n)$ be a point in D. If the limit

$$\lim_{\Delta x_i \to 0} \frac{\Delta_{x_i} z}{\Delta x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, (x_i + \Delta x_i), \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots x_n)}{\Delta x_i}$$

exists, then it is called the partial derivative of f at the point P_0 and is denoted by $(\partial f/\partial x_i)(P_0)$.

Remark 5

The definition of continuity, $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$ can be written in alternate forms. Set $x = x_0 + \Delta x$, $y = y_0 + \Delta y$. Define $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Then, $\Delta x \to 0$, $\Delta y \to 0$ implies that $\Delta\rho \to 0$. We note that $|\Delta x| < \Delta\rho$ and $|\Delta y| < \Delta\rho$.

The above definition of continuity is equivalent to the following forms:

(i) $\lim_{\Delta x \to 0, \Delta y \to 0} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0.$

(ii) $\lim_{\Delta\rho \to 0} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0.$

(iii) $\lim_{\Delta\rho \to 0} \Delta z = 0.$

Example 2.7 Find the first order partial derivatives of the following functions

(i) $f(x, y) = x^2 + y^2 + x$, (ii) $f(x, y) = y\, e^{-x}$, (iii) $f(x, y) = \sin(2x + 3y)$

at the point (x, y) from the first principles.

Solution we have

(i) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + y^2 + (x + \Delta x)] - [x^2 + y^2 + x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2x + 1)\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} [2x + 1 + \Delta x] = 2x + 1.$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{[x^2 + (y+\Delta y)^2 + x] - [x^2 + y^2 + x]}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2y\,\Delta y + (\Delta y)^2}{\Delta y} = \lim_{\Delta y \to 0} [2y + \Delta y] = 2y.$$

(ii) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{ye^{-(x+\Delta x)} - ye^{-x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-ye^{-x}(1 - e^{-\Delta x})}{\Delta x} = -ye^{-x} \lim_{\Delta x \to 0} \frac{1 - e^{-\Delta x}}{\Delta x} = -ye^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{(y+\Delta y)e^{-x} - ye^{-x}}{\Delta y} = e^{-x}.$$

(iii) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\sin(2(x+\Delta x) + 3y) - \sin(2x + 3y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\cos(2x + 3y + \Delta x)\sin \Delta x}{\Delta x}$$

$$= 2\cos(2x + 3y).$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{\sin(2x + 3(y+\Delta y)) - \sin(2x + 3y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2\cos(2x + 3y + 3\Delta y/2)\sin(3\Delta y/2)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} [3\cos(2x + 3y + 3\Delta y/2)] \frac{\sin(3\Delta y/2)}{(3\Delta y/2)} = 3\cos(2x + 3y).$$

Example 2.8 Show that the function

$$f(x, y) = \begin{cases} (x+y)\sin\left(\dfrac{1}{x+y}\right), & x + y \neq 0 \\ 0, & x + y = 0 \end{cases}$$

is continuous at (0, 0) but its partial derivatives f_x and f_y do not exist at (0, 0).

Solution We have

$$|f(x, y) - f(0, 0)| = \left|(x+y)\sin\left(\frac{1}{x+y}\right)\right| \le |x + y| \le |x| + |y| \le 2\sqrt{x^2 + y^2} < \varepsilon.$$

If we choose $\delta < \varepsilon/2$, then

$$|f(x, y) - 0| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim_{(x, y) \to (0,0)} f(x, y) = 0 = f(0, 0)$.

Hence, the given function is continuous at (0, 0).

Now, at (0, 0), the limit

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x \sin(1/\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \sin\left(\frac{1}{\Delta x}\right)$$

does not exist. Therefore, the partial derivative f_x does not exist at (0, 0).

Similarly at (0, 0), the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y \sin(1/\Delta y)}{\Delta y} = \lim_{\Delta y \to 0} \sin\left(\frac{1}{\Delta y}\right)$$

does not exist. Therefore, the partial derivative f_y does not exist at (0, 0).

Example 2.9 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{|x| + |y|}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at (0, 0) but its partial derivatives f_x and f_y do not exist at (0, 0).

Solution We have

$$|f(x, y) - f(0, 0)| = \left| \frac{x^2 + y^2}{|x| + |y|} \right| \leq \frac{[|x| + |y|]^2}{|x| + |y|} = |x| + |y| \leq 2\sqrt{x^2 + y^2} < \varepsilon.$$

Taking $\delta < \varepsilon/2$, we find that

$$|f(x, y) - 0| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim\limits_{(x, y) \to (0,0)} f(x, y) = 0 = f(0, 0)$.

Hence, the given function is continuous at (0, 0).

Now, at (0, 0) we have

$$\lim_{\Delta x \to 0} \frac{\Delta_x f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{|\Delta x|} = \begin{cases} 1, & \text{when } \Delta x > 0 \\ -1, & \text{when } \Delta x < 0. \end{cases}$$

Hence, the limit does not exist. Therefore, f_x does not exist at (0, 0).

Also at (0, 0), the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y f}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{|\Delta y|} = \begin{cases} 1, & \text{when } \Delta y > 0 \\ -1, & \text{when } \Delta y < 0 \end{cases}$$

does not exist. Therefore, f_y does not exist at (0, 0).

Example 2.10 Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + 2y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is not continuous at (0, 0) but its partial derivatives f_x and f_y exist at (0, 0).

Solution Choose the path $y = mx$. Since the limit

$$\lim_{(x, y) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{mx^2}{(1 + 2m^2)x^2} = \frac{m}{1 + 2m^2}$$

depends on m, the function is not continuous at (0, 0). We now have

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0.$$

Therefore, the partial derivatives f_x and f_y exist at (0, 0).

Theorem 2.1 (Sufficient condition for continuity) A sufficient condition for a function $f(x, y)$ to be continuous at a point (x_0, y_0) is that one of its first order partial derivatives exists and is bounded in the neighborhood of (x_0, y_0) and that the other exists at (x_0, y_0).

Proof Let the partial derivative f_x exist and be bounded in the neighborhood of the point (x_0, y_0) and f_y exist at (x_0, y_0). Since f_y exists at (x_0, y_0), we have

$$\lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = f_y(x_0, y_0).$$

Therefore, we can write

$$f(x_0, y_0 + \Delta y) - f(x_0, y_0) = \Delta y f_y(x_0, y_0) + \varepsilon_1 \Delta y \tag{2.15}$$

where ε_1 depends on Δy and tends to zero as $\Delta y \to 0$. Since f_x exists in the neighborhood of (x_0, y_0), we can write using the Lagrange mean value theorem

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) = \Delta x f_x(x_0 + \theta \Delta x, y_0 + \Delta y), 0 < \theta < 1. \tag{2.16}$$

Now, using Eqs. (2.15) and (2.16), we obtain

$$\begin{aligned} f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) &= [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)] + [f(x_0, y_0 + \Delta y) - f(x_0, y_0)] \\ &= \Delta x f_x(x_0 + \theta \Delta x, y_0 + \Delta y) + \Delta y f_y(x_0, y_0) + \varepsilon_1 \Delta y. \end{aligned} \tag{2.17}$$

Since f_x is bounded in the neighborhood of the point (x_0, y_0), we obtain from Eq. (2.17)

$$\lim_{\Delta x \to 0, \Delta y \to 0} f(x_0, \Delta x, y_0 + \Delta y) = f(x_0, y_0).$$

Hence, the function $f(x, y)$ is continuous at the point (x_0, y_0).

Geometrical interpretation of partial derivatives

Let $z = f(x, y)$ represent a surface as shown in Fig. 2.3. Let the plane $x = x_0$ = constant intersect the surface $z = f(x, y)$ along the curve $z = f(x_0, y)$. Let $P(x_0, y, 0)$ be a particular point in the x-y plane and $R(x_0, y, z)$ be the corresponding point on the surface, where $z = f(x_0, y)$. Let $Q(x_0, y + \Delta y, 0)$ be a point in the x-y plane in the neighborhood of P and $S(x_0, y + \Delta y, z + \Delta_y z)$ be the corresponding point on the surface $z = f(x, y)$. From Fig. 2.3, we find that $\Delta y = PQ = RS'$ and the function z is increased by $SS' = (z + \Delta_y z) - z = \Delta_y z$. Now, let θ^* be the angle which the chord RS makes with the positive y-axis. Then, from $\Delta RSS'$, we have

$$\tan \theta^* = \frac{SS'}{RS'} = \frac{\Delta_y z}{\Delta y}.$$

Let $\Delta y \to 0$. Then, $\Delta_y z \to 0$. Hence,

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \frac{\partial z}{\partial y} = \tan \theta$$

where in the limit, θ is the angle made by the tangent to the curve $z = f(x_0, y)$ at the point $R(x_0, y, z)$ on the surface $z = f(x, y)$ with the positive y-axis.

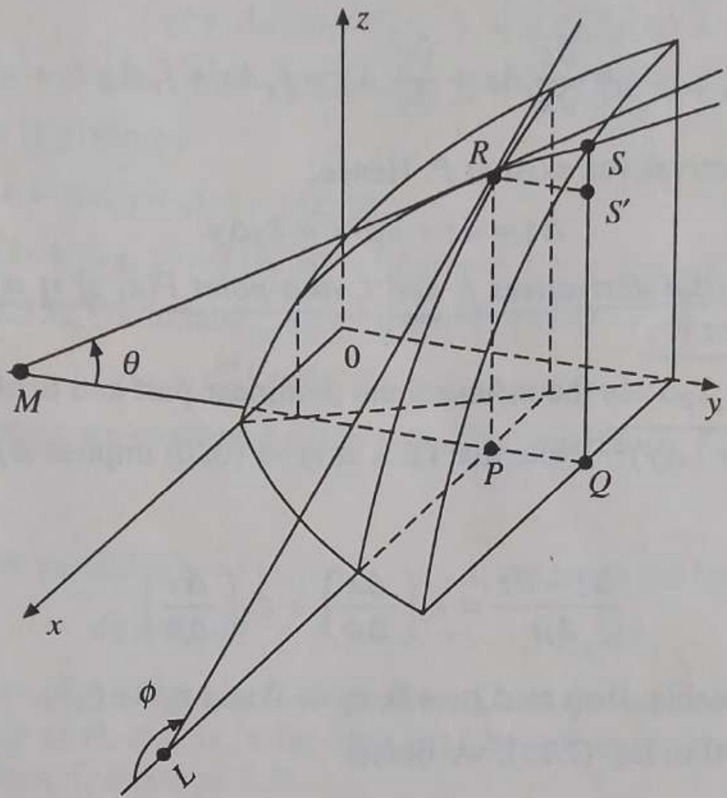


Fig. 2.3. Geometrical representation of partial derivatives.

Now, consider the intersection of the plane $y = y_0 =$ constant with the surface $z = f(x, y)$. Following the similar procedure, we obtain $\partial z/\partial x = \tan\phi$, where ϕ is the angle made by the tangent to the curve $z = f(x, y_0)$ at the point (x, y_0, z) on the surface $z = f(x, y)$ with the positive x-axis.

It can be observed that this representation of partial derivatives is a direct extension of the one dimensional case.

2.3.1 Total Differential and Differentiability

Let a function of two variables $z = f(x, y)$ be defined in some domain D in the x-y plane. Let $P(x, y)$ be any point in D and $(x + \Delta x, y + \Delta y)$ be a point in the neighborhood of (x, y), in D. Then,

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

is called the *total increment* in z corresponding to the increments Δx in x and Δy in y.

The function $z = f(x, y)$ is said to be *differentiable* at the point (x, y), if at this point Δz can be written as

i.e the value of f_x, f_y at x, y are to be written, & not the functions

f_x Total differential $\equiv dz$

$$\boxed{\Delta z = (\underset{f_x}{a}\,\Delta x + \underset{f_y}{b}\,\Delta y) + (\varepsilon_1 \Delta x + \varepsilon_2 \Delta y)} \tag{2.18}$$

where a, b are independent of Δx, Δy and $\varepsilon_1 = \varepsilon_1(\Delta x, \Delta y)$, $\varepsilon_2 = \varepsilon_2(\Delta x, \Delta y)$ are infinitesimals and functions of Δx, Δy such that $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

The first part $a\,\Delta x + b\,\Delta y$ in Eq. (2.18) which is linear in Δx and Δy is called the *total differential* or simply the differential of z at the point (x, y) and is denoted by dz or df. That is

$$dz = a\,\Delta x + b\Delta y \quad \text{or} \quad dz = a\,dx + b\,dy.$$

Let $\Delta y = 0$ in Eq. (2.18). Then, $\Delta z = a\,\Delta x + \varepsilon_1 \Delta x$. Dividing by Δx and taking limits as $\Delta x \to 0$ we obtain $a = \partial z/\partial x$. Similarly, letting $\Delta x = 0$ in Eq. (2.18), dividing by Δy and taking limits as $\Delta y \to 0$, we obtain $b = \partial z/\partial y$. Therefore,

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = f_x \Delta x + f_y \Delta_y \tag{2.19}$$

assuming that the partial derivatives exist at P. Hence,

$$\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y. \tag{2.20}$$

Therefore, *existence of partial derivatives f_x and f_y at a point $P(x, y)$ is a necessary condition for differentiability of $f(x, y)$ at P.*

The second part $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ is the infinitesimal nonlinear part and is of higher order relative to Δx, Δy or $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Note that $(\Delta x, \Delta y) \to (0, 0)$ implies $\Delta\rho \to 0$. Eq. (2.20) can be written as

$$\frac{\Delta z - dz}{\Delta\rho} = \varepsilon_1\left(\frac{\Delta x}{\Delta\rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\Delta\rho}\right) \tag{2.21}$$

Now, if $f(x, y)$ is differentiable, then as $\Delta\rho \to 0$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

Taking the limit as $\Delta\rho \to 0$ in Eq. (2.21), we obtain

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho \to 0}\left[\varepsilon_1\left(\frac{\Delta x}{\Delta\rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\Delta\rho}\right)\right] = 0 \tag{2.22}$$

since $|\Delta x/\Delta\rho| \le 1$ and $|\Delta y/\Delta\rho| \le 1$.

Therefore, to test differentiability at a point $P(x, y)$, we can use either of the following two approaches.

(i) Show that $\displaystyle\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta\rho} = 0$ $\quad \Delta\rho \to \sqrt{(\Delta x)^2 + (\Delta y)^2}$ (2.23)

(ii) Find the expressions for $\varepsilon_1(\Delta x, \Delta y)$, $\varepsilon_2(\Delta x, \Delta y)$ from Eq. (2.20) and then show that $\lim \varepsilon_1 \to 0$ and $\lim \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ or $\Delta\rho \to 0$.

ii) Show that both partial der' exist and one is continuous at (x, y)

Note that the function $f(x, y)$ may not be differentiable at a point $P(x, y)$, even if the partial derivatives f_x, f_y exist at P (see Example 2.12). However, if the first order partial derivatives are continuous at the point P, then the function is differentiable at P. We present this result in the following theorem.

Theorem 2.2 (Sufficient condition for differentiability) If the function $z = f(x, y)$ has continuous first order partial derivatives at a point $P(x, y)$ in D, then $f(x, y)$ is differentiable at P.

Proof Let $P(x, y)$ be a fixed point in D. By the Lagrange mean value theorem, we have

$$f(x + \Delta x, y) - f(x, y) = \Delta x\, f_x(x + \theta_1 \Delta x, y),\ 0 < \theta_1 < 1$$

and
$$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = \Delta y\, f_y(x + \Delta x, y + \theta_2 \Delta y),\ 0 < \theta_2 < 1.$$

Since f_x and f_y are continuous at (x, y), we can write

$$f_x(x + \theta_1 \Delta x, y) = f_x(x, y) + \varepsilon_1$$

and
$$f_y(x + \Delta x, y + \theta_2 \Delta y) = f_y(x, y) + \varepsilon_2$$

f ⇒ Both partial der exist, one

where ε_1, ε_2 are infinitesimals, are functions of Δx, Δy and tend to zero as $\Delta x \to 0$, $\Delta y \to 0$, that is, as $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2} \to 0$. Therefore, we have

$$f(x + \Delta x, y) - f(x, y) = \Delta x f_x(x, y) + \varepsilon_1 \Delta x \tag{2.24}$$

and

$$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = \Delta y\, f_y(x, y) + \varepsilon_2\, \Delta y \tag{2.25}$$

Now, the total increment is given by

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= [f(x + \Delta x, y) - f(x, y)] + [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)].\end{aligned}$$

Using Eqs. (2.24) and (2.25), we obtain the value @(x, y)

$$\Delta z = f_x\, \Delta x + f_y\, \Delta y + \varepsilon_1\, \Delta x + \varepsilon_2\, \Delta y \tag{2.26}$$

where the partial derivatives are evaluated at the point $P(x, y)$. Hence, $f(x, y)$ is differentiable at P.

Remark 6

(a) For a function of n variables $z = f(x_1, x_2, \ldots, x_n)$, we write the total differential as

$$dz = f_{x_1} dx_1 + f_{x_2}\, dx_2 + \ldots + f_{x_n} dx_n. \tag{2.27}$$

(b) Note that continuity of the first partial derivatives f_x and f_y at a point P is a sufficient condition for differentiability at P, that is, a function may be differentiable even if f_x and f_y are not continuous (Problem 5, Exercise 2.2).

not necess

(c) The conditions of Theorem 2.2 can be relaxed. It is sufficient that one of the first order partial derivatives is continuous at (x_0, y_0) and the other exists at (x_0, y_0).

Example 2.11 Find the total differential of the following functions

(i) $z = \tan^{-1}(x/y)$, $(x, y) \neq (0, 0)$, (ii) $u = \left(xz + \dfrac{x}{z}\right)^y$, $z \neq 0$.

Solution

(i) $$f(x, y) = \tan^{-1}\left(\frac{x}{y}\right),\ f_x = \frac{1}{1 + (x/y)^2}\left(\frac{1}{y}\right) = \frac{y}{x^2 + y^2}$$

and $$f_y = \frac{1}{1 + (x/y)^2}\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2 + y^2}.$$

Therefore, we obtain the total differential as

$$dz = f_x\, dx + f_y\, dy = \frac{1}{x^2 + y^2}(y\, dx - x\, dy).$$

(ii) $$f(x, y, z) = \left(xz + \frac{x}{z}\right)^y,\ f_x = y\left(xz + \frac{x}{z}\right)^{y-1}\left(z + \frac{1}{z}\right)$$

$$f_y = \left(xz + \frac{x}{z}\right)^y \ln\left(xz + \frac{x}{z}\right),\ f_z = y\left(xz + \frac{x}{z}\right)^{y-1}\left(x - \frac{x}{z^2}\right).$$

Therefore, we obtain the total differential as

$$du = \left(xz + \frac{x}{z}\right)^{y-1}\left[y\left(z + \frac{1}{z}\right)dx + xy\left(1 - \frac{1}{z^2}\right)dz\right] + \left[\left(xz + \frac{x}{z}\right)^y \ln\left(xz + \frac{x}{z}\right)\right]dy.$$

Example 2.12 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^3 + 2y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 &, (x, y) = (0, 0) \end{cases}$$

(i) is continuous at (0, 0),

(ii) possesses partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$,

(iii) is not differentiable at (0, 0).

Solution

(i) Let $x = r \cos \theta$ and $y = r \sin \theta$. We have

$$|f(x, y) - f(0, 0)| = \left| \frac{r^3 (\cos^3\theta + 2 \sin^3\theta)}{r^2} \right| \leq r[\,|\cos^3\theta| + 2\,|\sin^3\theta|\,]$$

$$\leq 3r = 3\sqrt{x^2 + y^2} < \varepsilon.$$

Taking $\delta < \varepsilon/3$, we find that

$$|f(x, y) - 0| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim\limits_{(x, y) \to f(0,0)} f(x, y) = 0 = f(0, 0)$.

Hence, $f(x, y)$ is continuous at (0, 0).

(ii) $$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1.$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2\Delta y - 0}{\Delta y} = 2.$$

Therefore, the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ exist.

(iii) We have $dz = \Delta x + 2\Delta y$. Using Eq. (2.20), we get

$$\Delta z = \Delta x + 2\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Let $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Now,

$(x+\Delta x, y+\Delta y) - f(x, y)$

$$\Delta z = f(\Delta x, \Delta y) - f(0, 0) = \frac{(\Delta x)^3 + 2(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2}$$

Hence

$z = f_x \Delta x + f_y \Delta y$

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho \to 0} \frac{1}{\Delta\rho} \left[\frac{(\Delta x)^3 + 2(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} - (\Delta x + 2\Delta y) \right]$$

$$= \lim_{\Delta\rho \to 0} -\left[\frac{\Delta x \Delta y(\Delta y + 2\Delta x)}{\{(\Delta x)^2 + (\Delta y)^2\}^{3/2}} \right]$$

$\frac{\Delta z - dz}{\Delta \ell}$

Let $\Delta x = r\cos\theta$ and $\Delta y = r\sin\theta$. As $(\Delta x, \Delta y) \to (0, 0)$, $\Delta\rho = r \to 0$ for arbitrary θ. Therefore,

$$\lim_{\Delta\rho\to 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{r\to 0} -[\cos\theta\sin\theta(\sin\theta + 2\cos\theta)]$$

$$= -[\cos\theta\sin\theta(\sin\theta + 2\cos\theta)]$$

The limit depends on θ and does not tend to zero for arbitrary θ. Hence, the given function is not differentiable. Alternately, we can write

$$\frac{\Delta z - dz}{\Delta\rho} = -\frac{1}{\Delta\rho}\left[\frac{\Delta x(\Delta y)^2 + 2(\Delta x)^2\Delta y}{(\Delta x)^2 + (\Delta y)^2}\right] = \varepsilon_1\left(\frac{\Delta x}{\Delta\rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\Delta\rho}\right)$$

where $$\varepsilon_1 = -\frac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \quad \text{and} \quad \varepsilon_2 = -\frac{2(\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2}.$$

Substituting $\Delta x = r\cos\theta$, $\Delta y = r\sin\theta$, we find that ε_1 and ε_2 depend on θ and do not tend to zero for arbitrary θ, in the limit as $r \to 0$.

Example 2.13 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x - y}, & (x, y) \neq (1, -1) \\ 0, & (x, y) = (1, -1) \end{cases}$$

is continuous and differentiable at $(1, -1)$.

Solution We have

$$\lim_{(x,y)\to(1,-1)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(1,-1)} (x + y) = 0 = f(1, -1).$$

Therefore, the function is continuous at $(1, -1)$.

The partial derivatives are given by

$$f_x(1, -1) = \lim_{\Delta x\to 0} \frac{f(1 + \Delta x, -1) - f(1, -1)}{\Delta x} = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[\frac{(1 + \Delta x)^2 - 1}{(1 + \Delta x) + 1} - 0\right] = \lim_{\Delta x\to 0} \frac{2 + \Delta x}{2 + \Delta x}$$

$$f_y(1, -1) = \lim_{\Delta y\to 0} \frac{f(1, -1 + \Delta y) - f(1, -1)}{\Delta y} = \lim_{\Delta y\to 0} \frac{1}{\Delta y}\left[\frac{1 - (-1 + \Delta y)^2}{1 - (-1 + \Delta y)} - 0\right] = \lim_{\Delta y\to 0} \frac{2 - \Delta y}{2 - \Delta y}$$

Therefore, the first order partial derivatives exist at $(1, -1)$.

Now, we have

$$f_x(x, y) = \frac{(x - y)(2x) - (x^2 - y^2)(1)}{(x - y)^2} = \frac{x^2 - 2xy + y^2}{(x - y)^2} = \frac{(x - y)^2}{(x - y)^2}, \ (x, y) \neq (1, -1)$$

and $$f_x(x, y) = 1, \ (x, y) = (1, -1).$$

Since $$\lim_{(x,y)\to(1,-1)} f_x(x, y) = \lim_{(x,y)\to(1,-1)} \frac{(x - y)^2}{(x - y)^2} = 1 = f_x(1, -1)$$

the partial derivative f_x is continuous at $(1, -1)$. Also $f_y(1, -1)$ exists. Hence, $f(x, y)$ is differentiable at $(1, -1)$.

Alternately, we can show that $\lim_{\Delta\rho \to 0} [(\Delta z - dz)/\Delta\rho] = 0$.

2.3.2 Approximation by Total Differentials

From Theorem 2.2, we have for a function $f(x, y)$ of two variables

$$f(x + \Delta x, y + \Delta y) - f(x, y) \approx f_x \, \Delta x + f_y \, \Delta y$$

or

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x \, \Delta x + f_y \, \Delta y \tag{2.28}$$

where the partial derivatives are evaluated at the given point (x, y). This result has applications in estimating errors in calculations.

Consider now a function of n variables $x_1, x_2, \ldots, x_n$. Let the function $z = f(x_1, x_2, \ldots, x_n)$ be differentiable at the point $P(x_1, x_2, \ldots, x_n)$. Let there be errors $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ in measuring the values of $x_1, x_2, \ldots, x_n$ respectively. Then, the computed value of z using the inexact values of the arguments will be obtained with an error

$$\Delta z = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) - f(x_1, x_2, \ldots, x_n). \tag{2.29}$$

When the errors $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ are small in magnitude, we obtain (using the Remark 6 (a), Eq. (2.27))

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) \approx f(x_1, x_2, \ldots, x_n) + f_{x_1} \Delta x_1 + f_{x_2} \Delta x_2 + \ldots + f_{x_n} \Delta x_n \tag{2.30}$$

where the partial derivatives are evaluated at the point $(x_1, x_2, \ldots, x_n)$. This is the generalization of the result for functions of two variables given in Eq. (2.28).

Since the partial derivatives and errors in arguments can be both positive and negative, we define the *absolute error* as (using Eq. (2.29))

$$|\Delta z| \approx |dz| = |df| = |f_{x_1} \Delta x_1 + f_{x_2} \Delta x_2 + \ldots + f_{x_n} \Delta x_n|.$$

Then,

$$|df| \le |f_{x_1}|\,|\Delta x_1| + |f_{x_2}|\,|\Delta x_2| + \ldots + |f_{x_n}|\,|\Delta x_n| \tag{2.31}$$

gives the *maximum absolute error* in z. If $\max |\Delta x_i| \le \Delta x$, then we can write

$$|df| \le \Delta x \, [\,|f_{x_1}| + |f_{x_2}| + \ldots + |f_{x_n}|\,].$$

The expression $|df|/|f|$ is called the *maximum relative error* and $[\,|df|/|f|\,] \times 100$ is called the *percentage error*.

The maximum relative error can also be written as

$$\frac{|df|}{|f|} \le \left|\frac{\partial f/\partial x_1}{f}\right| |\Delta x_1| + \left|\frac{\partial f/\partial x_2}{f}\right| |\Delta x_2| + \ldots + \left|\frac{\partial f/\partial x_n}{f}\right| |\Delta x_n|$$

$$\le \left|\frac{\partial}{\partial x_1}[\ln |f|]\right| |\Delta x_1| + \left|\frac{\partial}{\partial x_2}[\ln |f|]\right| |\Delta x_2| + \ldots + \left|\frac{\partial}{\partial x_n}[\ln |f|]\right| |\Delta x_n|.$$

Example 2.14 Find the total increment and the total differential of the function $z = x + y + xy$ at the point $(1, 2)$ for $\Delta x = 0.1$ and $\Delta y = -0.2$. Find the maximum absolute error and the maximum relative error.

Solution We are given that $f(x, y) = x + y + xy$, $(x, y) = (1, 2)$.

Therefore, $f(1, 2) = 5$, $f_x(1, 2) = 3$, $f_y(1, 2) = 2$. We have

$$\begin{aligned}\text{total increment} &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= [(x + \Delta x) + (y + \Delta y) + (x + \Delta x)(y + \Delta y)] - [x + y + xy]\\ &= \Delta x + \Delta y + x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y.\end{aligned}$$

At the point $(1, 2)$ with $\Delta x = 0.1$ and $\Delta y = -0.2$, we obtain

$$\text{total increment} = 0.1 - 0.2 + 1(-0.2) + 2(0.1) + (0.1)(-0.2) = -0.12$$

$$\text{total differential} = f_x(1, 2)\,\Delta x + f_y(1, 2)\,\Delta y = 3(0.1) + (2)(-0.2) = -0.1$$

$$\text{maximum absolute error} = |df| = \left|\frac{\partial f}{\partial x}\right| |\Delta x| + \left|\frac{\partial f}{\partial y}\right| |\Delta y| = 3(0.1) + 2(0.2) = 0.7$$

$$\text{maximum relative error} = \frac{|df|}{|f|} = \frac{0.7}{5} = 0.14.$$

Example 2.15 Using differentials, find an approximate value of

(i) $f(4.1, 4.9)$, where $f(x, y) = \sqrt{x^3 + x^2 y}$.

(ii) $f(2.1, 3.2)$, where $f(x, y) = x^y$, $(\log 2 = 0.3010)$.

Solution

(i) Let $(x, y) = (4, 5)$, $\Delta x = 0.1$, $\Delta y = -0.1$. We have

$$f(x, y) = \sqrt{x^3 + x^2 y},\quad f(4, 5) = 12,\quad f_x(x, y) = \frac{3x^2 + 2xy}{2\sqrt{x^3 + x^2 y}},\quad f_x(4, 5) = \frac{11}{3},$$

$$f_y(x, y) = \frac{x^2}{2\sqrt{x^3 + x^2 y}},\quad f_y(4, 5) = \frac{2}{3}.$$

Therefore,

$$\begin{aligned}f(4.1, 4.9) &\approx f(4, 5) + f_x(4, 5)\,\Delta x + f_y(4, 5)\,\Delta y\\ &= 12 + \left(\frac{11}{3}\right)(0.1) + \left(\frac{2}{3}\right)(-0.1) = 12.3.\end{aligned}$$

The exact value is $f(4.1, 4.9) = 12.3$.

(ii) Let $(x, y) = (2, 3)$, $\Delta x = 0.1$, $\Delta y = 0.2$. We have

$$f(x, y) = x^y,\quad f(2, 3) = 8,\quad f_x(x, y) = yx^{y-1},\quad f_x(2, 3) = 12,$$

$$f_y(x, y) = x^y \log x,\quad f_y(2, 3) = 8 \log 2 = 8(0.3010) = 2.408.$$

Therefore, $f(2.1, 3.2) \approx f(2, 3) + f_x(2, 3)\,\Delta x + f_y(2, 3)\,\Delta y$

$$= 8 + 12(0.1) + (2.408)\,(0.2) = 9.6816.$$

The exact value is $f(2.1, 3.2) = 10.7424$.

Example 2.16 Find the percentage error in the computed area of an ellipse when an error of 2% made in measuring the major and minor axes.

2%.

Solution Let the major and minor axes of the ellipse be $2a$ and $2b$ respectively. The errors Δa an Δb in computing the lengths of the semi major and minor axes are

$$\Delta a = a(0\,02) = 0.02\,a \text{ and } \Delta b = b(0.02) = 0.02\text{ b}.$$

The area of the ellipse is given by $A = \pi\, ab$. Therefore, we have the following:

Maximum absolute error in computing the area of ellipse is

$$|\,dA\,| = \left|\frac{\partial A}{\partial a}\right|\,|\,\Delta a\,| + \left|\frac{\partial A}{\partial b}\right|\,|\,\Delta b\,| = \pi b(0.02a) + \pi a(0.02b) = 0.04\,\pi ab.$$

Maximum relative error is

$$\left|\frac{dA}{A}\right| = (0.04\,\pi ab)\left(\frac{1}{\pi ab}\right) = 0.04.$$

Percentage error $= \left|\dfrac{dA}{A}\right| \times 100 = 4\%$.

2.3.3 Derivatives of Composite and Implicit Functions (*Chain Rule*)

Let $z = f(x, y)$ be a function of two independent variables x and y. Suppose that x and y are themselves functions of some independent variable t, say $x = \phi(t)$, $y = \psi(t)$. Then, $z = f[\phi(t), \psi(t)]$ is a composite function of the independent variable t. Now, assume that the partial derivatives f_x, f_y are continuous functions of x, y and $\phi(t)$, $\psi(t)$ are differentiable functions of t.

Let Δx, Δy and Δz be the increments respectively in x, y and z corresponding to the increment Δt in t. Then we have

$$\Delta z = \frac{\partial f}{\partial x}\,\Delta x + \frac{\partial f}{\partial y}\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y.$$

Dividing both sides by Δt, we get

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}. \tag{2.32}$$

Now as $\Delta t \to 0$; $\Delta x \to 0$, $\Delta y \to 0$ and $\varepsilon_1\left(\dfrac{\Delta x}{\Delta t}\right) \to 0$, $\varepsilon_2\left(\dfrac{\Delta y}{\Delta t}\right) \to 0$. Therefore, taking limits on both sides in Eq. (2.32) as $\Delta t \to 0$, we obtain

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{2.33}$$

Now, let x and y be functions of two independent variables u and v, say $x = \phi(u, v)$, $y = \psi(u, v)$. Then, $z = f[\phi(u, v), \psi(u, v)]$ is a composite function of two independent variables u and v. Assume

that the functions $f(x, y)$, $\phi(u, v)$, $\psi(u, v)$ have continuous partial derivatives with respect to their arguments. Now, consider v as a constant and give an increment Δu to u. Let $\Delta_u x$ and $\Delta_u y$ be the corresponding increments in x and y. Then, the increment Δz in z is given by (using Eq. (2.23))

$$\Delta z = \frac{\partial f}{\partial x} \Delta_u x + \frac{\partial f}{\partial y} \Delta_u y + \varepsilon_1 \Delta_u x + \varepsilon_2 \Delta_u y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta u \to 0$.
Dividing both sides by Δu, we get

$$\frac{\Delta z}{\Delta u} = \frac{\partial f}{\partial x} \frac{\Delta_u x}{\Delta u} + \frac{\partial f}{\partial y} \frac{\Delta_u y}{\Delta u} + \varepsilon_1 \frac{\Delta_u x}{\Delta u} + \varepsilon_2 \frac{\Delta_u y}{\Delta u}. \tag{2.34}$$

Taking limits on both sides in Eq. (2.34) as $\Delta u \to 0$, we obtain

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}. \tag{2.35}$$

Similarly, keeping u as constant and varying v, we obtain

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}. \tag{2.36}$$

The rules given in Eqs. (2.35) and (2.36) are called the *chain rules*. These rules can be easily extended to a function of n variables $z = f(x_1, x_2, \ldots, x_n)$. If the partial derivatives of f with respect to all its arguments are continuous and $x_1, x_2, \ldots, x_n$ are differentiable functions of some independent variable t, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} + \ldots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt}. \tag{2.37}$$

Example 2.17 Find df/dt at $t = 0$, where

(i) $f(x, y) = x \cos y + e^x \sin y$, $x = t^2 + 1$, $y = t^3 + t$.

(ii) $f(x, y, z) = x^3 + x z^2 + y^3 + xyz$, $x = e^t$, $y = \cos t$, $z = t^3$.

Solution

(i) When $t = 0$, we get $x = 1$, $y = 0$. Using the chain rule, we obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = (\cos y + e^x \sin y)(2t) + (-x \sin y + e^x \cos y)(3t^2 + 1).$$

Substituting $t = 0$, $x = 1$ and $y = 0$, we obtain $(df/dt) = e$.

(ii) When $t = 0$, we get $x = 1$, $y = 1$, $z = 0$. Using the chain rule, we obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

$$= (3x^2 + z^2 + yz)(e^t) + (3y^2 + xz)(-\sin t) + (2xz + xy)(3t^2).$$

Substituting $t = 0$, $x = 1$, $y = 1$, $z = 0$, we obtain $(df/dt) = 3$.

Example 2.18 If $z = f(x, y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$, then show that

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right].$$

Solution Using the chain rule, we obtain

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2e^{2u}\frac{\partial f}{\partial x} - 2e^{-2u}\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -2e^{-2v}\frac{\partial f}{\partial x} + 2e^{2v}\frac{\partial f}{\partial y}.$$

Therefore,

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2(e^{2u} + e^{-2v})\frac{\partial f}{\partial x} - 2(e^{-2u} + e^{2v})\frac{\partial f}{\partial y}$$

$$= 2x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}.$$

Change of variables

Suppose that $f(x, y)$ is a function of two independent variables x, y and x, y are functions of two new independent variables u, v given by $x = \phi(u, v)$, $y = \psi(u, v)$. By chain rule, we have

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

We want to determine $\partial f/\partial x$, $\partial f/\partial y$ in terms of $\partial f/\partial u$ and $\partial f/\partial v$. Solving the above system of equations by Cramer's rule, we get

$$\frac{\partial f/\partial x}{\dfrac{\partial f}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial f}{\partial v}\dfrac{\partial y}{\partial u}} = \frac{\partial f/\partial y}{\dfrac{\partial f}{\partial v}\dfrac{\partial x}{\partial u} - \dfrac{\partial f}{\partial u}\dfrac{\partial x}{\partial v}} = \frac{1}{\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}}.$$

The determinant

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}$$

is called the *Jacobian* of the variables of transformation. Similarly, we write

$$\frac{\partial f}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial y}{\partial u} = \frac{\partial(f, y)}{\partial(u, v)} = \begin{vmatrix} \partial f/\partial u & \partial f/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

and

$$\frac{\partial f}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial f}{\partial u}\frac{\partial x}{\partial v} = \frac{\partial(x, f)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial f/\partial u & \partial f/\partial v \end{vmatrix} = -\frac{\partial(f, x)}{\partial(u, v)}.$$

Hence, we obtain

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y)}{\partial(u, v)}\right] \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x)}{\partial(u, v)}\right]. \tag{2.38}$$

Similarly, if $f(x, y, z)$ is a function of three independent variables x, y, z and x, y, z are functions of three new independent variables u, v, w given by $x = F(u, v, w)$, $y = G(u, v, w)$, $z = H(u, v, w)$, then by chain rule, we have

Jacobian

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial w}.$$

Solving the above system of equations by Cramer's rule, we get

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y, z)}{\partial(u, v, w)}\right] = \frac{1}{J}\begin{vmatrix} \partial f/\partial u & \partial f/\partial v & \partial f/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

$$\frac{\partial f}{\partial y} = \frac{1}{J}\left[\frac{\partial(x, f, z)}{\partial(u, v, w)}\right] = -\frac{1}{J}\left[\frac{\partial(f, x, z)}{\partial(u, v, w)}\right] = -\frac{1}{J}\begin{vmatrix} \partial f/\partial u & \partial f/\partial v & \partial f/\partial w \\ \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

$$\frac{\partial f}{\partial y} = \frac{1}{J}\left[\frac{\partial(x, y, f)}{\partial(u, v, w)}\right] = \frac{1}{J}\left[\frac{\partial(f, x, y)}{\partial(u, v, w)}\right] = \frac{1}{J}\begin{vmatrix} \partial f/\partial u & \partial f/\partial v & \partial f/\partial w \\ \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \end{vmatrix} \qquad (2.39)$$

where
$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

is the Jacobian of the variables of transformation.

Example 2.19 If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2.$$

Solution The variables of transformation are r and θ. We have

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\frac{\partial(f, y)}{\partial(r, \theta)} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial \theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos\theta\frac{\partial f}{\partial r} - \sin\theta\frac{\partial f}{\partial \theta}$$

$$\frac{\partial(f, x)}{\partial(r, \theta)} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial \theta \\ \partial x/\partial r & \partial x/\partial \theta \end{vmatrix} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial \theta \\ \cos\theta & -r\sin\theta \end{vmatrix} = -r\sin\theta\frac{\partial f}{\partial r} - \cos\theta\frac{\partial f}{\partial \theta}.$$

Hence, using Eq. (2.38), we obtain

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y)}{\partial(r, \theta)}\right] = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x)}{\partial(r, \theta)}\right] = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}.$$

Squaring and adding, we obtain the required result.

Example 2.20 If $u = f(x, y, z)$ and $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial\phi}\right)^2.$$

Solution The variables of transformation are r, θ and ϕ. We have

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

$$\frac{\partial(f, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial\theta & \partial f/\partial\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin^2\theta\cos\phi\frac{\partial f}{\partial r} + r\sin\theta\cos\theta\cos\phi\frac{\partial f}{\partial\theta} - r\sin\phi\frac{\partial f}{\partial\phi}$$

$$\frac{\partial(f, x, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial\theta & \partial f/\partial\phi \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= -r^2\sin^2\theta\sin\phi\frac{\partial f}{\partial r} - r\sin\theta\cos\theta\sin\phi\frac{\partial f}{\partial\theta} - r\cos\phi\frac{\partial f}{\partial\phi}$$

$$\frac{\partial(f, x, y)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial\theta & \partial f/\partial\phi \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix}$$

$$= r^2\sin\theta\cos\theta\frac{\partial f}{\partial r} - r\sin^2\theta\frac{\partial f}{\partial\theta}.$$

Using Eq. (2.39), we obtain

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y, z)}{\partial(r, \theta, \phi)}\right] = \sin\theta\cos\phi\frac{\partial f}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial f}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial f}{\partial\phi}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x, z)}{\partial(r, \theta, \phi)}\right] = \sin\theta\sin\phi\frac{\partial f}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial f}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial f}{\partial\phi}$$

$$\frac{\partial f}{\partial z} = \frac{1}{J}\left[\frac{\partial(f, x, y)}{\partial(r, \theta, \phi)}\right] = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}.$$

Squaring and adding, we obtain the required result.

Derivative of implicit functions

The function $f(x, y) = 0$ defines implicitly a function $y = \phi(x)$ of one independent variable x. Then, we can determine dy/dx using the chain rule. From $f(x, y) = 0$, we get

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)},\ f_y(x, y) \neq 0. \tag{2.40}$$

The function $f(x, y, z) = 0$ defines one of the variables x, y, z implicitly in terms of the other two variables. Using differentials, we obtain

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0. \tag{2.41}$$

If we take y = constant, then $dy = 0$ and we get from Eq. (2.41)

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial z}dz = 0, \text{ or } \left(\frac{dz}{dx}\right)_y = -\frac{(\partial f/\partial x)}{(\partial f/\partial z)}. \tag{2.42}$$

If we take x = constant, then $dx = 0$ and we get from Eq. (2.41)

$$\frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0, \text{ or } \left(\frac{dy}{dz}\right)_x = -\frac{(\partial f/\partial z)}{(\partial f/\partial y)}. \tag{2.43}$$

If we take z = constant, then $dz = 0$ and we get from Eq. (2.41)

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0, \text{ or } \left(\frac{dx}{dy}\right)_z = -\frac{(\partial f/\partial y)}{(\partial f/\partial x)}. \tag{2.44}$$

Multiplying Eqs. (2.42), (2.43) and (2.44), we obtain

$$\left(\frac{dx}{dy}\right)_z\left(\frac{dy}{dz}\right)_x\left(\frac{dz}{dx}\right)_y = -1 \text{ or } \left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1. \tag{2.45}$$

Example 2.21 Find dy/dx, when

(i) $f(x, y) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0.$

(ii) $f(x, y) = \ln(x^2 + y^2) + \tan^{-1}(y/x) = 0.$

Solution

(i)
$$\frac{\partial f}{\partial x} = \frac{2x}{a^2} \text{ and } \frac{\partial f}{\partial y} = \frac{2y}{b^2}.$$

Therefore,
$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{b^2 x}{a^2 y},\ y \neq 0.$$

(ii)
$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2} + \frac{1}{1 + y^2/x^2}\left(-\frac{y}{x^2}\right) = \frac{2x}{x^2 + y^2} - \frac{y}{x^2 + y^2} = \frac{2x - y}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} + \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right) = \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{2y+x}{x^2+y^2}.$$

Therefore,
$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x-y}{2y+x} = \frac{y-2x}{2y+x}, \quad y \neq -\frac{x}{2}.$$

Exercises 2.2

1. Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

has partial derivatives $f_x(0, 0)$, $f_y(0, 0)$, but the partial derivatives are not continuous at (0, 0).

2. Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2+y^2}{x-y}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

possesses partial derivatives at (0, 0), though it is not continuous at (0, 0).

3. For the function

$$f(x, y) = \begin{cases} \dfrac{y(x^2-y^2)}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

compute $f_x(0, y)$, $f_y(x, 0)$, $f_x(0, 0)$ and $f_y(0, 0)$, if they exist.

4. Show that the function $f(x, y) = \sqrt{x^2+y^2}$ is not differentiable at (0, 0).

5. Show that the function

$$f(x, y) = \begin{cases} (x^2+y^2)\cos\left[\dfrac{1}{\sqrt{x^2+y^2}}\right], & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

is differentiable at (0, 0) and that f_x, f_y are not continuous at (0, 0). Does this result contradict Theorem 2.2?

Find the first order partial derivatives for the following functions at the specified point:

6. $f(x, y) = x^4 - x^2y^2 + y^4$ at (–1, 1).
7. $f(x, y) = \ln(x/y)$ at (2, 3).
8. $f(x, y) = x^2e^{y/x}$ at (4, 2).
9. $f(x, y) = x/\sqrt{x^2+y^2}$ at (6, 7).
10. $f(x, y) = \cot^{-1}(x+y)$ at (1, 2).
11. $f(x, y) = \ln\left[\dfrac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right]$ at (3, 4).
12. $f(x, y, z) = (x^2+y^2+z^2)^{-1/2}$ at (2, 1, 2).
13. $f(x, y, z) = e^{x/y} + e^{z/y}$ at (1, 1, 1).
14. $f(x, y, z) = (xy)^{\sin z}$ at (3, 5, $\pi/2$).
15. $f(x, y, z) = \ln(x + \sqrt{y^2+z^2})$ at (2, 3, 4).

Find dw/dt in the following problems.

16. $w = x^2 + y^2$, $x = (t^2 - 1)/t$, $y = t/(t^2 + 1)$ at $t = 1$.
17. $w = x^2 + y^2 + z^2$, $x = \cos t$, $y = \ln(t + 1)$, $z = e^t$ at $t = 0$.
18. $w = e^x \sin(y + 2z)$, $x = t$, $y = 1/t$, $z = t^2$.
19. $w = xy + yz + zx$, $x = t^2$, $y = te^t$, $z = te^{-t}$.
20. $w = z \ln y + y \ln z + xyz$, $x = \sin t$, $y = t^2 + 1$, $z = \cos^{-1} t$ at $t = 0$.

Verify the given results in the following problems:

21. If $z = f(ax + by)$, then $b \dfrac{\partial z}{\partial x} - a \dfrac{\partial z}{\partial y} = 0$.

22. If $z = \log[(x^2 - y^2)/(x^2 + y^2)]$, then $x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} = 0$.

23. If $u = f(x - y, y - z, z - x)$, then $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

24. If $z = f(x, y)$, $x = r \cosh \theta$, $y = r \sinh \theta$, then
$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

25. If $z = y + f(u)$, $u = \dfrac{x}{y}$, then $u \dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$.

26. If $w = f(u, v)$, $u = \sqrt{x^2 + y^2}$, $v = \cot^{-1}(y/x)$, then
$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \frac{1}{x^2 + y^2}\left[(x^2 + y^2)\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right].$$

27. If $z = f(x, y)$, $x = u \cos \alpha - v \sin \alpha$, $y = u \sin \alpha + v \cos \alpha$, where α is a constant, then
$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

28. If $z = \ln(u^2 + v)$, $u = e^{x + y^2}$, $v = x + y^2$, then $2y \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$.

29. If $w = \sqrt{x^2 + y^2 + z^2}$, $x = u \cos v$, $y = u \sin v$, $z = uv$, then
$$u \frac{\partial w}{\partial u} - v \frac{\partial w}{\partial v} = \frac{u}{\sqrt{1 + v^2}}.$$

30. If $w = \sin^{-1} u$, $u = (x^2 + y^2 + z^2)/(x + y + z)$, then
$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = \tan w.$$

Using implicit differentiation, obtain the following:

31. $\dfrac{dy}{dx}$, when $x^y + y^x = \alpha$, α any constant, $x > 0$, $y > 0$.

32. $\dfrac{dy}{dx}$, when $\cot^{-1}(x/y) + y^3 + 1 = 0$, $x > 0$, $y > 0$.

33. $\left(\dfrac{\partial z}{\partial x}\right)_y$ and $\left(\dfrac{\partial z}{\partial y}\right)_x$, when $\cos xy + \cos yz + \cos zx = 1$.

34. $\left(\frac{\partial z}{\partial x}\right)_y$ and $\left(\frac{\partial z}{\partial y}\right)_x$, when $x^3 + 3xy - 2y^2 + 3xz + z^2 = 0$.

35. $y\left(\frac{\partial x}{\partial y}\right)_z + z\left(\frac{\partial x}{\partial z}\right)_y$, when $f\left(\frac{z}{y}, \frac{x}{y}\right) = 0$.

Using differentials, obtain the approximate values of the following quantities:

36. $\sqrt{(298)^2 + (401)^2}$.

37. $(4.05)^{1/2}(7.97)^{1/3}$.

38. $\cos 44° \sin 32°$.

39. $\frac{1}{\sqrt{1.05}} + \frac{1}{\sqrt{3.97}} + \frac{1}{\sqrt{9.01}}$.

40. $\sin 26° \cos 57° \tan 48°$.

41. A certain function $z = f(x, y)$ has values $f(2, 3) = 5$, $f_x(2, 3) = 3$ and $f_y(2, 3) = 7$. Find an approximate value of $f(1.98, 3.01)$.

42. The radius r and the height h of a conical tank increases at the rate of $(dr/dt) = 0.2''/\text{hr}$ and $(dh/dt) = 0.1''/\text{hr}$. Find the rate of increase dV/dt in volume V when the radius is 5 feet and the height is 20 feet.

43. The dimensions of a rectangular block of wood are 60″, 80″ and 100″ with possible absolute error of 3″ in each measurement. Find the maximum absolute error and the percentage error in the surface area.

44. Two sides of a triangle are measured as 5 cm and 3 cm and the included angle as 30°. If the possible absolute errors are 0.2 cm in measuring the sides and 1° in the angle, then find the percentage error in the computed area of the triangle.

45. The sides of a rectangular box are found to be a feet, b feet and c feet with a possible error of 1% in magnitude in each of the measurements. Find the percentage error in the volume of the box caused by the errors in individual measurements.

46. The diameter and the altitude of a can in the shape of a right circular cylinder are measured as 6 cm and 8 cm respectively. The maximum absolute error in each measurement is 0.2 cm. Find the maximum absolute error and the percentage error in the computed value of the volume.

47. The power consumed in an electric resistor is given by $P = E^2/R$ (in watts). If $E = 80$ volts and $R = 5$ Ohms, by how much the power consumption will change if E is increased by 3 volts and R is decreased by 0.1 Ohms.

48. If two resistors with resistence R_1 and R_2 in Ohms are connected in parallel, then the resistence of the resulting circuit is $R = [(1/R_1) + (1/R_2)]^{-1}$. Find an approximate value of the percentage change in resistence that results by changing R_1 from 2 to 1.9 Ohms and R_2 from 6 to 6.2 Ohms.

49. Suppose that $u = xze^y$ and x, y, z can be measured with maximum absolute errors 0.1, 0.2 and 0.3 respectively. Find the percentage error in the computed value of u from the measured values $x = 3$, $y = \ln 2$ and $z = 5$.

50. If the radius r and the altitude h of a cone are measured with an absolute error of 1% in each measurement, then find the approximate percentage change in the lateral area of the cone if the measured values are $r = 3$ feet and $h = 4$ feet.

2.4 Higher Order Partial Derivatives

Let $z = f(x, y)$ be a function of two variables and let its first order partial derivatives exist at all the points in the domain of definition D of the function f. Then, the first order partial derivatives are also functions of x and y. We define the second order partial derivatives as

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = f_{xx}(x, y) = \lim_{\Delta x \to 0}\left[\frac{f_x(x + \Delta x, y) - f_x(x, y)}{\Delta x}\right]$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = f_{yx}(x, y) = \lim_{\Delta y \to 0}\left[\frac{f_x(x, y + \Delta y) - f_x(x, y)}{\Delta y}\right]$$

(differentiate partially first with respect to x and then with respect to y)

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = f_{xy}(x, y) = \lim_{\Delta x \to 0}\left[\frac{f_y(x + \Delta x, y) - f_y(x, y)}{\Delta x}\right]$$

(differentiate partially first with respect to y and then with respect to x)

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = f_{yy}(x, y) = \lim_{\Delta y \to 0}\left[\frac{f_y(x, y + \Delta y) - f_y(x, y)}{\Delta y}\right]$$

and vice a versa

if the limits exist. The derivatives f_{xy} and f_{yx} are called *mixed derivatives*. If f_{xy} and f_{yx} are continuous at a point $P(x, y)$, then at this point $f_{xy} = f_{yx}$. That is, the order of differentiation is immaterial in this case. There are four partial derivatives of second order for $f(x, y)$. If all the second order partial derivatives exist at all points in D, then these derivatives are also functions of x and y and can be further differentiated.

Example 2.22 Find all the second order partial derivatives of the function

$$f(x, y) = \ln(x^2 + y^2) + \tan^{-1}(y/x),\ (x, y) \neq (0, 0).$$

Solution We have

$$f_x(x, y) = \frac{2x}{x^2 + y^2} + \frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{2x - y}{x^2 + y^2}$$

$$f_y(x, y) = \frac{2y}{x^2 + y^2} + \frac{1}{1 + (y/x)^2}\left(\frac{1}{x}\right) = \frac{2y + x}{x^2 + y^2}$$

$$f_{yx}(x, y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{2x - y}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(-1) - (2x - y)(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 4xy}{(x^2 + y^2)^2}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{2y + x}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(1) - (2y + x)(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 4xy}{(x^2 + y^2)^2}$$

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{2x - y}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(2) - (2x - y)(2x)}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2 + 2xy}{(x^2 + y^2)^2}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{2y + x}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(2) - (2y + x)(2y)}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2 - 2xy}{(x^2 + y^2)^2}$$

We note that $f_{xy} = f_{yx}$.

Example 2.23 For the function

$$f(x, y) = \begin{cases} \dfrac{xy(2x^2 - 3y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Solution We obtain the required derivatives as

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0, \; f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{y[2(\Delta x)^2 - 3y^2]\Delta x}{[(\Delta x)^2 + y^2]\,\Delta x} = -3y$$

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{f(x, \Delta y) - f(x,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{x[2x^2 - 3(\Delta y)^2]\Delta y}{[x^2 + (\Delta y)^2]\,\Delta y} = 2x.$$

Now,

$$f_{xy}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)_{(0,0)} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x - 0}{\Delta x} = 2$$

$$f_{yx}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-3\Delta y - 0}{\Delta y} = -3.$$

Hence, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Example 2.24 Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ for the function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Also discuss the continuity of f_{xy} and f_{yx} at (0, 0).

Solution We have

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0, \; f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{y^3 \Delta x}{[\Delta x + y^2]\,\Delta x} = y$$

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{f(x, \Delta y) - f(x,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{x(\Delta y)^3}{[x + (\Delta y)^2]\,\Delta y} = 0$$

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x} = 0$$

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1.$$

Since $f_{xy}(0, 0) \neq f_{yx}(0, 0)$, f_{xy} and f_{yx} are not continuous at (0, 0).

Alternative We find that for $(x, y) \neq (0, 0)$

$$f_{yx}(x,y) = \frac{y^6 + 5xy^4}{(x + y^2)^3} = f_{xy}(x,y).$$

Along the path $x = my^2$, we obtain

$$\lim_{(x,y)\to(0,0)} f_{yx}(x, y) = \lim_{y\to 0} \frac{y^6(1+5m)}{y^6(1+m)^3} = \frac{1+5m}{(1+m)^3}.$$

Since the limit does not exist, f_{yx} is not continuous at (0, 0).

Example 2.25 For the implicit function $f(x, y) = 0$ of one independent variable x, obtain $y'' = d^2y/dx^2$. Assume that $f_{xy} = f_{yx}$.

Solution Taking the differential of $f(x, y) = 0$, we obtain : $f_x . dx + f_y . dy = 0$ Total d

$$y' = \frac{dy}{dx} = -\left(\frac{f_x}{f_y}\right).$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = -\frac{d}{dx}\left[\frac{f_x}{f_y}\right] = -\frac{f_y \frac{d}{dx}(f_x) - f_x \frac{d}{dx}(f_y)}{f_y^2}$$

$$= -\frac{f_y[f_{xx} + (f_{yx})y'] - f_x[f_{xy} + (f_{yy})y']}{f_y^2}$$

$$= -\frac{(f_y f_{xx} - f_x f_{xy}) + (f_y f_{yx} - f_x f_{yy})y'}{f_y^2}.$$

$\frac{d(f(x,y))}{dx} = \frac{f_x . dx + f ...}{dx} = f_x + f_y . \frac{dy}{dx}$

Substituting $y' = -f_x/f_y$, we obtain

$$\frac{d^2y}{dx^2} = -\frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{f_y^3}, \quad \text{since } f_{yx} = f_{xy}.$$

2.4.1 Homogeneous Functions

A function $f(x, y)$ is said to be *homogeneous* of degree n in x and y, if it can be written in any one of the following forms

(i) $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. → Easier (Replace x, y by λx & λy ... try to take out λ to get original fnctn) (2.46)

(ii) $f(x, y) = x^n g(y/x)$. (2.47)

(iii) $f(x, y) = y^n g(x/y)$. (2.48)

Similarly, a function $f(x, y, z)$ of three variables is said to be homogeneous, of degree n, if it can be written as $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$, or $f(x, y, z) = x^n g\left(\frac{y}{x}, \frac{z}{x}\right)$ etc.

Some examples of homogeneous functions are the following:

f	degree of homogeneity
$x^2 + xy$	2
$\tan^{-1}(y/x)$	0
$1/(x + y)$	-1
$1/(x^4 + y^4 + z^4)$	-4
$xyz/(x^4 + y^4 + z^4)$	-1
$\sqrt{x}/\sqrt{x^2 + y^2 + z^2}$	$-1/2$

The function $f(x, y) = (x^2 + y)/(x + y^2)$ is not homogeneous.

An important result concerning homogeneous functions is the following.

Theorem 2.4 **(Euler's theorem)** If $f(x, y)$ is a homogeneous function of degree n in x and y and h continuous first and second order partial derivatives, then

(i) $$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf. \quad (2.4$$

(ii) $$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x\, \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1) f. \quad (2.5$$

Proof Since $f(x, y)$ is a homogeneous function of degree n in x and y, we can wri $f(x, y) = x^n g(y/x)$.

Differentiating partially with respect to x and y, we get

$$\frac{\partial f}{\partial x} = nx^{n-1} g\left(\frac{y}{x}\right) + x^n g'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = nx^{n-1} g\left(\frac{y}{x}\right) - yx^{n-2} g'\left(\frac{y}{x}\right).$$

$$\frac{\partial f}{\partial y} = x^n g'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x^{n-1} g'\left(\frac{y}{x}\right).$$

Hence, we obtain

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nx^n g\left(\frac{y}{x}\right) - yx^{n-1} g'\left(\frac{y}{x}\right) + yx^{n-1} g'\left(\frac{y}{x}\right) = nx^n g\left(\frac{y}{x}\right) = nf.$$

Differentiating Eq. (2.49) partially with respect to x and y, we get

$$x \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y \frac{\partial^2 f}{\partial x\, \partial y} = n \frac{\partial f}{\partial x} \quad (2.51)$$

and $$x \frac{\partial^2 f}{\partial y\, \partial x} + \frac{\partial f}{\partial y} + y \frac{\partial^2 f}{\partial y^2} = n \frac{\partial f}{\partial y}. \quad (2.52)$$

Multiplying Eq. (2.51) by x and Eq. (2.52) by y and adding, we obtain

$$x^2 \frac{\partial^2 f}{\partial x^2} + \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}\right) + xy\left(\frac{\partial^2 f}{\partial x\, \partial y} + \frac{\partial^2 f}{\partial y\, \partial x}\right) + y^2 \frac{\partial^2 f}{\partial y^2} = n\left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}\right)$$

or $$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x\, \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1) f.$$

Example 2.26 If $u(x, y) = \cos^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, $0 < x, y < 1$, then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{2} \cot u.$$

Solution For all x, y, $0 < x, y < 1$, $(x + y)/[\sqrt{x} + \sqrt{y}] < 1$, so that $u(x, y)$ is defined. The given function can be written as

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x[1+y/x]}{\sqrt{x}\,[1+\sqrt{y/x}]} = \sqrt{x}\left[\frac{1+(y/x)}{1+\sqrt{y/x}}\right]$$

Therefore, $\cos u$ is a homogeneous function of degree 1/2. Using the Euler's theorem for $f = \cos u$ and $n = 1/2$, we obtain

$$x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u$$

or $$-x(\sin u)\frac{\partial u}{\partial x} - y(\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u, \text{ or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

Example 2.27 If $u(x, y) = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, $x > 0$, $y > 0$, then evaluate

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

Solution We have $u(\lambda x, \lambda y) = \lambda^2 u(x, y)$. Therefore, $u(x, y)$ is a homogeneous function of degree 2. Using Theorem 2.4 (ii) for $f = u$ and $n = 2$, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2(2-1)u = 2u.$$

Example 2.28 Let $u(x, y) = [x^3 + y^3]/[x + y]$, $(x, y) \neq (0, 0)$. Then evaluate

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial u}{\partial x}.$$

Solution We have $u(x, y) = \dfrac{x^2[1+(y/x)^3]}{[1+(y/x)]}$. Therefore, $u(x, y)$ is a homogeneous function of degree 2. Using Euler's theorem, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$$

Differentiating partially with respect to x, we obtain

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = 2\frac{\partial u}{\partial x}, \text{ or } x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial u}{\partial x} = 0.$$

Example 2.29 Let $f(x, y)$ and $g(x, y)$ be two homogeneous functions of degree m and n respectively where $m \neq 0$. Let $h = f + g$. If $x\dfrac{\partial h}{\partial x} + y\dfrac{\partial h}{\partial y} = 0$, then show that $f = \alpha g$ for some scalar α.

Solution Since f and g are homogeneous functions of degrees m and n respectively, we obtain on using Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = mf \text{ and } x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = ng.$$

Adding the two results, we get

$$x\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) + y\left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right) = mf + ng$$

or
$$x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = mf + ng = 0, \text{ where } h = f + g.$$

Therefore, $f = -\dfrac{n}{m}g = \alpha g$, where $\alpha = -\dfrac{n}{m}$ is a scalar.

2.4.2 Taylor's Theorem

In section 1.3.6 we have derived the Taylor's theorem in one variable. If $f(x)$ has continuo derivatives upto $(n + 1)$th order in some interval containing $x = a$, then

$$f(x) = f(a) + (x - a)f'(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x) \qquad (2.5$$

where $R_n(x)$ is the remainder term given by

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a + \theta(x-a)], a < \xi < x, 0 < \theta < 1. \quad (2.5$$

We now extend this theorem to functions of two variables.

Theorem 2.5 **(Taylor's theorem)** Let a function $f(x, y)$ defined in some domain D in $\mathbb{R}^2$ ha continuous partial derivatives upto $(n + 1)$th order in some neighborhood of a point $P(x_0, y_0)$ in I Then, for some point $(x_0 + h, y_0 + k)$ in this neighborhood, we have

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0)$$

$$+ \ldots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + R_n \qquad (2.55$$

where R_n is the remainder term given by

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h,\ y_0 + \theta k), 0 < \theta < 1. \qquad (2.56)$$

Proof Let $x = x_0 + th$, $y = y_0 + tk$, where the parameter t takes values in the interval $[0, 1]$. Define a function $\phi(t)$ as $\phi(t) = f(x, y) = f(x_0 + th, y_0 + tk)$.

Using the chain rule, we get

$$\phi'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f$$

$$\phi''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial f}{\partial y}\right)^2 f, \ldots, \phi^{(n+1)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f.$$

Using the Taylor's theorem for a function of one variable (see Eq. (2.53)) with $t = 1$ and $a = 0$, we obtain

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + \ldots + \frac{1}{n!}\phi^{(n)}(0) + \frac{1}{(n+1)!}\phi^{(n+1)}(\theta) \qquad (2.57)$$

where $$\phi(0) = f(x_0, y_0)$$

$$\phi(1) = f(x_0 + h, y_0 + k)$$

$$\phi^{(i)}(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(x_0, y_0), i = 1, 2, \ldots, n$$

$$\phi^{(n+1)}(\theta) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), 0 < \theta < 1.$$

Substituting the expressions for $\phi(1)$, $\phi(0)$, $\phi'(0)$, ..., $\phi^{(n)}(0)$ and $\phi^{(n+1)}(\theta)$ in Eq. (2.57), we obtain the Taylor's theorem for functions of two variables as given in Eqs. (2.55) and (2.56).

Substituting $x = x_0 + h$, $y = y_0 + k$ in Eq. (2.55), we can also write the Taylor's theorem as

$$f(x, y) = f(x_0, y_0) + \left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right] f(x_0, y_0)$$

$$+ \frac{1}{2!}\left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right]^2 f(x_0, y_0)$$

$$+ \cdots + \frac{1}{n!}\left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right]^n f(x_0, y_0) + R_n \quad (2.58)$$

where, $$R_n = \frac{1}{(n+1)!}\left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right]^{n+1} f(\xi, \eta) \quad (2.59)$$

and $$\xi = (1 - \theta)x_0 + \theta x, \eta = (1 - \theta) y_0 + \theta y, 0 < \theta < 1.$$

For $n = 1$, we get the *linear polynomial approximation* to $f(x, y)$ as

$$f(x, y) \approx f(x_0, y_0) + (x - x_0) f_x + (y - y_0) f_y \quad (2.60)$$

where the partial derivatives are evaluated at (x_0, y_0). This equation is same as the equation (2.28) which was obtained using differentials.

For $n = 2$, we get the *second degree (quadratic) polynomial approximation* to $f(x, y)$ as

$$f(x, y) \approx f(x_0, y_0) + (x - x_0)f_x + (y - y_0)f_y$$
$$+ \frac{1}{2}\left[(x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0) f_{xy} + (y - y_0)^2 f_{yy}\right] \quad (2.61)$$

where the partial derivatives are evaluated at (x_0, y_0).

Remark 7

(a) If we set $(x_0, y_0) = (0, 0)$ in Eq. (2.55), we obtain the *Maclaurin's theorem* for functions of two variables as

$$f(x, y) = f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0, 0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0)$$

$$+ \ldots + \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(0, 0) + R_n \quad (2.62)$$

where $$R_n = \frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f(\theta x, \theta y), 0 < \theta < 1.$$

(b) When $\lim_{n\to\infty} R_n = 0$, we obtain the *Taylor's series* expansion of the function $f(x, y)$ about the point (x_0, y_0).

(c) Taylor's theorem can be easily extended to functions of m variables $f(x_1, x_2, \ldots, x_m)$.

Error estimate

Since the point (ξ, η) or the value of θ in the error term given in Eq. (2.59) is not known, we cannot evaluate the error term exactly. However, it is possible to find a bound of the error term in a given rectangular region R: $|x - x_0| < \delta_1$, $|y - y_0| < \delta_2$. We assume that all the partial derivatives of the required order are continuous throughout this region.

For $n = 1$ (linear approximation), the error term is given by

$$R_1 = \frac{1}{2!}\left[(x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0) f_{xy} + (y - y_0)^2 f_{yy}\right] \quad (2.63)$$

where the partial dirivatives are evaluated at the point $(\xi, \eta) = [x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)]$, $0 < \theta < 1$. Hence, we get

$$|R_1| \le \frac{1}{2}\left[|x - x_0|^2 |f_{xx}| + 2|x - x_0||y - y_0||f_{xy}| + |y - y_0|^2 |f_{yy}|\right].$$

If we assume that

$B = \max\left[|f_{xx}|, |f_{xy}|, |f_{yy}|\right]$ for all (x, y) in R, then we obtain

$$|R_1| \le \frac{B}{2}\left[|x - x_0|^2 + 2|x - x_0||y - y_0| + |y - y_0|^2\right]$$

$$= \frac{B}{2}\left[|x - x_0| + |y - y_0|\right]^2 \le \frac{B}{2}[\delta_1 + \delta_2]^2. \quad (2.64)$$

This value of $|R_1|$ is called the *maximum absolute error* in the linear approximation of $f(x, y)$ about the point (x_0, y_0).

For $n = 2$ (quadratic approximation), the error term is given by

$$R_2 = \frac{1}{3!}\left[(x - x_0)^3 f_{xxx} + 3(x - x_0)^2 (y - y_0) f_{xxy} + 3(x - x_0)(y - y_0)^2 f_{xyy} + (y - y_0)^3 f_{yyy}\right] \quad (2.65)$$

where the partial derivatives are evaluated at the point

$$(\xi, \eta) = [x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)], 0 < \theta < 1.$$

From Eq. (2.65), we get

$$|R_2| \le \frac{1}{6}\left[|x - x_0|^3 |f_{xxx}| + 3|x - x_0|^2 |y - y_0||f_{xxy}| + 3|x - x_0||y - y_0|^2 |f_{xyy}| + |y - y_0|^3 |f_{yyy}|\right]$$

$$\le \frac{B}{6}\left[|x - x_0|^3 + 3|x - x_0|^2 |y - y_0| + 3|x - x_0||y - y_0|^2 + |y - y_0|^3\right]$$

$$= \frac{B}{6} [\,|x - x_0| + |y - y_0|\,]^3 \le \frac{B}{6} (\delta_1 + \delta_2)^3 \tag{2.66}$$

where $B = \max [\,|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\,]$ for all points (x, y) in R.

Remark 8

In a similar manner, we can obtain error estimates for approximations of functions of three or more variables. For example, if $f(x, y, z)$ is to be approximated by a first degree polynomial (linear approximation) about the point (x_0, y_0, z_0), then we have

$$f(x, y, z) \approx P_1(x, y, z) = f(x_0, y_0, z_0) + [(x - x_0) f_x + (y - y_0) f_y + (z - z_0) f_z]$$

where the partial derivatives are evaluated at (x_0, y_0, z_0). The error associated with this approximation is given by

$$R_1 = \frac{1}{2!} [(x - x_0)^2 f_{xx} + (y - y_0)^2 f_{yy} + (z - z_0)^2 f_{zz} + 2(x - x_0)(y - y_0) f_{xy}$$

$$+ 2(x - x_0)(z - z_0) f_{xz} + 2(y - y_0)(z - z_0) f_{yz}\,].$$

If we consider the region $R: |x - x_0| \le \delta_1, |y - y_0| < \delta_2, |z - z_0| < \delta_3$

and assume that $B = \max [\,|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|, |f_{yz}|\,]$

for all points (x, y, z) in this region, we can write

$$|R_1| \le \frac{B}{2} [\,|x - x_0| + |y - y_0| + |z - z_0|\,]^2 \le \frac{B}{2} (\delta_1 + \delta_2 + \delta_3)^2.$$

Example 2.30 Find the linear and the quadratic Taylor series polynomial approximations to the function $f(x, y) = 2x^3 + 3y^3 - 4x^2 y$ about the point $(1, 2)$. Obtain the maximum absolute error in the region $|x - 1| < 0.01$ and $|y - 2| < 0.1$.

Solution We have

$$f(x, y) = 2x^3 + 3y^3 - 4x^2y \quad ; \quad f(1, 2) = 18$$

$$f_x(x, y) = 6x^2 - 8xy \quad ; \quad f_x(1, 2) = -10$$

$$f_y(x, y) = 9y^2 - 4x^2 \quad ; \quad f_y(1, 2) = 32$$

$$f_{xx}(x, y) = 12x - 8y \quad ; \quad f_{xx}(1, 2) = -4$$

$$f_{xy}(x, y) = -8x \quad ; \quad f_{xy}(1, 2) = -8$$

$$f_{yy}(x, y) = 18y \quad ; \quad f_{yy}(1, 2) = 36$$

$$f_{xxx}(x, y) = 12, f_{xxy}(x, y) = -8, \quad f_{xyy}(x, y) = 0, f_{yyy}(x, y) = 18.$$

The linear approximation is given by

$$f(x, y) \approx f(1, 2) + [(x - 1) f_x(1, 2) + (y - 2) f_y(1, 2)]$$

$$= 18 + (x - 1)(-10) + (y - 2)(32) = 18 - 10(x - 1) + 32(y - 2).$$

Maximum absolute error in the linear approximation is given by

$$|R_1| \le \frac{B}{2} [\,|x - 1| + |y - 2|\,]^2 \le \frac{B}{2} [(0.01) + (0.1)]^2 = 0.00605\, B$$

where $B = \max [\,|f_{xx}|, |f_{xy}|, |f_{yy}|\,]$ in the given region $|x-1| < 0.01$, $|y-2| < 0.1$.

Now, $\max |f_{xx}| = \max |12x - 8y| = \max |12(x-1) - 8(y-2) - 4|$

$$\leq \max [\,12|x-1| + 8|y-2| + 4] = 4.92$$

$$\max |f_{xy}| = \max |-8x| = \max |8(x-1) + 8| \leq \max [8|x-1| + 8] = 8.08$$

$$\max |f_{yy}| = \max |18y| = \max [\,18(y-2) + 36] \leq \max [18|y-2| + 36\,] = 37.8.$$

Hence, $|B| = 37.8$ and $|R_1| \leq 0.00605(37.8) \approx 0.23$.

The quadratic approximation is given by

$$f(x, y) \approx f(1, 2) + [(x-1) f_x(1, 2) + (y-2) f_y(1, 2)]$$

$$+ \frac{1}{2} [(x-1)^2 f_{xx}(1, 2) + 2(x-1)(y-2) f_{xy}(1, 2) + (y-2)^2 f_{yy}(1, 2)]$$

$$= 18 - 10(x-1) + 32(y-2) + \frac{1}{2} [-4(x-1)^2 - 16(x-1)(y-2) + 36(y-2)^2]$$

$$= 18 - 10(x-1) + 32(y-2) - 2[(x-1)^2 + 4(x-1)(y-2) - 9(y-2)^2].$$

Using Eq. (2.66), the maximum absolute error in the quadratic approximation is given by

$$|R_2| \leq \frac{B}{6} [\,|x-1| + |y-2|\,]^3 \leq \frac{B}{6} (0.11)^3 = \frac{B}{6} (0.001331)$$

where $\quad B = \max [\,|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\,] = \max [\,12, 8, 0, 18\,] = 18.$

Hence, we obtain

$$|R_2| \leq \frac{18}{6} (0.001331) \approx 0.004.$$

Example 2.31 Expand $f(x, y) = 21 + x - 20y + 4x^2 + xy + 6y^2$ in Taylor series of maximum order about the point $(-1, 2)$.

Solution Since all the third order partial derivatives of $f(x, y)$ are zero, the maximum order of the Taylor series expansion of $f(x, y)$ about the point $(-1, 2)$ is two. We obtain

$$f(x,y) = f(-1,2) + \left[(x+1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y}\right] f(-1,2) + \frac{1}{2!}\left[(x+1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y}\right]^2 f(-1,2).$$

We have

$$f(-1, 2) = 6, \; f_x(x, y) = 1 + 8x + y, \; f_x(-1, 2) = -5,$$

$$f_y(x, y) = -20 + x + 12y, \; f_y(-1, 2) = 3,$$

$$f_{xx}(x, y) = 8, f_{xy}(x, y) = 1, \; f_{yy}(x, y) = 12.$$

Therefore,

$$f(x, y) = 6 - 5(x+1) + 3(y-2) + 4(x+1)^2 + (x+1)(y-2) + 6(y-2)^2.$$

This is an rearrangement of the terms in the given function.

Example 2.32 The function $f(x, y) = x^2 - xy + y^2$ is approximated by a first degree Taylor's polynomial about the point (2, 3). Find a square $|x - 2| < \delta$, $|y - 3| < \delta$ with centre at (2, 3) such that the error of approximation is less than or equal to 0.1 in magnitude for all points within this square.

Solution We have $f_x = 2x - y,\ f_y = 2y - x,\ f_{xx} = 2,\ f_{xy} = -1,\ f_{yy} = 2.$
The maximum absolute error in the first degree approximation is given by

$$|R_1| \le \frac{B}{2}[\,|x-2| + |y-3|\,]^2$$

where $B = \max[\,|f_{xx}|, |f_{xy}|, |f_{yy}|\,] = \max[2, 1, 2] = 2.$

We also have $|x - 2| < \delta$, $|y - 3| < \delta$. Therefore, we want to determine δ such that

$$|R_1| \le \frac{2}{2}[\delta + \delta]^2 < 0.1, \text{ or } 4\delta^2 < 0.1, \text{ or } \delta < \sqrt{0.025} \approx 0.1581.$$

Example 2.33 If $f(x, y) = \tan^{-1}(xy)$, find an approximate value of $f(1.1, 0.8)$ using the Taylor's series (i) linear approximation and (ii) quadratic approximation.

Solution Let $(x_0, y_0) = (1.0, 1.0)$, $h = 0.1$, $k = -0.2$. Then $f(1.1, 0.8) = f(1 + 0.1, 1 - 0.2)$.

(i) Using the Taylor series linear approximation, we have

$$f(1.1, 0.8) \approx f(1, 1) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(1, 1).$$

From $f(x, y) = \tan^{-1}(xy)$, we get

$$f(1, 1) = \tan^{-1}(1) = \pi/4 \approx 0.7854$$

$$f_x(x, y) = \frac{y}{1 + x^2y^2},\ f_x(1, 1) = \frac{1}{2},\ f_y(x, y) = \frac{x}{1 + x^2y^2},\ f_y(1, 1) = \frac{1}{2}.$$

Therefore,

$$f(1.1, 0.8) \approx 0.7854 + \left\{\frac{1}{2}(0.1) + \frac{1}{2}(-0.2)\right\} = 0.7354.$$

(ii) Using the Taylor series quadratic approximation, we have

$$f(1.1, 0.8) \approx f(1, 1) + (h f_x + k f_y)_{(1,1)} + \frac{1}{2}[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}]_{(1,1)}.$$

We have

$$f_{xx}(x, y) = -\frac{2xy^3}{(1 + x^2y^2)^2},\ f_{xx}(1, 1) = -\frac{1}{2};\ f_{yy}(x, y) = -\frac{2x^3y}{(1 + x^2y^2)^2},\ f_{yy}(1, 1) = -\frac{1}{2}$$

$$f_{xy}(x, y) = \frac{(1 + x^2y^2) - y(2x^2y)}{(1 + x^2y^2)^2} = \frac{1 - x^2y^2}{(1 + x^2y^2)^2},\ f_{xy}(1, 1) = 0.$$

Therefore, using the result of (i), we obtain

$$f(1.1, 0.8) \approx 0.7354 + \frac{1}{2}\left\{(0.01)\left(-\frac{1}{2}\right) + 2(0.1)(-0.2)(0) + (0.04)\left(-\frac{1}{2}\right)\right\}$$

$$= 0.7354 - 0.0125 = 0.7229.$$

The exact value of $f(1.1, 0.8)$ to four decimal places is 0.7217. Thus, the accuracy increases as the order of approximation increases.

Exercises 2.3

Find all the partial derivatives of the specified order for the following functions at the given point:

1. $f(x, y) = [x - y]/[x + y]$, second order at $(1, 1)$.
2. $f(x, y) = x \ln y$, third order at $(2, 3)$.
3. $f(x, y) = \ln[(1/x) - (1/y)]$, second order at $(1, 2)$.
4. $f(x, y) = e^x \ln y + (\cos y) \ln x$, third order at $(1, \pi/2)$.
5. $f(x, y) = e^{\sin(x/y)}$, second order at $(\pi/2, 1)$.
6. $f(x, y, z) = [x + y]/[x + z]$, second order at $(1, -1, 1)$.
7. $f(x, y, z) = e^{x^2+y^2+z^2}$, second order at $(-1, 1, -1)$.
8. $f(x, y, z) = \sin xy + \sin xz + \sin yz$, second order at $(1, \pi/2, \pi/2)$.
9. $f(x, y, z) = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$, second order at $(1, 2, 3)$.
10. $f(x, y, z) = x^x y^y z^z$, $\dfrac{\partial^2 f}{\partial x\, \partial y}$ at any point $(x, y, z) \neq (0, 0, 0)$.
11. For the function
$$f(x, y) = \begin{cases} \dfrac{x^2 y(x - y)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$
show that $f_{xy} \neq f_{yx}$ at $(0, 0)$.
12. Show that $f_{xy} = f_{yx}$ for all $(x, y) \neq (0, 0)$, when $f(x, y) = x^y$.
13. Show that $f_{xy} = f_{yx}$ for all $(x, y) \neq (0, 0)$, when $f(x, y) = \log[x + \sqrt{y^2 + x^2}]$.
14. Show that $f_{xyz} = f_{yzx}$ for all (x, y, z), when $f(x, y, z) = e^{xy} \sin z$.
15. Show that $f_{xyyz} = f_{yyxz}$ for all (x, y, z), when $f(x, y, z) = z^2 e^{x+y^2}$.
16. If $z = e^x \sin y + e^y \cos x$, where x and y are implicit functions of t defined by the equations $x^3 + x + e^t + t^2 + t - 1 = 0$ and $yt^3 + y^3 t + t + y = 0$, then find dz/dt at $t = 0$.
17. If x and y are defined as functions of u, v by the implicit equations $x^2 - y^2 + 2u^2 + 3v^2 - 1 = 0$ and $2x^2 - y^2 - u^2 + 4v^2 - 2 = 0$, then find $\partial x/\partial u$, $\partial y/\partial u$, $\partial^2 x/\partial u^2$ and $\partial^2 y/\partial u^2$.
18. If u and v are defined as functions of x and y by the implicit equations $4x^2 + 3y^2 - z^2 - u^2 + v^2 = 6$, $3x^2 - 2y^2 + z^2 + u^2 + 2v^2 = 14$, then find $(\partial u/\partial x)_{y,z}$ and $(\partial v/\partial y)_{x,z}$ at $x = 1$, $y = -1$, $z = 2$. Assume that $u > 0$, $v > 0$.
19. If $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = c$, c any constant, $|x| < 1$, $|y| < 1$, then find dy/dx and d^2y/dx^2.
20. Find dy/dx and d^2y/dx^2 at the point $(x, y) = (1, 1)$, for $e^y - e^x + xy = 1$.

21. If $z = u^v$, $u = (x/y)$, $v = xy$, then find $\partial^2 z/\partial x^2$.
22. If $u = \ln(1/r)$, $r = \sqrt{(x-a)^2 + (y-b)^2}$, then show that $u_{xx} + u_{yy} = 0$.
23. If $F = f(u, v)$, $u = y + ax$, $v = -y - ax$, a any constant, then show that $F_{xx} = a^2 F_{yy}$.
24. If $f(x, y) = x \log(y/x)$, $(x, y) \neq (0, 0)$, then show that $x^2 f_{xx} + 2xy\, f_{xy} + y^2 f_{yy} = 0$.
25. If $f(x, y) = y/(x^2 + y^2)$, $(x, y) \neq (0, 0)$, then show that $f_{xx} + f_{yy} = 0$.
26. Find α and β such that $u(x, y) = e^{\alpha x + \beta y}$ satisfies the equation $u_{xx} - 7u_{xy} + 12u_{yy} = 0$.
27. If $z = f(u, v)$, $u = x/(x^2 + y^2)$, $v = y/(x^2 + y^2)$, $(x, y) \neq (0, 0)$, then show that $z_{uu} + z_{vv} = (x^2 + y^2)^2 (z_{xx} + z_{yy})$.
28. If $x = r\cos\theta$, $y = r\sin\theta$, then show that

    (i) $\dfrac{\partial^2 \theta}{\partial x\, \partial y} = -\dfrac{\cos 2\theta}{r^2}$, (ii) $\dfrac{\partial^2 r}{\partial x\, \partial y} = -\dfrac{\sin 2\theta}{2r}$.

Using Euler's theorem, establish the following results.

29. If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.
30. If $u = \log\left[\dfrac{\sqrt{x^2 + y^2}}{x}\right]$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.
31. If $u = \sqrt{y^2 - x^2}\, \sin^{-1}\left(\dfrac{x}{y}\right)$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$.
32. If $u = \dfrac{y^3 - x^3}{y^2 + x^2}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$ and $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\, \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.
33. If $\tan u = \dfrac{x^3 + y^3}{x - y}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$ and

    $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u.$$
34. Obtain the Taylor's series expansion of the maximum order for the function $f(x, y) = x^2 + 3y^2 - 9x - 9y + 26$ about the point (2, 2).
35. Obtain the Taylor's linear approximation to the function $f(x, y) = 2x^2 - xy + y^2 + 3x - 4y + 1$ about the point (–1, 1). Find the maximum error in the region $|x + 1| < 0.1$, $|y - 1| < 0.1$.
36. Obtain the first degree Taylor's series approximation to the function $f(x, y) = e^y \ln(x + y)$ about the point (1, 0). Estimate the maximum absolute error over the rectangle $|x - 1| < 0.1$, $|y| < 0.1$.
37. Obtain the second order Taylor's series approximation to the function $f(x, y) = xy^2 + y\cos(x - y)$ about the point (1, 1). Find the maximum absolute error in the region $|x - 1| < 0.05$, $|y - 1| < 0.1$.
38. Expand $f(x, y) = \sqrt{x + y}$ in Taylor's series upto second order terms about the point (1, 3). Estimate the maximum absolute error in the region $|x - 1| < 0.2$, $|y - 3| < 0.1$.
39. Obtain the Taylor's series expansion, upto third degree terms, of the function $f(x, y) = e^{2x+y}$ about the point (0, 0). Obtain the maximum error in the region $|x| < 0.1$, $|y| < 0.2$.
40. Expand $f(x, y) = \sin(x + 2y)$ in Taylor's series upto third order terms about the point (0, 0). Find the maximum error over the rectangle $|x| < 0.1$, $|y| < 0.1$.

41. Expand $f(x, y) = \sin x \sin y$ in Taylor's series upto second order terms about the point $(\pi/4, \pi/4)$. Fin the maximum error in the region $|x - \pi/4| < 0.1$, $|y - \pi/4| < 0.1$.

42. Expand $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ in Taylor series upto first order terms about the point (2, 2, 1 Obtain the maximum error in the region $|x - 2| < 0.1$, $|y - 2| < 0.1$, $|z - 1| < 0.1$.

43. Expand $f(x, y, z) = \sqrt{xy + yz + xz}$ in Taylor's series upto first order terms about the point (1, 3, 3/2 Obtain the maximum error in the region $|x - 1| < 0.1$, $|y - 3| < 0.1$, $|z - 3/2| < 0.1$.

44. Expand $f(x, y, z) = e^z \sin(x + y)$ in Taylor's series upto second order terms about the point (0, 0, 0 Obtain the maximum error in the region $|x| < 0.1$, $|y| < 0.1$, $|z| < 0.1$.

45. Expand $f(x, y, z) = e^x \sin(yz)$ in Taylor's series upto second order terms about the point $(0, 1, \pi/2$ Obtain the maximum error in the region $|x| < 0.1$, $|y - 1| < 0.1$, $|z - \pi/2| < 0.1$.

2.5 Maximum and Minimum Values of a Function

Let a function $f(x, y)$ be defined and continuous in some closed and bounded region R. Let (a, b) b an interior point of R and $(a + h, b + k)$ be a point in its neighborhood and lies inside R. We defin the following.

(i) The point (a, b) is called a point of *relative* (or *local*) *minimum*, if

$$f(a + h, b + k) \geq f(a, b) \qquad (2.67a)$$

for all h, k. Then, $f(a, b)$ is called the *relative* (or *local*) *minimum* value.

(ii) The point (a, b) is called a point of *relative* (or *local*) *maximum*, if

$$f(a + h, b + k) \leq f(a, b) \qquad (2.67b)$$

for all h, k. Then $f(a, b)$ is called the *relative* (or *local*) *maximum* value.

A function $f(x, y)$ may also attain its minimum or maximum values on the boundary of the region The smallest and the largest values attained by a function over the entire region including th boundary are called the *absolute* (or *global*) *minimum* and *absolute* (or *global*) *maximum* value respectively.

The points at which minimum / maximum values of the function occur are also called *points o extrema* or the *stationary points* and the minimum and the maximum values taken together are calle the *extreme values* of the function.

We now present the necessary conditions for the existence of an extremum of a functon.

Theorem 2.6 (Necessary conditions for a function to have an extremum) Let the function $f(x, y)$ be continuous and possess first order partial derivatives at a point $P(a, b)$. Then, the necessary conditions for the existence of an extreme value of f at the point P are $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Proof Let $(a + h, b + k)$ be a point in the neighborhood of the point $P(a, b)$. Then, P will be a point of maximum, if

$$\Delta f = f(a + h, b + k) - f(a, b) \leq 0 \quad \text{for all} \quad h, k \qquad (2.68)$$

and a point of minimum, if

$$\Delta f = f(a + h, b + k) - f(a, b) \geq 0 \quad \text{for all} \quad h, k. \qquad (2.69)$$

Using the Taylor's series expansion about the point (a, b), we obtain

$$f(a+h, b+k) = f(a, b) + (hf_x + kf_y)_{(a,b)} + \frac{1}{2}[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}]_{(a,b)} + \ldots \tag{2.70}$$

Neglecting the second and higher order terms, we get

$$\Delta f \approx hf_x(a, b) + kf_y(a, b). \tag{2.71}$$

The sign of Δf in Eq. (2.71) depends on the sign of $hf_x(a, b) + kf_y(a, b)$ which is a function of h and k. Letting $h \to 0$, we find that Δf changes sign with k. Therefore, the function cannot have an extremum unless $f_y = 0$. Similarly, letting $k \to 0$, we find that the function f cannot have an extremum unless $f_x = 0$.

Therefore, the necessary conditions for the existence of an extremum at the point (a, b) is that

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0. \tag{2.72}$$

A point $P(a, b)$, where $f_x(a, b) = 0$ and $f_y(a, \mathrm{b}) = 0$ is called a *critical point* or a *stationary point*. A point P is also called a critical point when one or both of the first order partial derivatives do not exist at this point.

Remark 9

To find the minimum/maximum values of a function f, we first find all the critical points. We then examine each critical point to decide whether at this point the function has a minimum value or a maximum value using the sufficient conditions.

Theorem 2.7 (Sufficient conditions for a function to have a minimum/maximum) Let a function $f(x, y)$ be continuous and possess first and second order partial derivatives at a point $P(a, b)$. If $P(a, b)$ is a critical point, then the point P is a point of

$$\textit{relative minimum} \text{ if } rt - s^2 > 0 \text{ and } r > 0 \tag{2.73a}$$

$$\textit{relative maximum} \text{ if } rt - s^2 > 0 \text{ and } r < 0 \tag{2.73b}$$

where $r = f_{xx}(a, b)$, $s = f_{xy}(a, b)$ and $t = f_{yy}(a, b)$.

No conclusion about an extremum can be drawn if $rt - s^2 = 0$ and further investigation is needed. If $rt - s^2 < 0$, then the function f has no minimum or maximum at this point. In this case, the point P is called a *saddle point*.

Proof Let $(a+h, b+k)$ be a point in the neighborhood of the point $P(a, b)$. Since P is a critical point, we have $f_x(a, b) = 0$, and $f_y(a, b) = 0$. Neglecting the third and higher order terms in the Taylor's series expansion of $f(a+h, b+k)$ about the point (a, b), we get

$$\begin{aligned}\Delta f = f(a+h, b+k) - f(a, b) &\approx \frac{1}{2}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, \mathrm{b}) + k^2 f_{yy}(a, b)] \\ &= \frac{1}{2}[h^2 r + 2hks + k^2 t] = \frac{1}{2r}[h^2 r^2 + 2hkrs + k^2 rt] \\ &= \frac{1}{2r}[(hr + ks)^2 + k^2(rt - s^2)].\end{aligned} \tag{2.74}$$

Since $(hr + ks)^2 > 0$, the sufficient condition for the expression $(hr + ks)^2 + k^2(rt - s^2)$ to be positive is that $rt - s^2 > 0$.

Hence, if $rt - s^2 > 0$, then

$$\Delta f > 0 \text{ if } r > 0 \quad \text{and} \quad \Delta f < 0 \text{ if } r < 0.$$

Therefore, a sufficient condition for the critical point $P(a, b)$ to be a

$$\text{point of relative minimum is } rt - s^2 > 0 \text{ and } r > 0$$
$$\text{point of relative maximum is } rt - s^2 > 0 \text{ and } r < 0.$$

If $rt - s^2 < 0$, then the sign of Δf in Eq. (2.74) depends on h and k. Hence, no maximum/minimum of f can occur at $P(a, b)$ in this case.

If $rt - s^2 = 0$ or $r = t = s = 0$, no conclusion can be drawn and the terms involving higher order partial derivatives must be considered.

Remark 10

(a) We can also write Eq. (2.74) as

$$\Delta f = \frac{1}{2t}\,[k^2t^2 + 2hkst + h^2rt] = \frac{1}{2t}\,[(kt + hs)^2 + (rt - s^2)h^2].$$

Hence, a sufficient condition for a critical point $P(a, b)$ to be a

$$\text{point of relative minimum is } rt - s^2 > 0 \text{ and } t > 0$$
$$\text{point of relative maximum is } rt - s^2 > 0 \text{ and } t < 0.$$

From these conditions and Eqs. (2.73a, 2.73b), we find that when an extremum exists, then $rt - s^2 > 0$, and both r and t have the same sign either positive or negative.

(b) Alternate statement of Theorem 2.7

A real symmetric matrix $\mathbf{A} = (a_{ij})$ is called a positive definite matrix, if

$$\mathbf{x}^T \mathbf{A}\, \mathbf{x} > 0 \text{ for all real vectors } \mathbf{x} \neq \mathbf{0}$$

or
$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j > 0 \text{ for all } x_i, x_j \text{ (see section 3.5.3).}$$

A sufficient condition for the matrix **A** to be positive definite is that the minors of all its leading submatrices are positive. Now we state the result. Let

$$\mathbf{A} = \begin{bmatrix} r & s \\ s & t \end{bmatrix}$$

where $r = f_{xx}(a, b)$, $s = f_{xy}(a, b) = f_{yx}(a, b)$ and $t = f_{yy}(a, b)$. Then, the function $f(x, y)$ has a relative minimum at a critical point $P(a, b)$, if the matrix **A** is positive definite. Since all the leading minors of **A** are positive, we obtain the conditions $r > 0$ and $rt - s^2 > 0$.

The function $f(x, y)$ has a relative maximum at $P(a, b)$, if the matrix $\mathbf{B} = -\mathbf{A} = \begin{bmatrix} -r & -s \\ -s & -t \end{bmatrix}$ is positive definite. Since all the leading minors of **B** are positive, we obtain the conditions $-r > 0$ and $rt - s^2 > 0$, that is $r < 0$ and $rt - s^2 > 0$.

This alternative statement of the Theorem 2.7 is useful when we consider the extreme values of the functions of three or more variables. For example, for the function $f(x, y, z)$ of three variables, we have

$$\mathbf{A} = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

where $f_{yx} = f_{xy}, f_{zx} = f_{xz}, f_{zy} = f_{yz}$. The matrix **A** or the matrix $\mathbf{B} = -\mathbf{A}$ can be tested whether it is positive definite, to find the points of minimum/maximum. Therefore, a critical point (a point at which $f_x = 0 = f_y = f_z$)

(i) is a point of relative minimum if **A** is positive definite and f_{xx}, f_{yy}, f_{zz} are all positive.

(ii) is a point of relative maximum if $\mathbf{B} = -\mathbf{A}$ is positive definite (that is, the leading minors of **A** are alternately negative and positive) and f_{xx}, f_{yy}, f_{zz} are all negative.

Example 2.34 Find the relative maximum and minimum values of the function

$$f(x, y) = 2(x^2 - y^2) - x^4 + y^4.$$

Solution We have

$$f_x = 4x - 4x^3 = 0, \text{ or } x = 0, \pm 1$$

$$f_y = -4y + 4y^3 = 0, \text{ or } y = 0, \pm 1.$$

Hence, (0, 0), (0, ±1), (±1, 0), (±1, ±1) are the critical points. We find that

$$r = f_{xx} = 4 - 12x^2, \quad s = f_{xy} = 0, \quad t = f_{yy} = -4 + 12y^2$$

and $$rt - s^2 = -16(1 - 3x^2)(1 - 3y^2).$$

At the points (0, 1) and (0, –1), we have $rt - s^2 = 32 > 0$ and $r = 4 > 0$. Therefore, the points (0, 1) and (0, –1) are points of relative minimum and the minimum value at each point is –1.

At the points (–1, 0) and (1, 0), we have $rt - s^2 = 32 > 0$ and $r = -8 < 0$. The points (–1, 0) , (1, 0) are points of relative maximum and the maximum value at each point is 1.

At (0, 0), we have $rt - s^2 = -16 < 0$. At (±1, ±1), we have $rt - s^2 = -64 < 0$. Hence, the points (0, 0) (±1, ±1) are neither the points of maximum nor minimum.

Example 2.35 Find the absolute maximum and minimum values of

$$f(x, y) = 4x^2 + 9y^2 - 8x - 12y + 4$$

over the rectangle in the first quadrant bounded by the lines $x = 2$, $y = 3$ and the coordinate axes.

Solution The function f can attain maximum/minimum values at the critical points or on the boundary of the rectangle $OABC$ (Fig. 2.4).

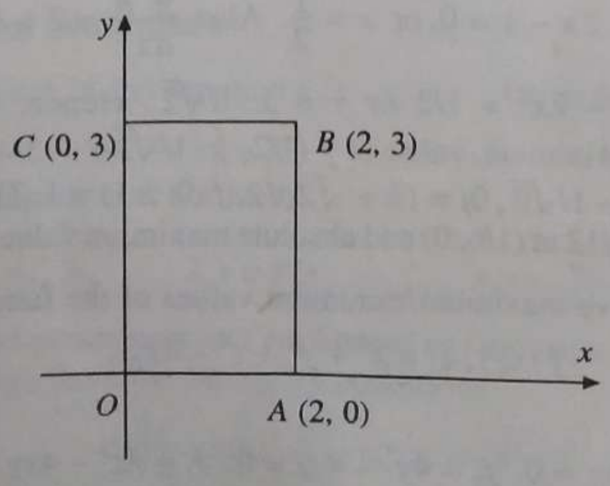


Fig. 2.4. Region in Example 2.35.

We have $f_x = 8x - 8 = 0$, $f_y = 18y - 12 = 0$. The critical point is $(x, y) = (1, 2/3)$. Now,

$$r = f_{xx} = 8,\ \ s = f_{xy} = 0,\ \ t = f_{yy} = 18,\ \ rt - s^2 = 144.$$

Since $rt - s^2 > 0$ and $r > 0$, the point $(1, 2/3)$ is a point of relative minimum. The minimum value is $f(1, 2/3) = -4$.

On the boundary line OA, we have $y = 0$ and $f(x, y) = f(x, 0) = g(x) = 4x^2 - 8x + 4$, which is a function of one variable. Setting $dg/dx = 0$, we get $8x - 8 = 0$ or $x = 1$. Now $d^2g/dx^2 = 8 > 0$. Therefore, at $x = 1$, the function has a minimum. The minimum value is $g(1) = 0$. Also, at the corners $(0, 0)$, $(2, 0)$, we have $f(0, 0) = g(0) = 4$, $f(2, 0) = g(2) = 4$.

Similarly, along the other boundary lines, we have the following results:

$x = 2$: $h(y) = 9y^2 - 12y + 4$; $dh/dy = 18y - 12 = 0$ gives $y = 2/3$; $d^2h/dy^2 = 18 > 0$. Therefore, $y = 2/3$ is a point of minimum. The minimum value is $f(2, 2/3) = 0$. At the corner $(2, 3)$, we have $f(2, 3) = 49$.

$y = 3$: $g(x) = 4x^2 - 8x + 49$; $dg/dx = 8x - 8 = 0$ gives $x = 1$; $d^2g/dx^2 = 8 > 0$. Therefore, $x = 1$ is a point of minimum. The minimum value is $f(1, 3) = 45$. At the corner point $(0, 3)$, we have $f(0, 3) = 49$.

$x = 0$: $h(y) = 9y^2 - 12y + 4$, which is the same case as for $x = 2$.

Therefore, the absolute minimum value is -4 which occurs at $(1, 2/3)$ and the absolute maximum value is 49 which occurs at the points $(2, 3)$ and $(0, 3)$.

Example 2.36 Find the absolute maximum and minimum values of the function

$$f(x, y) = 3x^2 + y^2 - x \text{ over the region } 2x^2 + y^2 \le 1.$$

Solution We have $f_x = 6x - 1 = 0$ and $f_y = 2y = 0$. Therefore, the critical point is $(x, y) = (1/6, 0)$.

Now,
$$r = f_{xx} = 6,\ \ s = f_{xy} = 0,\ \ t = f_{yy} = 2,\ \ rt - s^2 = 12 > 0.$$

Therefore, $(1/6, 0)$ is a point of minimum. The minimum value at this point is $f(1/6, 0) = -1/12$.

On the boundary, we have $y^2 = 1 - 2x^2$, $-1/\sqrt{2} \le x \le 1/\sqrt{2}$. Substituting in $f(x, y)$, we obtain

$$f(x, y) = 3x^2 + (1 - 2x^2) - x = 1 - x + x^2 = g(x)$$

which is a function of one variable. Setting $dg/dx = 0$, we get

$$\frac{dg}{dx} = 2x - 1 = 0, \text{ or } x = \frac{1}{2}. \text{ Also } \frac{d^2g}{dx^2} = 2 > 0.$$

For $x = 1/2$, we get $y^2 = 1 - 2x^2 = 1/2$ or $y = \pm 1/\sqrt{2}$. Hence, the points $(1/2, \pm 1/\sqrt{2})$ are points of minimum. The minimum value is $f(1/2, \pm 1/\sqrt{2}) = 3/4$. At the vertices, we have $f(1/\sqrt{2}, 0) = (3 - \sqrt{2})/2$, $f(-1/\sqrt{2}, 0) = (3 + \sqrt{2})/2$, $f(0, \pm 1) = 1$. Therefore, the given function has absolute minimum value $-1/12$ at $(1/6, 0)$ and absolute maximum value $(3 + \sqrt{2})/2$ at $(-1/\sqrt{2}, 0)$.

Example 2.37 Find the relative maximum/minimum values of the function

$$f(x, y, z) = x^4 + y^4 + z^4 - 4xyz.$$

Solution We have

$$f_x = 4x^3 - 4yz = 0,\ \ f_y = 4y^3 - 4xz = 0,\ \ f_z = 4z^3 - 4xy = 0.$$

Therefore,
$$x^3 = yz,\ \ y^3 = xz,\ \ z^3 = xy \ \text{ or } \ x^3y^3z^3 = x^2y^2z^2 \ \text{ or } \ x^2y^2z^2\,(xyz - 1) = 0.$$

Therefore, all points which satisfy $xyz = 0$ or $xyz = 1$ are critical points. The solutions of these equations are (0, 0, 0), (1, 1, 1), (±1, ±1, 1), (1, ±1, ±1), (±1, 1, ±1) with the same sign taken for the two coordinates. Now,

$$f_{xx} = 12x^2,\ f_{yy} = 12y^2,\ f_{zz} = 12z^2,\ f_{xy} = -4z,\ f_{xz} = -4y,\ f_{yz} = -4x.$$

At (0, 0, 0), all the second order partial derivatives are zero. Therefore, no conclusion can be drawn.

We have
$$\mathbf{A} = \begin{bmatrix} 12x^2 & -4z & -4y \\ -4z & 12y^2 & -4x \\ -4y & -4x & 12z^2 \end{bmatrix}$$

Depending on whether $\mathbf{A}$ or $\mathbf{B} = -\mathbf{A}$ is positive definite, we can decide the points of minimum or maximum. The leading minors are

$$M_1 = 12x^2,\ M_2 = \begin{vmatrix} 12x^2 & -4z \\ -4z & 12y^2 \end{vmatrix} = 16\,(9x^2y^2 - z^2)$$

and
$$M_3 = |\,\mathbf{A}\,| = 192x^2\,(9y^2z^2 - x^2) - 192z^4 - 64\,xyz - 64x\,yz - 192\,y^4$$
$$= 192\,[9x^2y^2z^2 - (x^4 + y^4 + z^4)] - 128xyz.$$

At all points (1, 1, 1), (±1, ±1, 1), (±1, 1, ±1), (1, ±1, ±1) with the same sign taken for two coordinates, we find that $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$. Hence, $\mathbf{A}$ is a positive definite matrix and the given function has relative minimum at all these points, since $f_{xx} > 0$, $f_{yy} > 0$, and $f_{zz} > 0$. The relative minimum value at all these points is same and is given by $f(1, 1, 1) = -1$.

Conditional maximum/minimum

In many practical problems, we need to find the maximum/minimum value of a function $f(x_1, x_2, \ldots, x_n)$ when the variables are not independent but are connected by one or more constraints of the form

$$\phi_i(x_1, x_2, \ldots, x_n) = 0,\ i = 1, 2, \ldots, k$$

where generally $n > k$. We present the Lagrange method of multipliers to find the solution of such problems.

2.5.1 Lagrange Method of Multipliers

We want to find the extremum of the function $f(x_1, x_2, \ldots, x_n)$ under the conditions

$$\phi_i(x_1, x_2, \ldots, x_n) = 0,\ i = 1, 2, \ldots, k. \tag{2.75}$$

We construct an auxiliary function of the form

$$F(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_k) = f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{k} \lambda_i\, \phi_i(x_1, x_2, \ldots, x_n) \tag{2.76}$$

where λ_i's are undetermined parameters and are known as *Lagrange multipliers*. Then, to determine the stationary points of F, we have the necessary conditions

$$\frac{\partial F}{\partial x_1} = 0 = \frac{\partial F}{\partial x_2} = \ldots = \frac{\partial F}{\partial x_n}$$

which give the equations

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{k} \lambda_i \frac{\partial \phi_i}{\partial x_j} = 0,\ j = 1, 2, \ldots, n. \tag{2.7}$$

From Eqs. (2.75) and (2.77), we obtain $(n + k)$ equations in $(n + k)$ unknowns $x_1, x_2, \ldots, x$ $\lambda_1, \lambda_2, \ldots, \lambda_k$. Solving these equations, we obtain the required stationary points $(x_1, x_2, \ldots, x_n)$ which the function f has an extremum. Further investigation is needed to determine the exact natu of these points.

Example 2.38 Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$.

Solution Consider the auxiliary function

$$F(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(xyz - a^3).$$

We obtain the necessary conditions for extremum as

$$\frac{\partial F}{\partial x} = 2x + \lambda yz = 0,\ \frac{\partial F}{\partial y} = 2y + \lambda xz = 0,\ \frac{\partial F}{\partial z} = 2z + \lambda xy = 0.$$

From these equations, we obtain

$$\lambda yz = -2x \text{ or } \lambda xyz = -2x^2$$
$$\lambda xz = -2y \text{ or } \lambda xyz = -2y^2$$
$$\lambda xy = -2z \text{ or } \lambda xyz = -2z^2.$$

Therefore, $x^2 = y^2 = z^2$. Using the condition $xyz = a^3$, we obtain the solutions as (a, a, a), $(a, -a, -a)$, $(-a, a, -a)$ and $(-a, -a, a)$. At each of these points, the value of the given function is $x^2 + y^2 + z^2 = 3a^2$.

Now, the arithmetic mean of x^2, y^2, z^2 is $AM = (x^2 + y^2 + z^2)/3$

the geometric mean of x^2, y^2, z^2 is $GM = (x^2y^2z^2)^{1/3} = a^2$.

Since, $AM \geq GM$, we obtain $x^2 + y^2 + z^2 \geq 3a^2$.

Hence, all the above points are the points of constrained minimum and the minimum value of $x^2 + y^2 + z^2$ is $3a^2$.

Example 2.39 Find the extreme values of $f(x, y, z) = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$.

Solution Consider the auxiliary function

$$F(x, y, z, \lambda_1, \lambda_2) = 2x + 3y + z + \lambda_1(x^2 + y^2 - 5) + \lambda_2(x + z - 1).$$

For the extremum, we have the necessary conditions

$$\frac{\partial F}{\partial x} = 2 + 2\lambda_1 x + \lambda_2 = 0;\ \frac{\partial F}{\partial y} = 3 + 2\lambda_1 y = 0,\ \frac{\partial F}{\partial z} = 1 + \lambda_2 = 0.$$

From these equations, we get

$$\lambda_2 = -1,\ 3 + 2\lambda_1 y = 0 \text{ and } 1 + 2\lambda_1 x = 0$$

or

$$x = -1/(2\lambda_1) \text{ and } y = -3/(2\lambda_1).$$

Substituting in the constraint $x^2 + y^2 = 5$, we get

$$\frac{1}{4\lambda_1^2} + \frac{9}{4\lambda_1^2} = 5 \text{ or } \lambda_1^2 = \frac{1}{2} \text{ or } \lambda_1 = \pm \frac{1}{\sqrt{2}}.$$

For $\lambda_1 = 1/\sqrt{2}$, we get $x = -\sqrt{2}/2,\ y = -3\sqrt{2}/2,\ z = 1 - x = (2 + \sqrt{2})/2$

and $f(x, y, z) = -\sqrt{2} - \dfrac{9\sqrt{2}}{2} + \dfrac{2 + \sqrt{2}}{2} = \dfrac{2 - 10\sqrt{2}}{2} = 1 - 5\sqrt{2}.$

For $\lambda_1 = -1/\sqrt{2}$, we get $x = \sqrt{2}/2,\ y = 3\sqrt{2}/2,\ z = 1 - x = (2 - \sqrt{2})/2$

and $f(x, y, z) = \sqrt{2} + \dfrac{9\sqrt{2}}{2} + \dfrac{2 - \sqrt{2}}{2} = \dfrac{2 + 10\sqrt{2}}{2} = 1 + 5\sqrt{2}.$

Example 2.40 Find the shortest distance between the line $y = 10 - 2x$ and the ellipse $(x^2/4) + (y^2/9) = 1$.

Solution Let (x, y) be a point on the ellipse and (u, v) be a point on the line. Then, the shortest distance between the line and the ellipse is the square root of the minimum value of

$$f(x, y, u, v) = (x - u)^2 + (y - v)^2$$

subject to the constraints

$$\phi_1(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \quad \text{and} \quad \phi_2(u, v) = 2u + v - 10 = 0.$$

We define the auxiliary function as

$$F(x, y, u, v, \lambda_1, \lambda_2) = (x - u)^2 + (y - v)^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + \lambda_2(2u + v - 10).$$

For extremum, we have the necessary conditions

$$\frac{\partial F}{\partial x} = 2(x - u) + \frac{x}{2}\lambda_1 = 0, \text{ or } \lambda_1 x = 4(u - x)$$

$$\frac{\partial F}{\partial y} = 2(y - v) + \frac{2y}{9}\lambda_1 = 0, \text{ or } \lambda_1 y = 9(v - y)$$

$$\frac{\partial F}{\partial u} = -2(x - u) + 2\lambda_2 = 0, \text{ or } \lambda_2 = x - u$$

$$\frac{\partial F}{\partial v} = -2(y - v) + \lambda_2 = 0, \text{ or } \lambda_2 = 2(y - v).$$

Eliminating λ_1 and λ_2 from the above equations, we get

$$4(u - x)y = 9(v - y)x \quad \text{and} \quad x - u = 2(y - v).$$

Dividing the two equations, we obtain $8y = 9x$. Substituting in the equation of the ellipse, we get

$$\frac{x^2}{4} + \frac{9x^2}{64} = 1, \text{ or } x^2 = \frac{64}{25}.$$

Therefore, $x = \pm 8/5$ and $y = \pm 9/5$. Corresponding to $x = 8/5$, $y = 9/5$, we get

$$\frac{8}{5} - u = 2\left(\frac{9}{5} - v\right), \quad \text{or} \quad 2v - u = 2, \quad \text{or} \quad u = 2v - 2.$$

Substituting in the equation of the line $2u + v - 10 = 0$, we get $u = 18/5$ and $v = 14/5$.

Hence, an extremum is obtained when $(x, y) = (8/5, 9/5)$ and $(u, v) = (18/5, 14/5)$. The dist between the two points is $\sqrt{5}$.

Corresponding to $x = -8/5$, $y = -9/5$, we get $u - 2v = 2$. Substituting in the equa $2u + v - 10 = 0$, we obtain $u = 22/5$, $v = 6/5$. Hence, another extremum is obtained w $(x, y) = (-8/5, -9/5)$ and $(u, v) = (22/5, 6/5)$. The distance between these two points is $3\sqrt{5}$.

Hence, the shortest distance between the line and the ellipse is $\sqrt{5}$.

Exercise 2.4

Test the following functions for relative maximum and minimum.

1. $xy + (9/x) + (3/y)$.
2. $\sqrt{a^2 - x^2 - y^2}$ $a > 0$.
3. $x^2 + 2bxy + y^2$.
4. $x^2 + xy + y^2 + (1/x) + (1/y)$.
5. $x^2 + 2/(x^2y) + y^2$.
6. $\cos 2x + \cos y + \cos (2x + y)$, $0 < x, y < \pi$.
7. $4x^2 + 4y^2 - z^2 + 12xy - 6y + z$.
8. $18xz - 6xy - 9x^2 - 2y^2 - 54z^2$.
9. $x^4 + y^4 + z^4 + 4xyz$.
10. $2 \ln(x + y + z) - (x^2 + y^2 + z^2)$, $x + y + z > 0$.

Find the relative and absolute maximum and minimum values for the following functions in the given clo region R in problemes **11** to **20**.

11. $x^2 - y^2 - 2v$, R: $x^2 + y^2 \le 1$.
12. xy , R: $x^2 + y^2 \le 1$.
13. $x + y$, R: $4x^2 + 9y^2 \le 36$.
14. $4x^2 + y^2 - 2x + 1$, R: $2x^2 + y^2 \le 1$.
15. $x^2 + y^2 - x - y + 1$, R: rectangular region; $0 \le x \le 2$, $0 \le y \le 2$.
16. $2x^2 + y^2 - 2x - 2y - 4$, R: triangular region bounded by the lines $x = 0$, $y = 0$ and $2x + y = 1$.
17. $x^3 + y^3 - xy$, R: triangular region bounded by the lines $x = 1$, $y = 0$ and $y = 2x$.
18. $4x^2 + 2y^2 + 4xy - 10x - 2y - 3$, R: rectangular region; $0 \le x \le 3$, $-4 \le y \le 2$.
19. $\cos x + \cos y + \cos (x + y)$, R: rectangular region; $0 \le x \le \pi$, $0 \le y \le \pi$.
20. $\cos x \cos y \cos (x + y)$, R: rectangular region; $0 \le x \le \pi$, $0 \le y \le \pi$.
21. Show that the necessary condition for the existence of an extreme value of $f(x, y)$ such that $\phi(x, y) =$ is that x, y satisfy the equation $f_x \phi_y - f_y \phi_x = 0$.
22. Find the smallest and the largest value of xy on the line segment $x + 2y = 2$, $x \ge 0$, $y \ge 0$.
23. Find the smallest and the largest value of $x + 2y$ on the circle $x^2 + y^2 = 1$.
24. Find the smallest and the largest value of $2x - y$ on the curve $x - \sin y = 0$, $0 \le y \le 2\pi$.
25. Find the extreme value of $x^2 + y^2$ when $x^4 + y^4 = 1$.
26. Find the points on the curve $x^2 + xy + y^2 = 16$, which are nearest and farthest from the origin.
27. Find the rectangle of constant perimeter whose diagonal is maximum.
28. Find the triangle whose perimeter is constant and has largest area.
29. Find a point on the plane $Ax + By + cz = D$ which is nearest to origin.
30. Find the extreme value of xyz, when $x + y + z = a$, $a > 0$.

31. Find the extreme value of $a^3x^2 + b^3y^2 + c^3z^2$ such that $x^{-1} + y^{-1} + z^{-1} = 1$, where $a > 0, b > 0, c > 0$.
32. Find the extreme value of $x^p + y^p + z^p$ on the surface $x^q + y^q + z^q = 1$, where $0 < p < q$, $x > 0, y > 0, z > 0$.
33. Find the extreme value of $x^3 + 8y^3 + 64z^3$, when $xyz = 1$.
34. Find the dimensions of a rectangular parallelopiped of maximum volume with edges parallel to the coordinate axes that can be inscribed in the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.
35. Divide a number into three parts such that the product of the first, square of the second and cube of the third is maximum.
36. Find the dimensions of a rectangular parallelopiped of fixed total edge length with maximum surface area.
37. Find the dimensions of a rectangular parallelopiped of greatest volume having constant surface area S.
38. A rectangular box without top is to have a given volume. How should the box be made so as to use the least material.
39. Find the dimensions of a right circular cone of fixed lateral area with minimum volume.
40. A tent is to be made in the form of a right circular cylinder surmounted by a cone. Find the ratios of the height H of the cylinder and the height h of the conical part to the radius r of the base, if the volume V of the tent is maximum for a given surface area S of the tent.
41. Find the maximum value of xyz under the constraints $x^2 + z^2 = 1$ and $y - x = 0$.
42. Find the extreme value of $x^2 + 2xy + z^2$ under the constraints $2x + y = 0$ and $x + y + z = 1$.
43. Find the extreme value of $x^2 + y^2 + z^2 + xy + xz + yz$ under the constraints $x + y + z = 1$ and $x + 2y + 3z = 3$.
44. Find the points on the ellipse obtained by the intersection of the plane $x + z = 1$ and the ellipsoid $x^2 + y^2 + 2z^2 = 1$ which are nearest and farthest from the origin.
45. Find the smallest and the largest distance between the points P and Q such that P lies on the plane $x + y + z = 2a$ and Q lies on the sphere $x^2 + y^2 + z^2 = a^2$, where a is any constant.

2.6 Multiple Integrals

In the previous chapter, we studied methods for evaluating the definite integral $\int_a^b f(x)dx$, where the integrand $f(x)$ is piecewise continuous on the interval $[a, b]$. In this section, we shall discuss methods for evaluating the double and triple integrals, that is integrals of the forms

$$\iint_R f(x, y)dx\,dy \quad \text{and} \quad \iiint_T f(x, y, z)dx\,dy\,dz.$$

We assume that the integrand f is continuous at all points inside and on the boundary of the region R or T. These integrals are called *multiple integrals*. The multiple integral over $\mathbb{R}^n$ is written as

$$\iint_R \cdots \int f(x_1, x_2, \ldots, x_n)dx_1 dx_2 \ldots dx_n.$$

2.6.1 Double Integrals

Let $f(x, y)$ be a continuous function in a simply connected, closed and bounded region R in a two dimensional space $\mathbb{R}^2$, bounded by a simple closed curve C (Fig. 2.5).

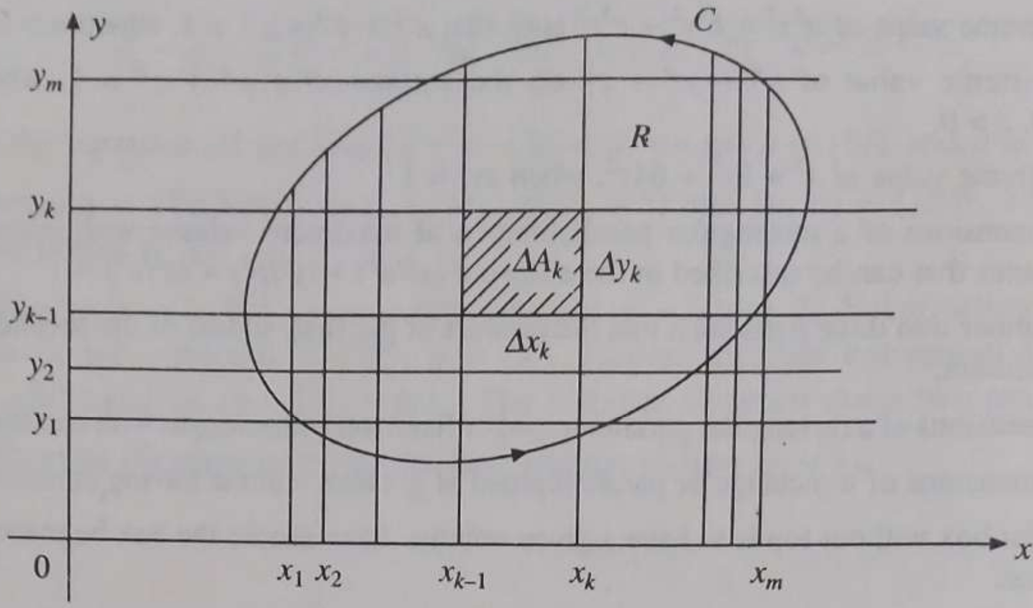


Fig. 2.5. Region *R* for double integral.

Subdivide the region R by drawing lines $x = x_k$, $y = y_k$, $k = 1, 2, \ldots, m$, parallel to the coordinate axes. Number the rectangles which are inside R from 1 to n. In each such rectangle, take an arbitrary point, say (ξ_k, η_k) in the kth rectangle and form the sum

$$J_n = \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta A_k$$

where $\Delta A_k = \Delta x_k \, \Delta y_k$ is the area of the kth rectangle and $d_k = \sqrt{(\Delta x_k)^2 + (\Delta y_x)^2}$ is the length of the diagonal of this rectangle. The maximum length of the diagonal, that is max d_k of the subdivisions is also called the *norm* of the subdivision. For different values of n, say $n_1, n_2, \ldots, n_m, \ldots$, we obtain a sequence of sums $J_{n_1}, J_{n_2}, \ldots, J_{n_m}, \ldots$. Let $n \to \infty$, such that the length of the largest diagonal $d_k \to 0$. If $\lim_{n\to\infty} J_n$ exists, independent of the choice of the subdivision and the point (ξ_k, η_k), then we say that $f(x, y)$ is integrable over R. This limit is called the *double integral* of $f(x, y)$ over R and is denoted by

$$J = \iint_R f(x, y)\,dx\,dy. \tag{2.78}$$

Evaluation of double integrals by two successive integrations

A double integral can be evaluated by two successive integrations. We evaluate it with respect to one variable (treating the other variable as constant) and reduce it to an integral of one variable. Thus, there are two possible ways to evaluate a double integral, which are the following:

$$J = \iint_R f(x, y)\,dy\,dx = \iint_R [f(x, y)dy]\,dx:$$ first integrate with respect to y and then integrate with respect to x.

or $$J = \iint_R f(x, y)\,dx\,dy = \iint_R [f(x, y)dx]\,dy:$$ first integrate with respect to x and then integrate with respect to y.

Let f be a continuous function over R. We consider the following cases.

Case 1 Let the region R be expressed in the form

$$R = \{(x, y) : \phi(x) \le y \le \psi(x),\ a \le x \le b\} \tag{2.79}$$

where $\phi(x)$ and $\psi(x)$ are integrable functions, such that $\phi(x) \le \psi(x)$ for all x in $[a, b]$. We write (Fig. 2.6)

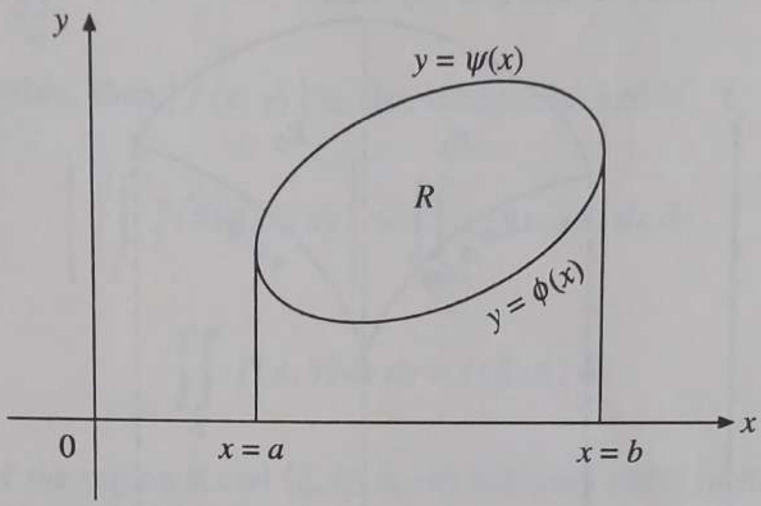


Fig. 2.6. Region of integration.

$$J = \int_{x=a}^{b} \left[\int_{y=\phi(x)}^{\psi(x)} f(x, y)dy \right] dx. \tag{2.80}$$

While evaluating the inner integral, x is treated as constant.

Case 2 Let the region R be expressed in the form

$$R = \{(x, y) : g(y) \le x \le h(y),\ c \le y \le d\} \tag{2.81}$$

where $g(y)$ and $h(y)$ are integrable functions, such that $g(y) \le h(y)$ for all y in $[c, d]$. We write (Fig. 2.7)

$$J = \int_{y=c}^{d} \left[\int_{x=g(y)}^{h(y)} f(x, y)dx \right] dy. \tag{2.82}$$

While evaluating the inner integral, y is treated as constant.

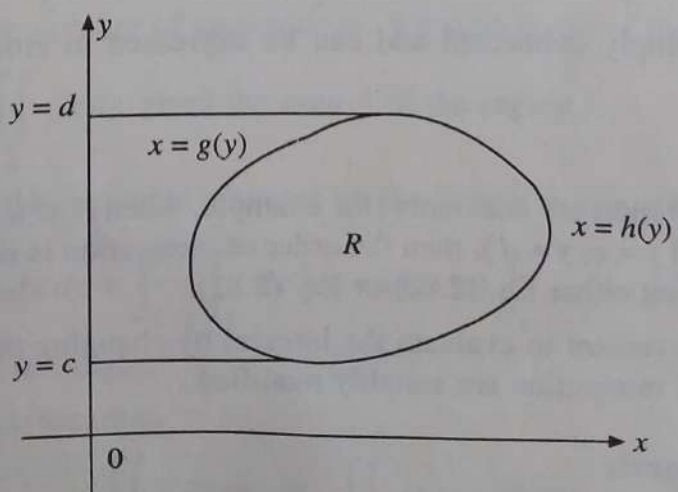


Fig. 2.7. Region of integration.

Often, the region R may be such that it cannot be represented in either of the forms given Eqs. (2.79) or (2.81). In such cases, the region R can be subdivided such that each of these can expressed in either of the forms given in Eqs. (2.79) or (2.81). For example, R may be expressed shown in Fig. 2.8 and we write $R = R_1 \cup R_2$ where R_1, R_2 have no common interior points.

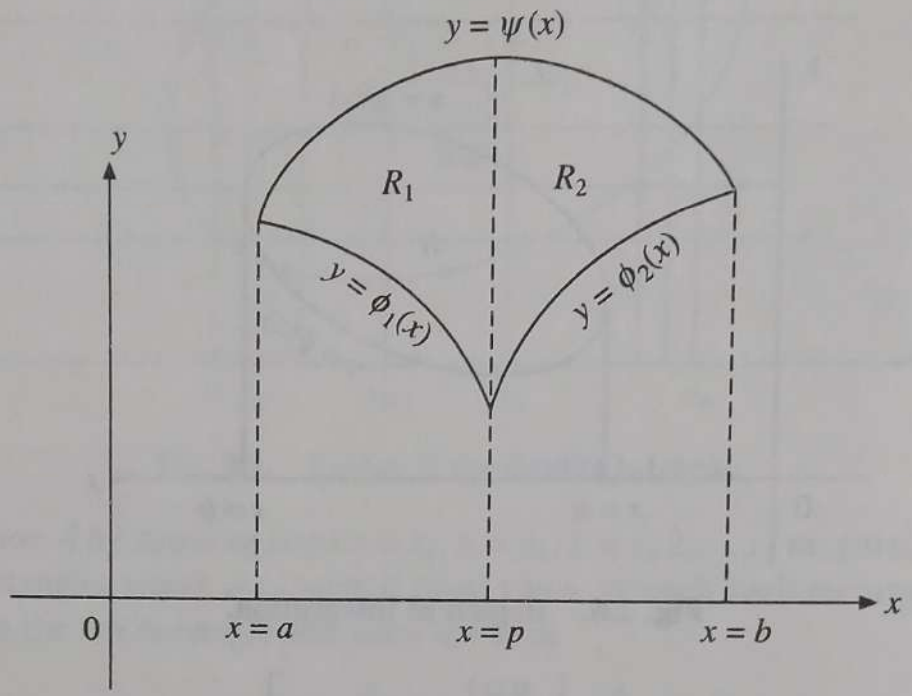


Fig. 2.8. Region of integration.

Then, we have

$$\iint_R f(x, y) dy\, dx = \iint_{R_1} f(x, y) dy\, dx + \iint_{R_2} f(x, y) dy\, dx$$

$$= \int_a^p \left[\int_{\phi_1(x)}^{\psi(x)} f(x, y)\, dy \right] dx + \int_p^b \left[\int_{\phi_2(x)}^{\psi(x)} f(x, y) dy \right] dx. \quad (2.83)$$

In the general case, the region R may be subdivided into a number of parts so that

$$\iint_R f(x, y) dy\, dx = \sum_{i=1}^{m} \left[\iint_{R_i} f(x, y)\, dy\, dx \right] \quad (2.84)$$

where each region R_i is simply connected and can be expressed in either of the forms given in Eqs. (2.79) or (2.81).

Remark 11

(a) If the limits of integration are constants (for example, when R is a rectangle bounded by the lines $x = a$, $x = b$ and $y = c$, $y = d$), then the order of integration is not important. The integral can be evaluated using either Eq. (2.80) or Eq. (2.82).

(b) Sometimes, it is convenient to evaluate the integral by changing the order of integration. In such cases, limits of integration are suitably modified.

Properties of double integrals

1. If $f(x, y)$ and $g(x, y)$ are integrable functions, then

$$\iint_R [f(x, y) \pm g(x, y)]dx\,dy = \iint_R f(x, y)dx\,dy \pm \iint_R g(x, y)dx\,dy.$$

2. $\iint_R kf(x, y)dx\,dy = k\iint_R f(x, y)dx\,dy$, where k is any real constant.

3. When $f(x, y)$ is integrable, then $|f(x, y)|$ is also integrable, and

$$\left|\iint_R f(x, y)dx\,dy\right| \le \iint_R |f(x, y)|\,dx\,dy. \tag{2.85}$$

4.
$$\iint_R f(x, y)dx\,dy = f(\xi, \eta)\,A \tag{2.86}$$

where A is the area of the region R and (ξ, η) is any arbitrary point in R. This result is called the *mean value theorem* of the double integrals.
If $m \le f(x, y) \le M$ for all (x, y) in R, then

$$mA \le \iint_R f(x, y)dx\,dy \le MA. \tag{2.87}$$

5. If $0 < f(x, y) \le g(x, y)$ for all (x, y) in R, then

$$\iint_R f(x, y)dx\,dy \le \iint_R g(x, y)dx\,dy. \tag{2.88}$$

6. If $f(x, y) \ge 0$ for all (x, y) in R, then

$$\iint_R f(x, y)dx\,dy \ge 0. \tag{2.89}$$

Application of double integrals

Double integrals have large number of applications. We state some of them.

1. If $f(x, y) = 1$, then $\iint_R dx\,dy$ gives the *area* A of the region R.

   For example, if R is the rectangle bounded by the lines $x = a$, $x = b$, $y = c$ and $y = d$, then

$$A = \int_c^d \int_a^b dx\,dy = \int_c^d \left[\int_a^b dx\right]dy = (b-a)\int_c^d dy = (b-a)(d-c)$$

   gives the area of the rectangle.

2. If $z = f(x, y)$ is a surface, then

$$\iint_R z\,dx\,dy \quad \text{or} \quad \iint_R f(x, y)dx\,dy$$

gives the *volume* of the region beneath the surface $z = f(x, y)$ and above the x-y plane.

For example, if $z = \sqrt{a^2 - x^2 - y^2}$ and $R : x^2 + y^2 \le a^2$, then

$$V = \iint_R \sqrt{a^2 - x^2 - y^2}\, dx\, dy$$

gives the volume of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$.

3. Let $f(x, y) = \rho(x, y)$ be a density function (mass per unit area) of a distribution of mass in the x-y plane. Then

$$M = \iint_R f(x, y)\, dx\, dy \tag{2.90}$$

give the total *mass* of R.

4. Let $f(x, y) = \rho(x, y)$ be a density function. Then

$$\bar{x} = \frac{1}{M} \iint_R x f(x, y)\, dx\, dy, \quad \bar{y} = \frac{1}{M} \iint_R y f(x, y)\, dx\, dy \tag{2.91}$$

give the coordinates of the *centre of gravity* $(\bar{x}, \bar{y})$ of the mass M in R.

5. Let $f(x, y) = \rho(x, y)$ be a density function. Then

$$I_x = \iint_R y^2 f(x, y)\, dx\, dy \quad \text{and} \quad I_y = \iint_R x^2 f(x, y)\, dx\, dy \tag{2.92}$$

give the *moments of inertia* of the mass in R about the x-axis and the y-axis respectively, whereas $I_0 = I_x + I_y$ is called the moment of inertia of the mass in R about the origin. Similarly,

$$I_y = \iint_R (x - a)^2 f(x, y)\, dx\, dy \quad \text{and} \quad I_x = \iint_R (y - b)^2 f(x, y)\, dx\, dy \tag{2.93}$$

give the moment of inertia of the mass in R about the lines $x = a$ and $y = b$ respectively.

6. $\frac{1}{A} \iint_R f(x, y)\, dx\, dy$ gives the *average value* of $f(x, y)$ over R, where A is the area of the region R.

Example 2.41 Evaluate the double integral $\iint_R xy\, dx\, dy$, where R is the region bounded by the x-axis, the line $y = 2x$ and the parabola $y = x^2/(4a)$.

Solution The points of intersection of the curves $y = 2x$ and $y = x^2/(4a)$ are $(0, 0)$ and $(8a, 16a)$. The region

$$R = \{(x, y) : (x^2/4a) \le y \le 2x,\ 0 \le x \le 8a\}$$

is given in Fig. 2.9.

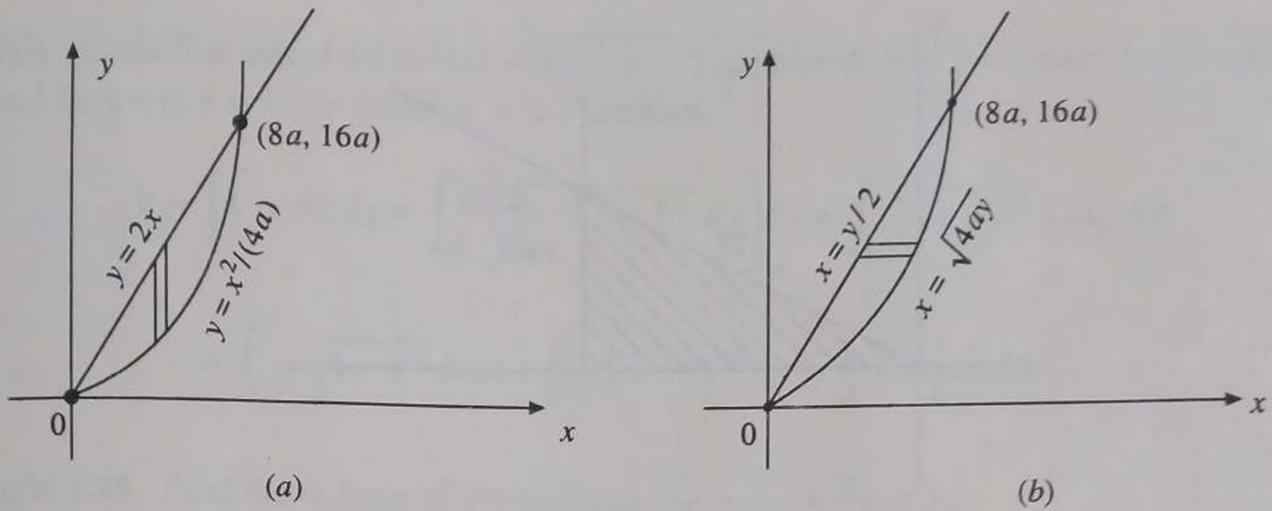


Fig. 2.9. Region in Example 2.41.

We evaluate the double integral as

$$I = \iint_R xy\,dx\,dy = \int_0^{8a}\left[\int_{x^2/(4a)}^{2x} x\,y\,dy\right]dx = \int_0^{8a}\left[\frac{xy^2}{2}\right]_{x^2/(4a)}^{2x} dx$$

$$= \int_0^{8a} \frac{x}{2}\left(4x^2 - \frac{x^4}{16a^2}\right)dx = \left[\frac{x^4}{2} - \frac{x^6}{192a^2}\right]_0^{8a} = 4096\left[\frac{1}{2} - \frac{64}{192}\right]a^4 = \frac{2048}{3}a^4.$$

Alternative We can evaluate the integral as

$$I = \iint_R xy\,dx\,dy = \int_0^{16a}\left[\int_{y/2}^{\sqrt{4ay}} xy\,dx\right]dy = \int_0^{16a}\left[\frac{1}{2}yx^2\right]_{y/2}^{\sqrt{4ay}} dy$$

$$= \frac{1}{2}\int_0^{16a} y\left(4ay - \frac{y^2}{4}\right)dy = \frac{1}{2}\left[\frac{4ay^3}{3} - \frac{y^4}{16}\right]_0^{16a} = \frac{4096\,a^3}{2}\left[\frac{4a}{3} - \frac{16a}{16}\right] = \frac{2048}{3}a^4.$$

Example 2.42 Evaluate the double integral $\iint_R e^{x^2}\,dx\,dy$, where the region R is given by $R : 2y \le x \le 2$ and $0 \le y \le 1$.

Solution The integral cannot be evaluated by integrating first with respect to x. We try to evaluate it by integrating it first with respect to y. The region of integration is given in Fig. 2.10. We have

$$I = \int_0^2\left[\int_0^{x/2} e^{x^2}\,dy\right]dx = \int_0^2\left[y\,e^{x^2}\right]_0^{x/2} dx$$

$$= \frac{1}{2}\int_0^2 x\,e^{x^2}\,dx = \left[\frac{1}{4}e^{x^2}\right]_0^2 = \frac{1}{4}(e^4 - 1).$$

Example 2.43 Evaluate the integral $\int_0^2\int_0^{y^2/2} \frac{y}{\sqrt{x^2+y^2+1}}\,dx\,dy.$

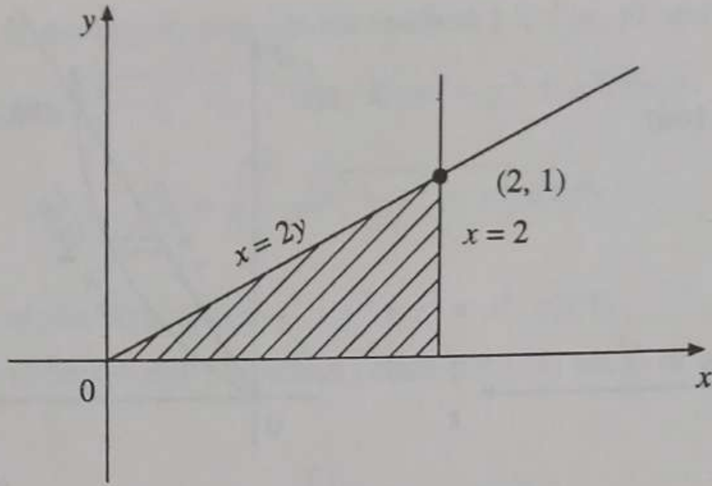


Fig. 2.10. Region in Example 2.42.

Solution Because of the form of the integrand, it would be easier to integrate it first with respect to y. The point of intersection of the line $y = 2$ and the curve $y^2 = 2x$ is (2, 2). The region of integration is given in Fig. 2.11.

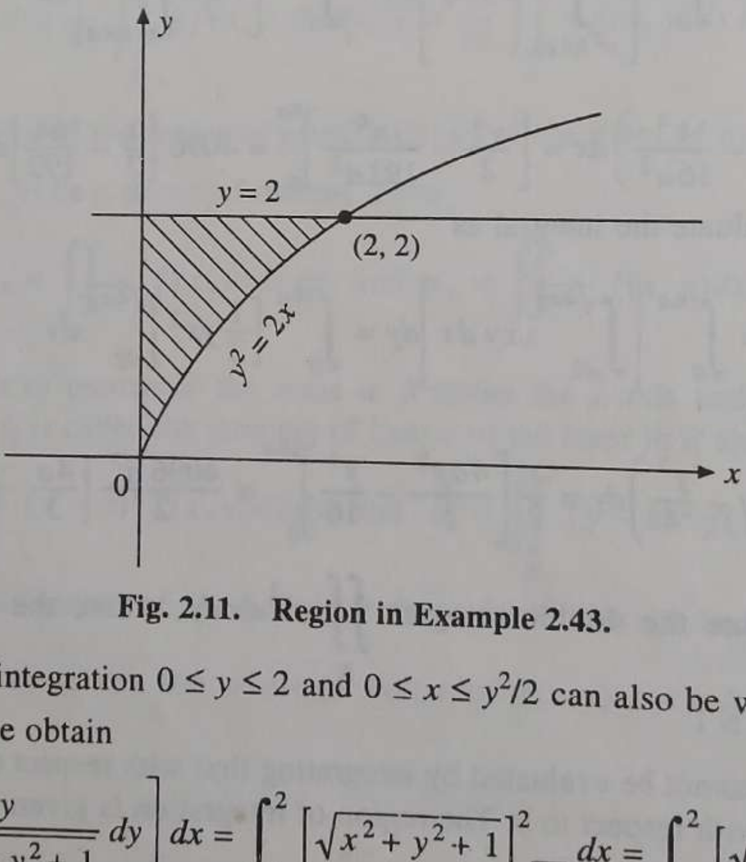


Fig. 2.11. Region in Example 2.43.

The given region of integration $0 \le y \le 2$ and $0 \le x \le y^2/2$ can also be written as $0 \le x \le 2$ and $\sqrt{2x} \le y \le 2$. Hence, we obtain

$$I = \int_0^2 \left[\int_{\sqrt{2x}}^2 \frac{y}{\sqrt{x^2 + y^2 + 1}}\, dy\right] dx = \int_0^2 \left[\sqrt{x^2 + y^2 + 1}\right]_{\sqrt{2x}}^2 dx = \int_0^2 \left[\sqrt{x^2 + 5} - (x + 1)\right] dx$$

?, ?

$$= \left[\frac{x\sqrt{x^2 + 5}}{2} + \frac{5}{2}\ln (x + \sqrt{x^2 + 5}) - \frac{1}{2}(x + 1)^2\right]_0^2$$

$$= 3 + \frac{5}{2}(\ln 5 - \ln \sqrt{5}) - \frac{1}{2}(9 - 1) = \frac{5}{4}\ln 5 - 1.$$

Example 2.44 The cylinder $x^2 + z^2 = 1$ is cut by the planes $y = 0$, $z = 0$ and $x = y$. Find the volume of the region in the first octant.

Solution In the first octant we have $z = \sqrt{1-x^2}$. The projection of the surface in the x-y plane is bounded by $x = 0$, $x = 1$, $y = 0$ and $y = x$. Therefore,

$$V = \iint_R z\,dx\,dy = \int_0^1 \left[\int_0^x \sqrt{1-x^2}\,dy\right]dx = \int_0^1 \sqrt{1-x^2}\,[y]_0^x\,dx$$

$$= \int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{3}\left[(1-x^2)^{3/2}\right]_0^1 = \frac{1}{3} \text{ cubic units.}$$

Example 2.45 Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Solution We have volume = 8 (volume in the first octant). The projection of the surface $z = c\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}$ in the x-y plane is the region in the first quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Therefore,

$$V = 8\int_0^a \left[\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dy\right]dx = 8c\int_0^a \left[\int_0^{bk} \sqrt{k^2 - \frac{y^2}{b^2}}\,dy\right]dx$$

where $k^2 = 1 - (x^2/a^2)$. Setting $y = b\,k\sin\theta$, we obtain

$$V = 8c\int_0^a \left[\int_0^{\pi/2} \sqrt{k^2 - k^2\sin^2\theta}\,(bk\cos\theta)\,d\theta\right]dx = 8bc\int_0^a \left[\int_0^{\pi/2} k^2\cos^2\theta\,d\theta\right]dx$$

$$= 4bc\left(\frac{\pi}{2}\right)\int_0^a \left(1 - \frac{x^2}{a^2}\right)dx = \frac{2\pi bc}{a^2}\int_0^a (a^2 - x^2)\,dx$$

$$= \frac{2\pi bc}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{4\pi abc}{3} \text{ cubic units.}$$

Example 2.46 Find the centre of gravity of a plate whose density $\rho(x, y)$ is constant and is bounded by the curves $y = x^2$ and $y = x + 2$. Also, find the moments of inertia about the axes.

Solution The mass of the plate is given by (see Eq. 2.90)

$$M = \iint_R \rho(x, y)\,dx\,dy = k\iint_R dx\,dy \qquad (\rho(x, y) = k \text{ constant}).$$

The boundary of the plate is given in Fig. 2.12. The line $y = x + 2$ intersects the parabola $y = x^2$ at the points $(-1, 1)$ and $(2, 4)$. The limits of integration can be written as $-1 \le x \le 2$, $x^2 \le y \le x + 2$. Therefore,

$$M = k\int_{-1}^2 \left[\int_{x^2}^{x+2} dy\right]dx = k\int_{-1}^2 (x + 2 - x^2)\,dx$$

$$= k\left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2} = k\left(-\frac{9}{3} + \frac{3}{2} + 6\right) = \frac{9}{2}k.$$

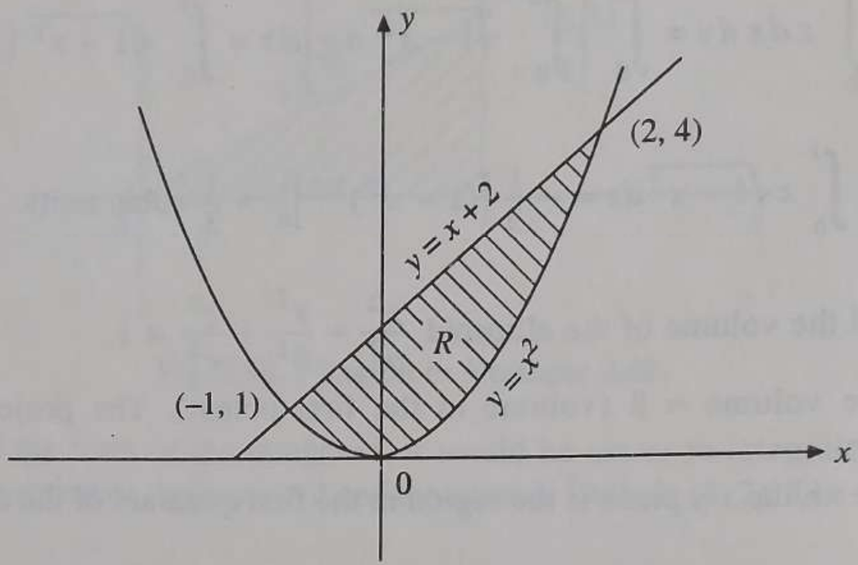


Fig. 2.12. Region in Example 2.46.

The centre of gravity $(\bar{x}, \bar{y})$ is given by (see Eq. 2.91)

$$\bar{x} = \frac{1}{M}\iint_R x\,\rho(x, y)\,dx\,dy = \frac{2}{9}\int_{-1}^{2}\left[\int_{x^2}^{x+2} dy\right] x\,dx$$

$$= \frac{2}{9}\int_{-1}^{2} x(x + 2 - x^2)\,dx = \frac{2}{9}\left[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\right]_{-1}^{2} = \frac{1}{2}.$$

$$\bar{y} = \frac{1}{M}\iint_R y\,\rho(x, y)\,dx\,dy = \frac{2}{9}\int_{-1}^{2}\left[\int_{x^2}^{x+2} y\,dy\right]dx = \frac{2}{9}\int_{-1}^{2}\left[\frac{y^2}{2}\right]_{x^2}^{x+2} dx$$

$$= \frac{1}{9}\int_{-1}^{2} [(x + 2)^2 - x^4]\,dx = \frac{1}{9}\left[\frac{(x + 2)^3}{3} - \frac{x^5}{5}\right]_{-1}^{2}$$

$$= \frac{1}{9}\left[\frac{1}{3}(64 - 1) - \frac{1}{5}(32 + 1)\right] = \frac{1}{9}\left[21 - \frac{33}{5}\right] = \frac{8}{5}.$$

Therefore, the centre of gravity is located at (1/2, 8/5).

Moment of inertia about the x-axis is given by (see Eq. 2.92)

$$I_x = \iint_R y^2\rho(x, y)\,dx\,dy = k\int_{-1}^{2}\left[\int_{x^2}^{x+2} y^2\,dy\right]dx = k\int_{-1}^{2}\left[\frac{y^3}{3}\right]_{x^2}^{x+2} dx$$

$$= \frac{k}{3}\int_{-1}^{2} [(x + 2)^3 - x^6]\,dx = \frac{k}{3}\left[\frac{(x + 2)^4}{4} - \frac{x^7}{7}\right]_{-1}^{2}$$

$$= \frac{k}{3}\left(\frac{255}{4} - \frac{129}{7}\right) = \frac{423}{28}k.$$

Moment of inertia about the y-axis is given by (see Eq. 2.92)

$$I_y = \iint_R x^2 \rho(x, y) dx\, dy = k \int_{-1}^{2} \left[\int_{x^2}^{x+2} dy \right] x^2 dx = k \int_{-1}^{2} x^2 (x + 2 - x^2) dx$$

$$= k \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^5}{5} \right]_{-1}^{2} = k \left[\frac{15}{4} + 6 - \frac{33}{5} \right] = \frac{63}{20} k.$$

2.6.2 Triple Integrals

Let $f(x, y, z)$ be a continuous function defined over a closed and bounded region T in $\mathbb{R}^3$. Divide the region T into a number of parallelopipeds by drawing planes parallel to the coordinate planes. Number the parallelopipeds inside T from 1 to n and form the sum

$$J_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta V_k$$

where (x_k, y_k, z_k) is an arbitrary point in the kth parallelopiped and ΔV_k is its volume. For different values of n, say $n_1, n_2, \ldots, n_m, \ldots$, we obtain a sequence of sums $J_{n_1}, J_{n_2}, \ldots, J_{n_m}, \ldots$ The length of the diagonal of the kth parallelopiped is $d_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2 + (\Delta z_k)^2}$. Let $n \to \infty$ such that $\max d_k \to 0$. If $\lim_{n\to\infty} J_n$ exists, independent of the choice of the subdivision and the point (x_k, y_k, z_k), then we say that $f(x, y, z)$ is integrable over T. This limit is called the *triple integral* of $f(x, y, z)$ over T and is denoted by

$$J = \iiint_T f(x, y, z)\, dx\, dy\, dz. \tag{2.94}$$

Triple integrals satisfy properties similar to double integrals.

Application of triple integrals

1. If $f(x, y, z) = 1$, then the triple integral

$$V = \iiint_T dx\, dy\, dz \tag{2.95}$$

gives the volume of the region T.

2. If $f(x, y, z) = \rho(x, y, z)$ is the density of a mass, then the triple integral

$$M = \iiint_T f(x, y, z) dx\, dy\, dz \tag{2.96}$$

gives the *mass* of the solid.

3.
$$\bar{x} = \frac{1}{M} \iiint_T x f(x, y, z)\, dx\, dy\, dz, \quad \bar{y} = \frac{1}{M} \iiint_T y f(x, y, z)\, dx\, dy\, dz,$$

$$\bar{z} = \frac{1}{M} \iiint_T z f(x, y, z)\, dx\, dy\, dz \tag{2.97}$$

give the coordinates of the *centre of mass* (or the *centre of gravity*) of the solid of mass M in T, where $f(x, y, z) = \rho(x, y, z)$ is the density function.

4. $$I_x = \iiint_T (y^2 + z^2) f(x, y, z)\, dx\, dy\, dz, \quad I_y = \iiint_T (x^2 + z^2) f(x, y, z)\, dx\, dy\, dz,$$

$$I_z = \iiint_T (x^2 + y^2) f(x, y, z)\, dx\, dy\, dz \tag{2.98}$$

give the *moments of inertia* of the mass in T about the x-axis, y-axis and z-axis respectively where $f(x, y, z) = \rho(x, y, z)$ is the density function.

Evaluation of triple integrals

We evaluate the triple integral by three successive integrations. If the region T can be described by

$$x_1 \le x \le x_2,\ \ y_1(x) \le y \le y_2(x),\ \ z_1(x, y) \le z \le z_2(x, y)$$

then we evaluate the triple integral as

$$\int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dy\, dx = \int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz \right] dy \right] dx \tag{2.99}$$

We note that there are six possible ways in which a triple integral can be evaluated (order of variables of integration). We choose the one which is simple to use.

Example 2.47 Evaluate the triple integral $\iiint_T y\, dx\, dy\, dz$, where T is the region bounded by the surfaces $x = y^2$, $x = y + 2$, $4z = x^2 + y^2$ and $z = y + 3$.

Solution The variable z varies from $(x^2 + y^2)/4$ to $y + 3$. The projection of T on the x-y plane is the region bounded by the curves $x = y^2$ and $x = y + 2$. These curves intersect at the points $(1, -1)$ and $(4, 2)$. Also, $y^2 \le y + 2$ for $-1 \le y \le 2$. Hence, the required region can be written as

$$-1 \le y \le 2,\ \ y^2 \le x < y + 2 \ \text{ and } \ [(x^2 + y^2)/4] \le z \le y + 3.$$

Therefore, we can evaluate the triple integral as

$$J = \int_{-1}^{2} \left[\int_{y^2}^{y+2} \left[\int_{(x^2+y^2)/4}^{(y+3)} y\, dz \right] dx \right] dy = \int_{-1}^{2} \left[\int_{y^2}^{y+2} y \left\{ y + 3 - \frac{x^2 + y^2}{4} \right\} dx \right] dy$$

$$= \int_{-1}^{2} \left[\left(y^2 + 3y - \frac{y^3}{4} \right) x - \frac{x^3 y}{12} \right]_{y^2}^{y+2} dy$$

$$= \int_{-1}^{2} \left[\left(y^2 + 3y - \frac{y^3}{4} \right) (y + 2 - y^2) - \frac{1}{12} y \{(y + 2)^3 - y^6\} \right] dy$$

$$= \int_{-1}^{2} \left[\frac{y^7}{12} + \frac{y^5}{4} - \frac{4y^4}{3} - 3y^3 + 4y^2 + \frac{16y}{3} \right] dy$$

$$= \left[\frac{y^8}{96} + \frac{y^6}{24} - \frac{4y^5}{15} - \frac{3y^4}{4} + \frac{4y^3}{3} + \frac{8y^2}{3}\right]_{-1}^{2} = \frac{837}{160}.$$

Example 2.48 Evaluate the integral $\iiint_T z\,dx\,dy\,dz$, where T is the region bounded by the cone $z^2 = x^2\tan^2\alpha + y^2\tan^2\beta$ and the planes $z = 0$ to $z = h$ in the first octant.

Solution The required region can be written as

$$0 \le z \le \sqrt{x^2\tan^2\alpha + y^2\tan^2\beta},\ 0 \le y \le (\sqrt{h^2 - x^2\tan^2\alpha})\cot\beta,\ 0 \le x \le h\cot\alpha$$

Therefore,

$$J = \int_0^{h\cot\alpha}\left[\int_0^{(\sqrt{h^2-x^2\tan^2\alpha})\cot\beta} \frac{1}{2}(x^2\tan^2\alpha + y^2\tan^2\beta)dy\right]dx$$

$$= \frac{1}{2}\int_0^{h\cot\alpha}\left[x^2(h^2 - x^2\tan^2\alpha)^{1/2}\tan^2\alpha + \frac{1}{3}(h^2 - x^2\tan^2\alpha)^{3/2}\right]\cot\beta\,dx.$$

Substituting $x\tan\alpha = h\sin\theta$, we obtain

$$J = \frac{\cot\beta}{2}\int_0^{\pi/2}\left[h^2\sin^2\theta(h\cos\theta) + \frac{1}{3}(h^3\cos^3\theta)\right]h\cot\alpha\cos\theta\,d\theta$$

$$= \frac{1}{2}h^4\cot\beta\cot\alpha\left[\int_0^{\pi/2}(\sin^2\theta\cos^2\theta + \frac{1}{3}\cos^4\theta)d\theta\right]$$

$$= \frac{1}{2}h^4\cot\beta\cot\alpha\left[\int_0^{\pi/2}(\sin^2\theta - \sin^4\theta + \frac{1}{3}\cos^4\theta)d\theta\right]$$

$$= \frac{1}{2}h^4\cot\beta\cot\alpha\left[\frac{\pi}{4} - \frac{3\pi}{16} + \frac{\pi}{16}\right] = \frac{h^4\pi}{16}\cot\alpha\cot\beta.$$

Example 2.49 Find the volume of the solid in the first octant bounded by the paraboloid $z = 36 - 4x^2 - 9y^2$.

Solution We have

$$V = \iiint_T dz\,dy\,dx.$$

The projection of the paraboloid (in the first octant) in the x-y plane is the region in the first quadrant of the ellipse $4x^2 + 9y^2 = 36$.

Therefore, the region T is given by

$$0 \le z \le 36 - 4x^2 - 9y^2,\ 0 \le y \le \frac{1}{3}\sqrt{36 - 4x^2},\ 0 \le x \le 3.$$

Hence,

$$V = \int_0^3 \left[\int_0^{(2\sqrt{9-x^2}/3)} (36 - 4x^2 - 9y^2)\,dy \right] dx$$

$$= \int_0^3 [4(9 - x^2)y - 3y^3]_0^{(2\sqrt{9-x^2}/3)}\, dx$$

$$= \int_0^3 \left[\frac{8}{3}(9 - x^2)^{3/2} - \frac{8}{9}(9 - x^2)^{3/2} \right] dx = \frac{16}{9} \int_0^3 (9 - x^2)^{3/2}\, dx.$$

Substituting $x = 3 \sin \theta$, we obtain

$$V = \frac{16}{9} \int_0^{\pi/2} (27 \cos^3\theta)(3 \cos \theta)\, d\theta = 144 \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= 144 \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right) = 27\pi \text{ cubic units.}$$

Example 2.50 Find the volume of the solid enclosed between the surfaces $x^2 + y^2 = a^2$ an $x^2 + z^2 = a^2$.

Solution We have the region as

$$-\sqrt{a^2 - x^2} \le z \le \sqrt{a^2 - x^2}, \quad -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}, \quad -a \le x \le a.$$

Therefore,

$$V = \int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} dz\, dy\, dx = 8 \int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2}\, dy\, dx$$

$$= 8 \int_0^a (a^2 - x^2)\, dx = 8 \left(a^2 x - \frac{x^3}{3} \right)_0^a = \frac{16a^3}{3} \text{ cubic units.}$$

2.6.3 Change of Variables in Integrals

In the case of definite integrals $\int_a^b f(x)\,dx$ of one variable, we have seen that the evaluation of the integral is often simplified by using some substitution and thus changing the variable of integration. Similarly, the double and triple integrals can be evaluated by using some substitutions and changing the variables of integration.

Double integrals

Let the variables x, y defined in a region R of the x-y plane be transformed as

$$x = x(u, v),\ y = y(u, v). \tag{2.100}$$

We assume that the functions $x(u, v)$, $y(u, v)$ are defined and have continuous partial derivatives in

the region R^* of interest in the u-v plane. We also assume that the inverse functions $u = u(x, y)$, $v = v(x, y)$ are defined and are continuous in the region of interest in the x-y plane, so that the mapping is one-to-one. Since the function $f(x, y)$ is continuous in R, the function $f[x(u, v), y(u, v)]$ is also continuous in R^*. Then, the double integral transforms as

$$\iint_R f(x, y)dx\,dy = \iint_{R^*} f[x(u, v), y(u, v)]\,|J|\,du\,dv = \iint_{R^*} F(u, v)du\,dv \tag{2.101}$$

where
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

is the *Jacobian* of the variables of transformation.

For example, if we change the cartesian coordinates to *polar* coordinates, we have

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ r_1 \le r \le r_2,\ \ \theta_1 \le \theta \le \theta_2$$

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \tag{2.102}$$

Therefore,

$$\iint_R f(x, y)dx\,dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \iint_{R^*} F(r, \theta)\,r\,dr\,d\theta$$

where R^* is the region corresponding to R is the r-θ plane.

Triple integrals

Analogous to double integrals, we define x, y, z as functions of three new variables

$$x = x(u, v, w),\ y = y(u, v, w),\ z = z(u, v, w). \tag{2.103}$$

Then,

$$\iiint_T f(x, y, z)dx\,dy\,dz = \iiint_{T^*} f[x(u, v, w), y(u, v, w), z(u, v, w)]\,|J|\,du\,dv\,dw \tag{2.104}$$

where
$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

is the Jacobian of the variables of transformation.

For example, if we change the cartesian coordinates to *cylindrical* coordinates, we have

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \tag{2.105}$$

and
$$\iiint_T f(x, y, z)dx\,dy\,dz = \iiint_{T^*} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

If we change the cartesian coordinates to *spherical* coordinates, we have (Fig. 2.13)

$$x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$$

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \phi & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \phi & \partial y/\partial \theta \\ \partial z/\partial r & \partial z/\partial \phi & \partial z/\partial \theta \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi[r^2\sin\phi\cos\phi\cos^2\theta + r^2\sin\phi\cos\phi\sin^2\theta] + r\sin\phi[r\sin^2\phi\cos^2\theta + r\sin^2\phi\sin^2\theta]$$

$$= r^2[\sin\phi\cos^2\phi + \sin^3\phi] = r^2\sin\phi \qquad (2.1$$

and $$\iiint_T f(x, y, z)dx\,dy\,dz = \iiint_{T^*} F(r, \theta, \phi)\, r^2\sin\phi\, dr\, d\theta\, d\phi.$$

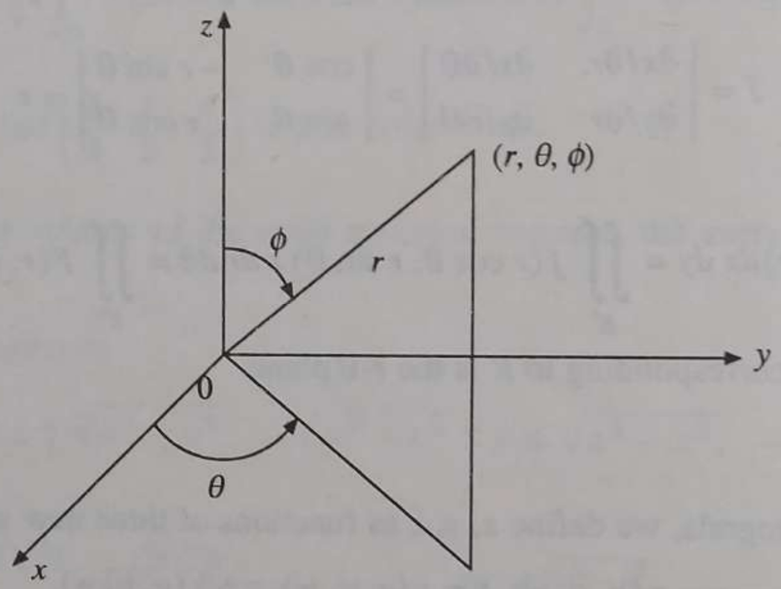


Fig. 2.13. Spherical coordinates.

Example 2.51 Evaluate the integral $\iint_R (a^2 - x^2 - y^2)\, dx\, dy$, where R is the region $x^2 + y^2 \le a$

Solution We can evaluate the integral directly by writing it as

$$I = \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2)dy\right]dx.$$

However, it is easier to evaluate, if we change to polar coordinates. Transforming cartesian coordinate to polar coordinates, we have (see Eq. 2.102)

$$x = r\cos\theta,\ y = r\sin\theta,\ J = r.$$

Therefore,

$$I = \int_0^a \int_0^{2\pi} (a^2 - r^2)\, r\, dr\, d\theta = \int_0^a \left[\int_0^{2\pi} d\theta\right](a^2 r - r^3)dr$$

$$= 2\pi \int_0^a (a^2 r - r^3) dr = 2\pi \left(\frac{a^2 r^2}{2} - \frac{r^4}{4} \right)_0^a = \frac{\pi a^4}{2}.$$

*

Example 2.52 Evaluate the integral $\iint_R (x - y)^2 \cos^2 (x + y) dx\, dy$, where R is the rhombus with successive vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

Solution The region R is given in Fig. 2.14. The equations of the sides AB, BC, CD and DA are respectively

$$x - y = \pi, \quad x + y = 3\pi, \quad x - y = -\pi \quad \text{and} \quad x + y = \pi.$$

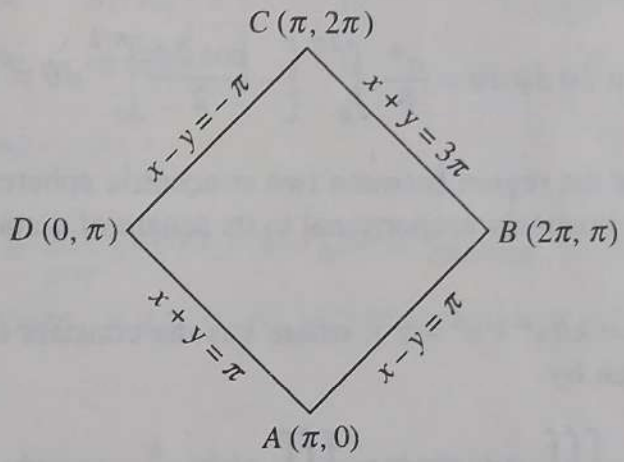


Fig. 2.14. Region in Example 2.52.

Substitute $y - x = u$ and $y + x = v$. Then, $-\pi \le u \le \pi$ and $\pi \le v \le 3\pi$. We obtain

$$x = (v - u)/2, \quad y = (v + u)/2$$

and
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\frac{1}{2}, \ |J| = \frac{1}{2}.$$

Therefore,

$$I = \iint_R (x - y)^2 \cos^2 (x + y) dx\, dy = \frac{1}{2} \int_{\pi}^{3\pi} \int_{-\pi}^{\pi} u^2 \cos^2 v\, du\, dv$$

$$= \frac{\pi^3}{3} \int_{\pi}^{3\pi} \cos^2 v\, dv = \frac{\pi^3}{6} \int_{\pi}^{3\pi} (1 + \cos 2v)\, dv = \frac{\pi^4}{3}.$$

Example 2.53 Evaluate the integral $\iint_R \sqrt{x^2 + y^2}\, dx\, dy$ by changing to polar coordinates, where R is the region in the x-y plane bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Solution Using $x = r \cos \theta$, $y = r \sin \theta$, we get $dx\, dy = r\, dr\, d\theta$, and

$$I = \int_0^{2\pi} \int_2^3 r(r\, dr\, d\theta) = \int_0^{2\pi} \left[\frac{r^3}{3} \right]_2^3 d\theta = \frac{19}{3} \int_0^{2\pi} d\theta = \frac{38\pi}{3}.$$

Example 2.54 Evaluate the integral $\iiint_T z\,dx\,dy\,dz$, where T is the hemisphere of radius $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

Solution Changing to spherical coordinates

$$x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi/2,$$

we obtain $dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$ (see Eq. 2.106). Therefore,

$$I = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (r\cos\phi)\,r^2\sin\phi\,dr\,d\phi\,d\theta = \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\,d\theta$$

$$= \frac{a^4}{8}\int_0^{2\pi}\int_0^{\pi/2}\sin 2\phi\,d\phi\,d\theta = \frac{a^4}{8}\int_0^{2\pi}\left[-\frac{\cos 2\phi}{2}\right]_0^{\pi/2} d\theta = \frac{a^4}{8}\int_0^{2\pi} d\theta = \frac{\pi a^4}{4}.$$

Example 2.55 A solid fills the region between two concentric spheres of radii a and b, $0 < a <$ The density at each point is inversely proportional to its square of distance from the origin. Find t total mass.

Solution The density is $\rho = k/(x^2 + y^2 + z^2)$, where k is the constant of proportionality. Therefor the mass of the solid is given by

$$M = \iiint_T \rho\,dx\,dy\,dz = \iiint_T \frac{k}{x^2 + y^2 + z^2}\,dx\,dy\,dz$$

where $a^2 < x^2 + y^2 + z^2 < b^2$. Changing to spherical coordinates, we obtain

$$x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi,\ x^2 + y^2 + z^2 = r^2,\ a \leq r \leq b,$$

$$dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi.$$

Therefore,

$$M = k\int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{r^2\sin\phi}{r^2}\,dr\,d\phi\,d\theta = k(b-a)\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta$$

$$= k(b-a)\int_0^{2\pi}[-\cos\phi]_0^{\pi}\,d\theta = 2k(b-a)\int_0^{2\pi} d\theta = 4\pi k(b-a).$$

2.6.4 Dirichlet Integrals

Let T be a closed region in the first octant in $\mathbb{R}^3$, bounded by the surface $(x/a)^p + (y/b)^q + (z/c)^r = 1$ and the coordinate planes. Then, an integral of the form

$$I = \iiint_T x^{\alpha-1}\,y^{\beta-1}\,z^{\gamma-1}\,dx dy dz \tag{2.107}$$

is called a Dirichlet integral, where all the constants α, β, γ, a, b, c and p, q, r are assumed to be positive.

We now show that

$$I = \iiint_T x^{\alpha-1} y^{\beta-1} z^{\gamma-1} dx\,dy\,dz = \frac{a^\alpha b^\beta c^\gamma}{pqr} \frac{\Gamma(\alpha/p)\Gamma(\beta/q)\Gamma(\gamma/r)}{\Gamma\left(\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} + 1\right)}. \tag{2.108}$$

Let $\left(\frac{x}{a}\right)^p = u, \left(\frac{y}{b}\right)^q = v, \left(\frac{z}{c}\right)^r = w$, or $x = au^{1/p}$, $y = bv^{1/q}$, $z = cw^{1/r}$.

The Jacobian of the transformation is given by

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} (a/p)u^{(1/p)-1} & 0 & 0 \\ 0 & (b/q)v^{(1/q)-1} & 0 \\ 0 & 0 & (c/r)w^{(1/r)-1} \end{vmatrix}$$

$$= \frac{abc}{pqr} u^{(1/p)-1} v^{(1/q)-1} w^{(1/r)-1}$$

and $dxdydz = |J|\, dudvdw = \frac{abc}{pqr} u^{(1/p)-1} v^{(1/q)-1} w^{(1/r)-1}\, dudvdw.$

Now, $x \ge 0, y \ge 0, z \ge 0$ gives $u \ge 0, v \ge 0, w \ge 0$ respectively.

Hence, we obtain

$$I = \iiint_R \left[au^{(1/p)}\right]^{\alpha-1} \left[bv^{(1/q)}\right]^{\beta-1} \left[cw^{(1/r)-1}\right] \frac{abc}{pqr} u^{(1/p)-1} v^{(1/q)-1} w^{(1/r)-1}\, dudvdw$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pqr} \iiint_R u^{(\alpha/p)-1} v^{(\beta/q)-1} w^{(\gamma/r)-1} dudvdw$$

where R is the region in the uvw-space bounded by the plane $u + v + w = 1$ and the uv, vw and uw coordinate planes, (Fig. 2.15), that is, R is defined by

$$0 \le w \le 1 - u - v, \quad 0 \le v \le 1 - u, \quad 0 \le u \le 1.$$

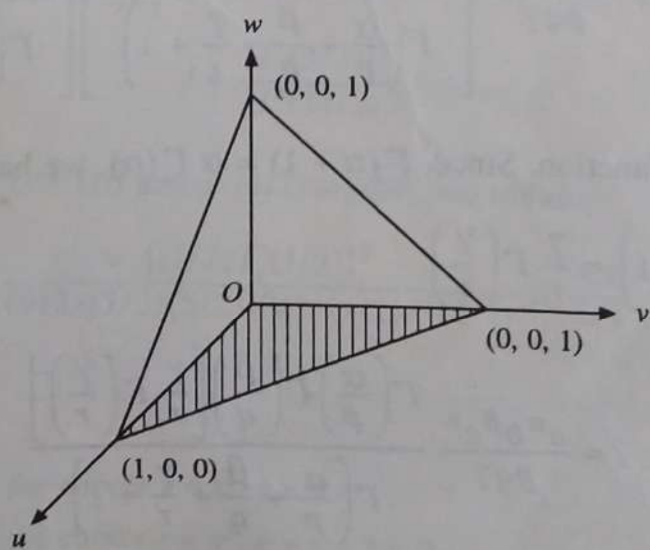


Fig. 2.15. Dirichlet integral.

Therefore, we get

$$I = \frac{a^\alpha b^\beta c^\gamma}{pqr} \int_{u=0}^{1} \int_{v=0}^{1-u} \int_{w=0}^{1-u-v} u^{(\alpha/p)-1} v^{(\beta/q)-1} w^{(\gamma/r)-1} du dv dw$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pqr} \int_{u=0}^{1} \int_{v=0}^{1-u} u^{(\alpha/p)-1} v^{(\beta/q)-1} \left[\frac{w^{(\gamma/r)}}{(\gamma/r)} \right]_0^{1-u-v} du dv$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \int_{u=0}^{1} \int_{v=0}^{1-u} u^{(\alpha/p)-1} v^{(\beta/q)-1} (1-u-v)^{(\gamma/r)} du dv$$

Substituting $v = (1-u)\,t$, $dv = (1-u)dt$, we obtain

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \int_{u=0}^{1} \int_{t=0}^{1} u^{(\alpha/p)-1} (1-u)^{[(\beta/q)+(\gamma/r)]}\, t^{(\beta/q)-1} (1-t)^{(\gamma/r)} du dt.$$

Since the limits are constants, we can write

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \left[\int_0^1 u^{(\alpha/p)-1} (1-u)^{[(\beta/q)+(\gamma/r)]} du \right] \left[\int_0^1 t^{(\beta/q)-1} (1-t)^{(\gamma/r)} dt \right]$$

Using the definition of Beta function

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx = \beta(m, n)$$

we obtain

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \beta\left(\frac{\alpha}{p}, \frac{\beta}{q} + \frac{\gamma}{r} + 1 \right) \beta\left(\frac{\beta}{q}, \frac{\gamma}{r} + 1 \right)$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \left[\frac{\Gamma\left(\frac{\alpha}{p}\right) \Gamma\left(\frac{\beta}{q} + \frac{\gamma}{r} + 1 \right)}{\Gamma\left(\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} + 1 \right)} \right] \left[\frac{\Gamma\left(\frac{\beta}{q}\right) \Gamma\left(\frac{\gamma}{r} + 1 \right)}{\Gamma\left(\frac{\beta}{q} + \frac{\gamma}{r} + 1 \right)} \right]$$

where $\Gamma(x)$ is the Gamma function. Since, $\Gamma(\alpha + 1) = \alpha\, \Gamma(\alpha)$, we have

$$\Gamma\left(\frac{\gamma}{r} + 1 \right) = \frac{\gamma}{r} \Gamma\left(\frac{\gamma}{r} \right)$$

Hence,

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \frac{\Gamma\left(\frac{\alpha}{p}\right) \Gamma\left(\frac{\beta}{q}\right) \left[\frac{\gamma}{r} \Gamma\left(\frac{\gamma}{r} \right) \right]}{\Gamma\left(\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} + 1 \right)}$$

$$= \frac{a^{\alpha} b^{\beta} c^{\gamma}}{pqr} \frac{\Gamma(\alpha/p)\Gamma(\beta/q)\Gamma(\gamma/r)}{\Gamma\left(\frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r}+1\right)}$$

which is the required result.

Example 2.56 Evaluate the Dirichlet integral

$$I = \iiint_T x^3 y^3 z^3 dx dy dz$$

where T is the region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the coordinate planes.

Solution Comparing the given integral with Eq. (2.107), we get

$$\alpha = \beta = \gamma = 4, p = q = r = 2, a = b = c = 1.$$

Substituting in Eq. (2.108), we obtain

$$I = \frac{1}{8} \frac{[\Gamma(2)]^3}{\Gamma(7)} = \frac{1}{8(6!)} = \frac{1}{5760}$$

since $\Gamma(n+1) = n!$, when n is an integer.

Example 2.57 Evaluate the Dirichlet integral

$$I = \iiint_T x^{1/2} y^{1/2} z^{1/2} dx dy dz$$

where T is the region in the first octant bounded by the plane $x + y + z = 1$ and the coordinate planes.

Solution Comparing the given integral with Eq. (2.107), we get

$$\alpha = \beta = \gamma = 3/2, p = q = r = 1, a = b = c = 1.$$

Substituting in Eq. (2.108), we obtain

$$I = \frac{[\Gamma(3/2)]^3}{\Gamma(11/2)}.$$

Using the results, $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ and $\Gamma(1/2) = \sqrt{\pi}$, we obtain

$$I = \frac{[(1/2)\Gamma(1/2)]^3}{(9/2)(7/2)(5/2)(3/2)(1/2)\Gamma(1/2)} = \frac{4\pi}{945}.$$

Exercises 2.5

1. Find the area bounded by the curves $y = x^2$, $y = 4 - x^2$.
2. Find the area bounded by the curves $x = y^2$, $x + y - 2 = 0$.
3. Find the area bounded by the curves $y^2 = 4 - 2x$, $x \geq 0$, $y \geq 0$.

4. Find the area bounded by the curves $x^2 = y^3$, $x = y$.
5. By changing to polar coordinates, find the area bounded by the curves $x^2 + y^2 = 2y$, $x^2 + y^2 = 4y$, x and $x = 0$.

Change the order of integration and evaluate the following double integrals.

6. $\displaystyle\int_{y=0}^{1}\int_{x=y}^{\sqrt{2-y^2}} \frac{y\,dx\,dy}{\sqrt{x^2+y^2}}$.

7. $\displaystyle\int_{y=0}^{1}\int_{x=0}^{y+4} \frac{2y+1}{x+1}\,dx\,dy$.

8. $\displaystyle\int_{y=0}^{1}\int_{x=y}^{y^{1/3}} e^{x^2}\,dx\,dy$.

9. $\displaystyle\int_{x=0}^{2}\int_{y=0}^{x^2/2} \frac{x}{\sqrt{x^2+y^2+1}}\,dy\,dx$.

10. $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1-x} e^{y/(x+y)}\,dy\,dx$ (use the substitution $x + y = u$ and $y = u\,v$).

11. Find the volume of the solid which is below the plane $z = 2x + 3$ and above the x-y plane and bound by $y^2 = x$, $x = 0$ and $x = 2$.
12. Find the volume of the solid which is below the plane $z = x + 3y$ and above the ellipse $25x^2 + 16y^2 = 4$ $x \geq 0$, $y \geq 0$.
13. Find the volume of the solid which is bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y + z =$ and $z = 0$.
14. Find the volume of the solid which is bounded by the paraboloid $z = 9 - x^2 - 4y^2$ and the coordina planes $x \geq 0$, $y \geq 0$, $z \geq 0$.
15. Find the volume of the solid which is enclosed between the cylinders $x^2 + y^2 = 2ay$ and $z^2 = 2ay$.
16. Find the volume of the solid which is bounded by the surfaces $2z = x^2 + y^2$ and $z = x$.
17. Find the volume of the solid which is bounded by the surfaces $z = 0$, $3z = x^2 + y^2$ and the cylinde $x^2 + y^2 = 9$.
18. Find the volume of the solid which is in the first octant bounded by the cylinders $x^2 + y^2 = a^2$ an $y^2 + z^2 = a^2$.
19. Find the volume of the solid which is bounded by the paraboloid $4z = x^2 + y^2$, the cone $z^2 = x^2 + y$ and the cylinder $x^2 + y^2 = 2x$.
20. Find the volume of the solid which is common to the right circular cylinders $x^2 + z^2 = 1$, $y^2 + z^2 = 1$ and $x^2 + y^2 = 1$.
21. Find the volume of the solid which is above the cone $z^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + (z - a)^2 = a^2$.
22. Find the volume of the solid which is below the surface $z = 4x^2 + 9y^2$ and above the square with vertices at (0, 0), (2, 0), (2, 2) and (0, 2).
23. Find the volume of the solid which is bounded by the paraboloids $z = x^2 + y^2$ and $z = 4 - 3(x^2 + y^2)$.
24. Find the volume of the solid which is bounded by $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ and the coordinate planes.
25. Find the volume of the solid which is contained between the cone $z^2 = 2(x^2 + y^2)$ and the hyperboloid $z^2 = x^2 + y^2 + a^2$.
26. Find the volume of the region under the cone $z = 3r$ and over the rose petal with boundary $r = \sin 4\theta$, $0 \leq \theta \leq \pi/4$.
27. Find the volume of the portion of the unit sphere which lies inside the right circular cone having its vertex at the origin and making an angle α with the positive z-axis.

28. Find the volume of the region under the plane $z = 1 + 3x + 2y$, $z \geq 0$ and above the region bounded by $x = 1$, $x = 2$, $y = x^2$, and $y = 2x^2$.

29. Find the volume of the portion of the sphere $x^2 + y^2 + z^2 \leq 2ay$ between the planes $y = 0$ and $y = a$.

30. Find the moment of inertia about the axes, of the circular lamina $x^2 + y^2 \leq a^2$, when the density function is $\rho = \sqrt{x^2 + y^2}$.

31. Find the total mass and the centre of gravity of the region bounded by $x^{2/3} + y^{2/3} = a^{2/3}$, $x \geq 0$, $y \geq 0$, when the density is constant k.

32. Show that $I = \iint\limits_R \dfrac{dx\,dy}{(x^2 + y^2)^p}$, p integer, R: $x^2 + y^2 \geq 1$ converges for $p > 1$.

Hence, evaluate the integral.

Evaluate the following integrals (change the variables if necessary) in the given region.

33. $\iint\limits_R (x^2 + y^2)\,dx\,dy$, boundary of R: triangle with vertices (0, 0), (1, 0), (1, 1).

34. $\iint\limits_R x^2\,dx\,dy$, boundary of R : $y = x^2$, $y = x + 2$.

35. $\iint\limits_R (x^2 + y^2)\,dx\,dy$, R: $0 \leq y \leq \sqrt{1 - x^2}$, $0 \leq x \leq 1$.

36. $\iint\limits_R \sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}\,dx\,dy$, boundary of R : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

37. $\iint\limits_R e^{2(x^2 + y^2)}\,dx\,dy$, R : $x^2 + y^2 \geq 4$, $x^2 + y^2 \leq 25$, $y = x$, $x \geq 0$, $y \geq 0$.

38. $\iint\limits_R x^3 y^3\,dx\,dy$, R : $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$.

39. $\iint\limits_R xy\,dx\,dy$, R : $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x \geq 0$, $y \geq 0$.

40. $\iint\limits_R (1 - x^2 - y^2)\,dx\,dy$, boundary of R : the square with vertices $(\pm 1, 0)$, $(0, \pm 1)$

(change coordinates : $x - y = u$, $x + y = v$).

41. $\iint\limits_R (x + y)^2\,dx\,dy$, boundary of R : parallelogram with sides $x + y = 1$, $x + y = 4$, $x - 2y = -2$, $x - 2y = 1$, (change coordinates: $x + y = u$, $x - 2y = v$).

42. $\iint\limits_R (4 - 3x^2 - y^2)\,dx\,dy$, boundary of R : $x = 0$, $y = 0$, $x + y - 2 = 0$.

43. $\iint_R xy\,dx\,dy$, region (in polar coordinates) $R : r = \sin 2\theta,\ 0 \le \theta \le \pi/2$.

44. $\iiint_T x^2 y^2 z\,dx\,dy\,dz$, $T : x^2 + y^2 \le 1,\ 0 \le z \le 1$.

45. $\iiint_T \dfrac{dx\,dy\,dz}{(x + y + z + 1)^3}$, boundary of $T : x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$.

46. $\iiint_T (x + 3y - 2z)dx\,dy\,dz$, $T : 0 \le y \le x^2,\ 0 \le z \le x + y,\ 0 \le x \le 1$.

47. $\iiint_T x\,dx\,dy\,dz$, boundary of $T : y = x^2,\ y = x + 2,\ 4z = x^2 + y^2,\ z = x + 3$.

48. $\iiint_T (2x - y - z)dx\,dy\,dz$, $T : 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le x + y$.

49. $\iiint_T \dfrac{dx\,dy\,dz}{(x^2 + y^2 + z^2)^{3/2}}$, boundary of $T : x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 + z^2 = b^2,\ a > b$.

50. $\iiint_T z\,dx\,dy\,dz$, boundary of $T : z^2 = x^2 + y^2,\ x^2 + y^2 + z^2 = 1$.

51. $\iiint_T \sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}}\,dx\,dy\,dz$, boundary of $T : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

52. $\iiint_T \sqrt{x^2 + y^2 + z^2}\,dx\,dy\,dz$, $T : x^2 + y^2 + z^2 \le y$.

53. $\iiint_T (x^2 + y^2)dx\,dy\,dz$, boundary of T : the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ and the coordinate planes.

54. $\iiint_T (y^2 + z^2)dx\,dy\,dz$, boundary of $T : y^2 + z^2 \le a^2,\ 0 \le x \le h$.

55. $\iiint_T x^2 y\,dx\,dy\,dz$, $T : x^2 + y^2 \le 1,\ 0 \le z \le 1$.

Evaluate the following Dirichlet integrals.

56. $\iiint_T xyz\,dx\,dy\,dz$, T: Region bounded by $x + y + z = 2$ and the coordinate planes.

57. $\iiint\limits_T xy^2z^3\,dx\,dy\,dz$, T: Region bounded by $x + y + z = 1$ and the coordinate planes.

58. $\iiint\limits_T \sqrt{xyz}\,dx\,dy\,dz$, T: Region bounded by $x^3 + y^3 + z^3 = 8$ and the coordinate planes.

59. $\iiint\limits_T xy^{1/2}z\,dx\,dy\,dz$, T: Region bounded by $x + y^3 + z^4 = 1$.

60. $\iiint\limits_T x^2y\,dx\,dy\,dz$, T: Region bounded by $\dfrac{x^2}{1} + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$.

2.7 Answers and Hints

Exercise 2.1

1. $|f(x, y) - 1| = |(x-1)^2 + (y-1)^2 + 2(x-1) + 2(y-1)|$
$$< |x-1|^2 + |y-1|^2 + 2|x-1| + 2|y-1| < \varepsilon$$
   (i) if $|x-1| < \delta$, $|y-1| < \delta$ is used, we get $2\delta^2 + 4\delta < \varepsilon$ or $\delta < [\sqrt{(\varepsilon+2)/2} - 1]$
   (ii) if $\delta^2 < \delta$ is used, we get $\delta < \varepsilon/6$
   (iii) if $(x-1)^2 + (y-1)^2 < \delta^2$ and $|x-1| < \delta$, $|y-1| < \delta$ is used, we get $\delta < \sqrt{\varepsilon+4} - 2$.
2. $|f(x, y) - 7| = |(x-2)^2 - (y-1)^2 + 6(x-2) - 2(y-1)|$
$$< |x-2|^2 + |y-1|^2 + 6|x-2| + 2|y-1| < \varepsilon.$$
   (i) if $|x-2| < \delta$, $|y-1| < \delta$ is used, we get $2\delta^2 + 8\delta < \varepsilon$, or $\delta < \sqrt{(\varepsilon+8)/2} - 2$.
   (ii) if $\delta^2 < \delta$ is used, we get $\delta < \varepsilon/10$.
   (iii) if $(x-2)^2 + (y-1)^2 < \delta^2$ and $|x-2| < \delta$, $|y-1| < \delta$ is used, we get $\delta < \sqrt{\varepsilon+16} - 4$.
3. $\left|\dfrac{x+y}{x^2+y^2+1}\right| < |x+y| < |x| + |y| < 2\sqrt{x^2+y^2} < \varepsilon$. Take $\delta < \varepsilon/2$.
4. Let $x = r\cos\theta$, $y = r\sin\theta$. Therefore
$$\left|\frac{x^3+y^3}{x^2+y^2}\right| < |r(\cos^3\theta + \sin^3\theta)| < 2r < \varepsilon. \text{ Take } \delta < \varepsilon/2.$$
5. $|f(x, y) - 0| < |x| + |y| < 2\sqrt{x^2+y^2} < \varepsilon$. Take $\delta < \varepsilon/2$.
6. $|f(x, y) - 0| < x^2 + y^2 < \varepsilon$. Take $\delta < \sqrt{\varepsilon}$.
7. Choose the path $y = mx$. Limit does not exist.
8. Factorize and cancel $x - y$; 1.
9. $[1 + (x/y)]^y = [[1 + (x/y)]^{y/x}]^x$; e^{α}.
10. 0.
11. 1/2.
12. 1.
13. Limit does not exist.
14. Limit does not exist.

15. Let $x = r\cos\theta$, $y = r\sin\theta$; $\frac{1}{r}\left(\frac{\cos^2\theta}{\cos^3\theta+\sin^3\theta}\right) \to \infty$ as $r \to 0$. Limit does not exist.

16. Choose the path $y = mx^2$. Limit does not exist.

17. Choose the path $z = x^2$, $y = mx$. Limit does not exist.

18. Choose the path $y = mx$, $z = mx$. Limit does not exist.

19. Choose the path $z = \sqrt{x}$, $y = mx$. Limit does not exist.

20. Choose the path $z = 0$, $y = mx$. Limit does not exist.

21. Choose the path $y = mx$. Discontinuous.

22. Limit is 0 for $x > 0$ and 1 for $x < 0$. Discontinuous.

23. Discontinuous.

24. Choose the path $y = mx$. Discontinuous.

25. Choose the path $y = mx$. Discontinuous.

26. Cancel $(x - y)$. Discontinuous.

27. Let $x = r\cos\theta$, $y = r\sin\theta$. Continuous.

28. Choose the path $y^2 = mx$. Discontinuous.

29. Since $x^2 + y^2 \geq 2|x||y|$, we have $\frac{1}{\sqrt{x^2+y^2}} \leq \frac{1}{\sqrt{2|xy|}}$. Therefore, $|f(x, y)| \leq \frac{\left|\sin\sqrt{|xy|} - \sqrt{|xy|}\right|}{\sqrt{2}\sqrt{|xy|}}$.
Continuous.

30. Since $2 \leq 3 + \sin x \leq 4$, we have $[1/(3 + \sin x)] \leq 1/2$. Therefore, $|f(x, y)| \leq [(2x^2 + y^2)/2] \leq x^2 + y^2$. Continuous.

31. The function is not defined along the path $y = -x$. Discontinuous.

32. $\left|\frac{x^5 - y^5}{x^2 + y^2}\right| \leq \frac{|x|^5 + |y|^5}{x^2 + y^2} \leq \frac{(x^2+y^2)^{5/2} + (x^2+y^2)^{5/2}}{x^2+y^2}$. Continuous.

33. Function is unbounded in any neighborhood of $x = -1$. Discontinuous.

34. Since $|x|, |y|, |z|$ are all $\leq \sqrt{x^2+y^2+z^2}$, $|f| \leq \sqrt{x^2+y^2+z^2}$. Continuous.

35. The function is unbounded along $x = \sqrt{3}z$. Discontinuous.

Exercise 2.2

1. $f_x(0, 0) = 0, f_y(0, 0) = 0$. For $(x, y) \neq (0, 0)$, find f_x, f_y and choose the path $y = mx$. The limits do not exist as $(x, y) \to (0, 0)$.

2. $f(x, y)$ is unbounded as $(x, y) \to (0, 0)$, for example along $x = y$; $f_x(0, 0) = 1, f_y(0, 0) = -1$.

3. $f_x(0, 0) = 0, f_y(0, 0) = -1, f_x(0, y) = 0, f_y(x, 0) = 1$.

4. $f_x(0, 0) = 1, f_y(0, 0) = 1$, $dz = \Delta x + \Delta y$, $\lim_{\Delta\rho\to 0} [(\Delta z - dz)/\Delta\rho]$ does not exist.

5. $f_x(0, 0) = 0 = f_y(0, 0)$, $dz = 0$. $\lim_{\Delta\rho\to 0} [(\Delta z - dz)/\Delta\rho] = 0$.

No contradiction since continuity of f_x, f_y is only a sufficient condition.

In problems **6** to **15**, f_x, f_y and f_z are given in that order at the given point.

6. $-2, 2$.

7. $1/2, -1/3$.

8. $6e^{1/2}, 4e^{1/2}$.

9. $49/(85)^{3/2}, -42/(85)^{3/2}$.

10. $-1/10, -1/10$.

11. $f(x, y) = 2 \ln [\sqrt{x^2 + y^2} - x] - 2 \ln y, -2/5, 3/10.$

12. $-2/27, -1/27, -2/27.$

13. $e, -2e, e.$

14. 5, 3, 0.

15. $1/7, 3/35, 4/35.$

16. 0.

17. 2.

18. $e^x[\sin (y + 2z) + \{(4t^3 - 1)/t^2\} \cos (y + 2z)].$

19. $2(y + z)t + (x + z)(t + 1) e^t + (x + y)(1 - t)e^{-t}.$

20. $(\pi/2) - (2/\pi).$

23. Set $s = x - y$, $v = y - z$, $w = z - x$.

31. $-[yx^{y-1} + y^x \ln y]/[xy^{x-1} + x^y \ln x].$

32. $y/[x + 3y^2(x^2 + y^2)].$

33. $\left(\dfrac{\partial z}{\partial x}\right)_y = -\left(\dfrac{\partial f/\partial x}{\partial f/\partial z}\right) = -\dfrac{y(\sin xy) + z(\sin xz)}{y(\sin yz) + x(\sin xz)}, \left(\dfrac{\partial z}{\partial y}\right)_x = -\left(\dfrac{\partial f/\partial y}{\partial f/\partial z}\right) = -\dfrac{x(\sin xy) + z(\sin yz)}{y(\sin yz) + x(\sin xz)}.$

34. $\left(\dfrac{\partial z}{\partial x}\right)_y = -\left(\dfrac{\partial f/\partial x}{\partial f/\partial z}\right) = -\dfrac{3x^2 + 3y + 3z}{3x + 2z}, \left(\dfrac{\partial z}{\partial y}\right)_x = -\left(\dfrac{\partial f/\partial y}{\partial f/\partial z}\right) = -\dfrac{3x - 4y}{3x + 2z}.$

35. Let $u = z/y$, $v = x/y$; then $f(u, v) = 0$; x.

36. 499.6.

37. 4.02.

38. $\dfrac{1}{2\sqrt{2}}\left[1 + \dfrac{\pi}{180}(2\sqrt{3} + 1)\right].$

39. 1.81.

40. $\dfrac{1}{720}[180 + \pi(6 - \sqrt{3})] \approx 0 \cdot 2686.$

41. 5.01.

42. $V = \pi r^2 h/3$, $dV/dt = 85\pi/72 \approx 3.71$ ft^3/hr.

43. $S = 2(xy + xz + yz)$, max. absolute error = 2880 in^2, max. relative error = 0.0766 in, percentage error ≈ 7.66%.

44. $A = \dfrac{1}{2} xy \sin \alpha$, percentage error ≈ 13.7%.

45. $V = abc$, percentage error = 3%.

46. $V = \pi r^2 h$, percentage error ≈ 9.2%.

47. 121.6 watts.

48. 2.92%.

49. 29.33%.

50. Lateral length $l = \sqrt{r^2 + h^2}$, lateral area $= \pi r l$, $dr = r/100$, $dh = h/100$, $dl = \sqrt{(dr)^2 + (dh)^2} = 1/20$, percentage error = 2%.

Exercise 2.3

1. At (1, 1): $f_{xx} = -1/2, f_{xy} = 0, f_{yy} = 1/2.$

2. At (2, 3): $f_{xxx} = 0, f_{xxy} = 0, f_{xyy} = -1/9, f_{yyy} = 4/27.$

3. At (1, 2): $f_{xx} = 0, f_{xy} = 1, f_{yy} = -3/4.$

4. At $(1, \pi/2)$: $f_{xxx} = e \ln (\pi/2), f_{xxy} = (2e/\pi) + 1, f_{xyy} = -4e/\pi^2, f_{yyy} = 16e/\pi^3.$

5. At $(\pi/2, 1)$: $f_{xx} = -e, f_{xy} = \pi e/2, f_{yy} = -\pi^2 e/4.$

6. At $(1, -1, 1)$: $f_{xx} = -1/2, f_{xy} = -1/4, f_{xz} = -1/4, f_{yy} = 0, f_{yz} = -1/4, f_{zz} = 0.$

7. At $(-1, 1, -1)$: $f_{xx} = 6e^3, f_{xy} = -4e^3, f_{xz} = 4e^3, f_{yy} = 6e^3, f_{yz} = -4e^3, f_{zz} = 6e^3.$

8. At $(1, \pi/2, \pi/2)$: $f_{xx} = -\pi^2/2, f_{xy} = -\pi/2, f_{xz} = -\pi/2, f_{yy} = -[1 + (\pi^2 S/4)], f_{yz} = -[(\pi^2 S/4) - c],$ $f_{zz} = -[1 + (\pi^2 S/4)], S = \sin (\pi^2/4), c = \cos (\pi^2/4).$

9. At (1, 2, 3) : $f_{xx} = 6, f_{xy} = -1/4, f_{xz} = -1, f_{yy} = 1/4, f_{yz} = -1/9, f_{zz} = 4/27.$

10. $f_{xy} = f \ln (ex) \ln (ey).$

11. $f_x(0, 0) = 0, f_y(0, 0) = 0, f_x(0, y) = 0, f_y(x, 0) = x, f_{xy}(0, 0) = 1, f_{yx}(0, 0) = 0.$

12. $f_{xy}(x, y) = f_{yx}(x, y) = x^{y-1}(1 + y \ln x).$

13. $f_{xy}(x, y) = f_{yx}(x, y) = -y/(x^2 + y^2)^{3/2}.$

14. $(1 + xy)(\cos z) e^{xy}.$

15. $4(1 + 2y^2)\, z\, e^{x+y^2}.$

16. For $t = 0$, we get $x = 0, y = 0, dz/dt = -2.$

17. $\partial x/\partial u = 3u/x,\ \partial y/\partial u = 5u/y;\ \partial^2 x/\partial u^2 = 3(x^2 - 3u^2)/x^3,\ \partial^2 y/\partial u^2 = 5(y^2 - 5u^2)/y^3.$

18. For $x = 1, y = -1, z = 2$, we get $u = 1, v = 2$; $(\partial u/\partial x)_{y,z} = 5/3$; $(\partial v/\partial y)_{x,z} = 1/6.$

19. $\dfrac{dy}{dx} = -\sqrt{\dfrac{1-y^2}{1-x^2}},\ \dfrac{d^2 y}{dx^2} = -\dfrac{c}{(1-x^2)^{3/2}}.$

20. $dy/dx = (e - 1)/(e + 1),\ d^2y/dx^2 \doteq 2(e^2 + 1)/(e + 1)^3.$

21. $\dfrac{\partial z}{\partial x} = u^v (v/u)^{1/2} \ln(eu),\ \dfrac{\partial^2 z}{\partial x^2} = u^{v-1}[1 + v(\ln eu)^2].$

26. $\alpha = 3\beta$ or $\alpha = 4\beta$ and $\beta \neq 0$ arbitrary.

27. Note that $u_x^2 + u_y^2 = v_x^2 + v_y^2 = 1/(x^2 + y^2)^2$, $u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$. We have

$$z_{xx} + z_{yy} = f_u(u_{xx} + u_{yy}) + f_v(v_{xx} + v_{yy}) + f_{uu}(u_x^2 + u_y^2) + f_{vv}(v_x^2 + v_y^2).$$

28. Use $x^2 + y^2 = r^2$, $\theta = \tan^{-1}(y/x)$ and differentiate.

29. $\sin u = (x^2 + y^2)/(x + y)$ is a homogeneous function of degree 1.

30. $e^u = [\sqrt{x^2 - y^2}/x]$ is a homogeneous function of degree 0.

31. u is a homogeneous function of degree 1.

32. u is a homogeneous function of degree 1.

33. $w = \tan u$ is a homogeneous function of degree 2.

34. $f(x, y) = 6 - 5(x - 2) + 3(y - 2) + (x - 2)^2 + 3(y - 2)^2.$

35. $f(x, y) \approx -2 - 2(x - 1) - (y - 1)$; $B = 4$; $|E| \le 0.08.$

36. $f(x, y) \approx (x - 1) + y$; $B = 4.6912$; $|E| \le 0.0938.$

37. $f(x, y) \approx 2 + [(x - 1) + 3(y - 1)] + \frac{1}{2}[-(x - 1)^2 + 6(x - 1)(y - 1) + (y - 1)^2]$; $B = 5.1$; $|E| \le 0.0029.$

38. $f(x, y) \approx 2 + \frac{1}{4}[(x - 1) + (y - 3)] - \frac{1}{64}[(x - 1)^2 + 2(x - 1)(y - 3) + (y - 3)^2]$; $B = 0.0142$, $|E| \le 0.64 \times 10^{-4}.$

39. $f(x, y) \approx 1 + (2x + y) + \frac{1}{2}(2x + y)^2 + \frac{1}{6}(2x + y)^3$; $B \doteq 23.87$; $|E| \le 0.008.$

40. $f(x, y) \approx (x + 2y) - \frac{1}{6}(x + 2y)^3$; $B = 16[\sin(0.3)] = 4.7283$; $|E| \le 0.315 \times 10^{-3}.$

41. $f(x, y) \approx \frac{1}{2} + \frac{1}{2}\left[\left(x - \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)\right] - \frac{1}{4}\left[\left(x - \frac{\pi}{4}\right)^2 - 2\left(x - \frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)^2\right]$; $B = 1$; $|E| \le 0.0013.$

42. $f(x, y, z) \approx 3 + \frac{2}{3}[(x - 2) + (y - 2) + (z - 1)]$; $B = 0.3872$; $|E| \le 0.017.$

43. $f(x, y, z) \approx 3 + \frac{3}{4}(x - 1) + \frac{5}{12}(y - 3) + \frac{2}{3}\left(z - \frac{3}{2}\right)$; $B = 0.3985$; $|E| \le 0.0179.$

44. $f(x, y, z) \approx x + y + xz + yz$; $B = 1.11$; $|E| \le 0.005$.

45. $f(x, y, z) \approx 1 + x + \frac{1}{2}\left[x^2 - \frac{\pi^2}{4}(y-1)^2 - \left(z - \frac{\pi}{2}\right)^2 - \pi(y-1)\left(z - \frac{\pi}{2}\right)\right]$; $|B| = 7.0817$; $|E| \le 0.0319$.

Exercises 2.4

1. minimum value 9 at (3, 1).
2. maximum value a at (0, 0).
3. minimum value 0 at (0, 0) if $|b| < 1$.
4. minimum value $(3)^{4/3}$ at $(3^{-1/3}, 3^{-1/3})$
5. minimum value $5(2)^{-2/5}$ at $(\pm 2^{3/10}, 2^{-1/5})$.
6. minimum value $-3/2$ at $(\pi/3, 2\pi/3)$.
7. The matrix **A** or the matrix **B** = – **A** is not positive definite. The function has no relative minimum or maximum.
8. The matrix **B** = – **A** is positive definite and $f_{xx}, f_{yy}, f_{zz} < 0$ at (0, 0, 0). Maximum value is 0.
9. **A** is positive definite and $f_{xx}, f_{yy}, f_{zz} > 0$ at (–1, –1, –1), (–1, 1, 1), (1, –1, 1), (1, 1, –1). Minimum value is –1 at all these points.
10. **B** = – **A** is positive definite and $f_{xx}, f_{yy}, f_{zz} < 0$ at $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Maximum value is (log 3) –1.
11. No relative maximum and minimum. Absolute minimum value – 3 at (0, 1). Absolute maximum value 3/2 at $(\pm\sqrt{3}/2, -1/2)$.
12. No relative maximum and minimum. Absolute maximum value 1/2 at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. Absolute minimum – 1/2 at $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$.
13. No relative maximum and minimum. Absolute maximum value $\sqrt{13}$ at $(9/\sqrt{13}, 4/\sqrt{13})$. Absolute minimum value $-\sqrt{13}$ at $(-9/\sqrt{13}, -4/\sqrt{13})$.
14. Relative minimum value 3/4 at (1/4, 0). Minimum value 3/2 on the boundary at $(1/2, \pm 1/\sqrt{2})$. Absolute minimum value 3/4 at (1/4, 0).
15. Absolute minimum value 1/2 at (1/2, 1/2). Absolute maximum value 5 at (2, 2).
16. Absolute minimum value – 93/18 at (1/6, 2/3). Absolute maximum value – 4 at (0, 0).
17. Absolute minimum value – 1/27 at (1/3, 1/3). Absolute maximum value 7 at (1, 2).
18. Absolute minimum value – 23/2 at (2, –3/2). Absolute maximum value 37 at (0, – 4).
19. Absolute minimum value – 3/2 at $(2\pi/3, 2\pi/3)$. Absolute maximum value 3 at (0, 0).
20. Absolute maximum value 1 at (0, 0), $(0, \pi)$, $(\pi, 0)$ and (π, π). Absolute minimum value –1/8 at $(\pi/3, \pi/3)$, $(2\pi/3, 2\pi/3)$.
21. $F = f(x, y) + \lambda\phi(x, y) \Rightarrow f_x + \lambda\phi_x = 0$ and $f_y + \lambda\phi_y = 0$. Eliminate λ.
22. $\lambda = -1/2$, $(x, y) = (1, 1/2)$; maximum value is 1/2; minimum value is 0.
23. $\lambda = \sqrt{5}/2$, $(x, y) = (-1/\sqrt{5}, -2/\sqrt{5})$, minimum value is $-\sqrt{5}$;
    $\lambda = -\sqrt{5}/2$, $(x, y) = (1/\sqrt{5}, 2/\sqrt{5})$, maximum value is $\sqrt{5}$.
24. Maximum value $(3\sqrt{3} - \pi)/3$ at $(\sqrt{3}/2, \pi/3)$. Minimum value $-(3\sqrt{3} + 5\pi)/3$ at $(-\sqrt{3}/2, 5\pi/3)$.
25. Extreme value is $\sqrt{2}$.
26. The points (4, – 4), (– 4, 4) are farthest, $d^2 = 32$. The points $(4/\sqrt{3}, 4/\sqrt{3})$, $(-4/\sqrt{3}, -4/\sqrt{3})$ are nearest, $d^2 = 32/3$.

27. Rectangle must be a square.
28. Triangle must be an equilateral triangle.
29. The point is $(AD/t, BD/t, CD/t)$, $t = A^2 + B^2 + C^2$.
30. Extreme value is $a^3/27$ at $(a/3, a/3, a/3)$.
31. Extreme value is $(a + b + c)^3$ at $(t/a, t/b, t/c)$, $t = a + b + c$.
32. Extreme value is $3^{(q-p)/q}$ at (t, t, t), $t = 3^{-1/q}$.
33. Extreme value is 24 at (2, 1, 1/2).
34. Maximise $V = 8xyz$ such that $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$. We get $(x, y, z) = (2a/\sqrt{3}, 2b/\sqrt{3}, 2c/\sqrt{3})$.
35. Maximise xy^2z^3 such that $x + y + z = a$, a constant. We get $x = a/6$, $y = a/3$, $z = a/2$.
36. Maximise $2(xy + xz + yz)$ such that $4(x + y + z) = a$, a constant. We get $x = y = z = a/12$, that is the parallelopiped is a cube.
37. Maximise $V = xyz$ such that $xy + xz + yz = S/2$, we get $x = y = z = \sqrt{S/6}$.
38. Minimise $S = xy + 2xz + 2yz$ such that $xyz = a$. We get $x = y = (2a)^{1/3}$ and $z = x/2$.
39. Maximise $V = \pi r^2 h/3$ such that $\pi r l = a$ where $l = \sqrt{r^2 + h^2}$. We get $h = \sqrt{2}r$.
40. Maximise $V = \pi r^2 H + (\pi r^2 h)/3$ such that $2\pi r H + \pi r l = S$, $l = \sqrt{r^2 + h^2}$. We get $h/r = 2/\sqrt{5}$ and $H/r = 1/\sqrt{5}$.
41. Maximum value is $2/(3\sqrt{3})$ at $(\pm 2/\sqrt{3}, \pm 2/\sqrt{3}, 1/\sqrt{3})$.
42. Extreme value is 3/2 at (1/2, –1, 3/2).
43. Extreme value is 11/12 at (–1/6, 1/3, 5/6).
44. Farthest point (1, 0, 0), $d = 1$; nearest point (1/3, 0, 2/3) $d = \sqrt{5}/3$.
45. The coordinates of the points P and Q are obtained as $(2a/3, 2a/3, 2a/3)$ and $(\pm a/\sqrt{3}, \pm a/\sqrt{3}, \pm a/\sqrt{3})$. Shortest distance : $d^2 = a^2(7 - 4\sqrt{3})/3$; Largest distance : $d^2 = a^2(7 + 4\sqrt{3})/3$.

Exercise 2.5

1. Curves intersect at $x = \pm\sqrt{2}$, $y = 2$; $16\sqrt{2}/3$.
2. Curves intersect at (1, 1) and (4, –2); 9/2.
3. 8/3.
4. Curves intersect at (0, 0) and (1, 1); 1/10.
5. R: $\pi/4 \le \theta \le \pi/2$, $2\sin\theta \le r \le 4\sin\theta$; $3(\pi + 2)/4$.
6. $I = \int_{x=0}^{1}\int_{y=0}^{x} f(x, y)\,dy\,dx + \int_{x=1}^{\sqrt{2}}\int_{0}^{\sqrt{2-x^2}} f(x, y)\,dy\,dx$, where $f(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}$; $(2 - \sqrt{2})/2$.
7. $I = \int_{x=0}^{4}\int_{y=0}^{1} f(x, y)\,dy\,dx + \int_{x=4}^{5}\int_{y=x-4}^{1} f(x, y)\,dy\,dx$, where $f(x, y) = \dfrac{2y + 1}{x + 1}$; $20\ln(5) - 18\ln(6) + (7/2)$.
8. $I = \int_{x=0}^{1}\int_{y=x^3}^{x} e^{x^2}\,dy\,dx = (e - 2)/2$.
9. $I = \int_{y=0}^{2}\int_{x=\sqrt{2y}}^{2} \dfrac{x}{\sqrt{x^2 + y^2 + 1}}\,dx\,dy = \dfrac{1}{4}(5\ln 5 - 4)$.
10. $I = \int_{0}^{1}\int_{0}^{1} ue^{v}\,du\,dv = \dfrac{1}{2}(e - 1)$.
11. $14\sqrt{2}/5$.

12. 380/3.
13. π.
14. $81\pi/16$.
15. $128\,a^3/15$.
16. $\pi/4$.
17. $27\pi/2$.
18. $2a^3/3$.
19. $(256 - 27\pi)/72$.
20. $8(2 - \sqrt{2})$.
21. πa^3.
22. 208/3.
23. 2π.
24. $a^3/90$.
25. $4\pi a^3(\sqrt{2}-1)/3$.
26. 1/3.
27. $2\pi(1 - \cos\alpha)/3$.
28. 1931/60.
29. $2\pi a^3/3$.
30. $I_y = a^5\pi/5 = I_x$.
31. $M = 3\pi k\,a^2/32$, $\bar{x} = \bar{y} = 8ka^3/(105\,M)$.
32. Evaluate the integral over $1 \le x^2 + y^2 \le a^2$ and take the limit as $a \to \infty$. $I = \pi/(p - 1)$.
33. 1/3.
34. 63/20.
35. $\pi/8$.
36. $2\pi ab/3$.
37. $(e^{50} - e^8)\,\pi/16$.
38. 1/96.
39. $a^4/280$.
40. 4/3.
41. 21.
42. 8/3.
43. 1/15.
44. $\pi/48$.
45. $(8 \ln 2 - 5)/16$.
46. 11/42.
47. 837/160.
48. 8/35.
49. $4\pi \ln(a/b)$.
50. $\pi/8$.
51. $\pi^2 abc/4$.
52. $\pi/10$.
53. $abc(a^2 + b^2)/60$.
54. $\pi h a^4/2$.
55. 0.

In problems **56** to **60** compare the given integral with Eq. (2.107).

56. $\alpha = \beta = \gamma = 2$, $a = b = c = 2$, $p = q = r = 1$; $I = 4/45$.

57. $\alpha = 2$, $\beta = 3$, $\gamma = 4$, $a = b = c = 1$, $p = q = r = 1$, $I = 12/9!$.

58. $\alpha = \beta = \gamma = 3/2$, $a = b = c = 1$, $p = q = r = 3$, $I = 64\sqrt{2}\,\pi/81$.

59. $\alpha = 2$, $\beta = 3/2$, $\gamma = 2$, $a = b = c = 1$, $p = 1$, $q = 3$, $r = 4$, $I = \pi/288$.

60. $\alpha = 3$, $\beta = 2$, $\gamma = 1$, $a = 1$, $b = 2$, $c = 3$, $p = q = r = 2$, $I = \pi/8$.

Chapter 3

Matrices and Eigenvalue Problems

3.1 Introduction

In modern mathematics, matrix theory occupies an important place and has applications in almost all branches of engineering and physical sciences. Matrices of order $m \times n$ form a vector space and they define linear transformations which map vector spaces consisting of vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ into another vector space consisting of vectors in $\mathbb{R}^m$ or $\mathbb{C}^m$ under a given set of rules of vector addition and scalar multiplication. A matrix does not denote a number and no value can be assigned to it. The usual rules of arithmetic operations do not hold for matrices. The rules defining the operations on matrices are usually called its algebra. In this chapter, we shall discuss the matrix algebra and its use in solving linear system of algebraic equations $\mathbf{Ax} = \mathbf{b}$ and solving the eigenvalue problem $\mathbf{Ax} = \lambda \mathbf{x}$.

3.2 Matrices

An $m \times n$ matrix is an arrangement of mn objects (not necessarily distinct) in m *rows* and n *columns* in the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}. \tag{3.1}$$

We say that the matrix is of *order* $m \times n$ (m by n). The objects $a_{11}, a_{12}, \dots, a_{mn}$ are called the *elements* of the matrix. Each element of the matrix can be a real or a complex number or a function of one or more variables or any other object. The element a_{ij} which is common to the ith row and the jth column is called its *general element*. The matrices are usually denoted by boldface uppercase letters $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, ... etc. When the order of the matrix is understood, we can simply write $\mathbf{A} = (a_{ij})$. If all the elements of a matrix are real, it is called a *real matrix*, whereas if one or more elements of a matrix are complex, it is called a *complex matrix*. We define the following particular types of matrices.

Row vector A matrix of order $1 \times n$, that is, it has one row and n columns is called a *row* vector or a row matrix of order n and is written as

$$[a_{11} \;\; a_{12} \;\; \dots \;\; a_{1n}], \quad \text{or} \quad [a_1 \;\; a_2 \;\; \dots \;\; a_n]$$

in which a_{1j} (or a_j) is the jth element.

Column vector A matrix of order $m \times 1$, that is, it has m rows and one column is called a *column vector* or a *column matrix* of order m and is written as

$$\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in which b_{j1} (or b_j) is the jth element.

The number of elements in a row/column vector is called its *order*. The vectors are usually denoted by boldface lower case letters **a**, **b**, **c**, . . . etc. If a vector has n elements and all its elements are real numbers, then it is called an *ordered n-tuple* in $\mathbb{R}^n$, whereas if one or more elements are complex numbers, then it is called an ordered n-tuple in $\mathbb{C}^n$.

Rectangular matrix A matrix **A** of order $m \times n$, $m \neq n$ is called a *rectangular matrix*.

Square matrices A matrix **A** of order $m \times n$ in which $m = n$, that is number of rows is equal to the number of columns is called a square matrix of order n. The elements a_{ii}, that is the elements a_{11}, a_{22}, . . . , a_{nn} are called the *diagonal elements* and the line on which these elements lie is called the *principal diagonal* or the *main diagonal* of the matrix. The elements a_{ij}, when $i \neq j$ are called the *off-diagonal elements*. The sum of the diagonal elements of a square matrix is called the *trace* of the matrix.

Null matrix A matrix **A** of order $m \times n$ in which all the elements are zero is called a *null matrix* or a *zero matrix* and is denoted by **0**.

Equal matrices Two matrices $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{p\times q}$ are said to be equal, when

(i) they are of the same order, that is $m = p$, $n = q$ and

(ii) their corresponding elements are equal, that is $a_{ij} = b_{ij}$ for all i, j.

Diagonal matrix A square matrix **A** in which all the off-diagonal elements a_{ij}, $i \neq j$ are zero is called a diagonal matrix. For example

$$\mathbf{A} = \begin{bmatrix} a_{11} & & & \mathbf{0} \\ & a_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & a_{nn} \end{bmatrix} \text{ is a } \textit{diagonal matrix} \text{ of order } n.$$

A diagonal matrix is denoted by **D**. It is also written as diag $[a_{11} \quad a_{22} \quad \cdots \quad a_{nn}]$.

If all the elements of a diagonal matrix of order n are equal, that is $a_{ii} = \alpha$ for all i, then the matrix

$$\mathbf{A} = \begin{bmatrix} \alpha & & & \mathbf{0} \\ & \alpha & & \\ & & \ddots & \\ \mathbf{0} & & & \alpha \end{bmatrix} \text{ is called a } \textit{scalar matrix} \text{ of order } n.$$

If all the elements of a diagonal matrix of order n are 1, then the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$ is called an *unit matrix* or an *identity matrix* of order n.

An identity matrix is denoted by **I**.

Submatrix A matrix obtained by omitting some rows and/or columns from a given matrix **A** is called a *submatrix* of **A**. As a convention, the given matrix **A** is also taken as a submatrix of **A**.

3.2.1 Matrix Algebra

The basic operations allowed on matrices are

(i) multiplication of a matrix by a scalar,

(ii) addition/subtraction of two matrices,

(iii) multiplication of two matrices.

Note that there is no concept of dividing a matrix by a matrix. Therefore, the operation **A/B** where **A** and **B** are matrices is not defined.

Multiplication of a matrix by a scalar

Let α be a scalar (real or complex) and $\mathbf{A} = (a_{ij})$ be a given matrix of order $m \times n$. Then

$$\mathbf{B} = \alpha\mathbf{A} = \alpha(a_{ij}) = (\alpha a_{ij}) \quad \text{for all } i \text{ and } j. \tag{3.2}$$

The order of the new matrix **B** is same as that of the matrix **A**.

Addition/subtraction of two matrices

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be two matrices of the same order. Then

$$\mathbf{C} = (c_{ij}) = \mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}), \quad \text{for all } i \text{ and } j \tag{3.3a}$$

and

$$\mathbf{D} = (d_{ij}) = \mathbf{A} - \mathbf{B} = (a_{ij}) - (b_{ij}) = (a_{ij} - b_{ij}), \quad \text{for all } i \text{ and } j. \tag{3.3b}$$

The order of the new matrix **C** or **D** is the same as that of the matrices **A** and **B**. Matrices of the same order are said to be *conformable* for addition/subtraction.

If $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_p$ are p matrices which are conformable for addition and $\alpha_1, \alpha_2, \ldots, \alpha_p$ are any scalars, then

$$\mathbf{C} = \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2 + \ldots + \alpha_p \mathbf{A}_p \tag{3.4}$$

is called a linear combination of the matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_p$. The order of the matrix **C** is same as that of $\mathbf{A}_i$, $i = 1, 2, \ldots, p$.

Properties of the matrix addition and scalar multiplication

Let **A**, **B**, **C** be the matrices which are conformable for addition and α, β be scalars. Then

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. (commutative law)

2. (**A** + **B**) + **C** = **A** + (**B** + **C**) (associative law).
3. **A** + **0** = **A** (**0** is the null matrix of the same order as **A**).
4. **A** + (− **A**) = **0**.
5. $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$.
6. $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$.
7. $\alpha(\beta\mathbf{A}) = \alpha\beta\mathbf{A}$.
8. $1 \times \mathbf{A} = \mathbf{A}$ and $0 \times \mathbf{A} = \mathbf{0}$.

Multiplication of two matrices

The product **AB** of two matrices **A** and **B** is defined only when the number of columns in **A** is equal to the number of rows in **B**. Such matrices are said to be *conformable* for multiplication. Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix and $\mathbf{B} = (b_{ij})$ be an $n \times p$ matrix. Then the product matrix

$$\mathbf{C} = (c_{ij}) = \mathbf{AB} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2p} \\ \vdots & & & & & \\ b_{n1} & b_{n2} & \dots & b_{nj} & \dots & b_{np} \end{bmatrix}}_{n \times p}$$

is a matrix of order $m \times p$. The general element of the product matrix **C** is given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}. \tag{3.5}$$

In the product **AB**, **B** is said to be pre-multiplied by **A** or **A** is said to be post-multiplied by **B**.

If **A** is a row matrix of order $1 \times n$ and **B** is a column matrix of order $n \times 1$, then **AB** is a matrix of order 1×1, that is a single element, and **BA** is a matrix of order $n \times n$.

Remark 1

(a) It is possible that for two given matrices **A** and **B**, the product matrix **AB** is defined but the product matrix **BA** may not be defined. For example, if **A** is a 2×3 matrix and **B** is a 3×4 matrix, then the product matrix **AB** is defined and is a matrix of order 2×4, whereas the product matrix **BA** is not defined.

(b) If both the product matrices **AB** and **BA** are defined, then both the matrices **AB** and **BA** are square matrices. In general $\mathbf{AB} \neq \mathbf{BA}$. Thus, the matrix product is not commutative. If **AB** = **BA**, then the matrices **A** and **B** are said to *commute* with each other.

(c) If **AB** = **0**, then it does not always imply that either **A** = **0** or **B** = **0**. For example, let

$$\mathbf{A} = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$$

then $$\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{BA} = \begin{bmatrix} 0 & 0 \\ ax + by & 0 \end{bmatrix} \neq \mathbf{AB}.$$

(d) If $\mathbf{AB} = \mathbf{AC}$, it does not always imply that $\mathbf{B} = \mathbf{C}$.

(e) Define $\mathbf{A}^k = \mathbf{A} \times \mathbf{A} \ldots \times \mathbf{A}$ (k times). Then, a matrix $\mathbf{A}$ such that $\mathbf{A}^k = \mathbf{0}$ for some positive integer k is said to be *nilpotent*. The smallest value of k for which $\mathbf{A}^k = \mathbf{0}$ is called the *index of nilpotency* of the matrix $\mathbf{A}$.

(f) If $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A}$ is called an *idempotent matrix.*

Properties of matrix multiplication

1. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices of orders $m \times n$, $n \times p$ and $p \times q$ respectively, then
$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \qquad \text{(associative law)}$$
is a matrix of order $m \times q$.
2. If $\mathbf{A}$ is a matrix of order $m \times n$ and $\mathbf{B}, \mathbf{C}$ are matrices of order $n \times p$, then
$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \qquad \text{(left distributive law).}$$
3. If $\mathbf{A}, \mathbf{B}$ are matrices of order $m \times n$ and $\mathbf{C}$ is a matrix of order $n \times p$, then
$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC} \qquad \text{(right distributive law).}$$
4. If $\mathbf{A}$ is a matrix of order $m \times n$ and $\mathbf{B}$ is a matrix of order $n \times p$, then
$$\alpha(\mathbf{AB}) = \mathbf{A}(\alpha\mathbf{B}) = (\alpha\mathbf{A})\mathbf{B}$$
for any scalar α.

3.2.2 Some Special Matrices

We now define some special matrices.

Transpose of a matrix The matrix obtained by interchanging the corresponding rows and columns of a given matrix $\mathbf{A}$ is called the *transpose matrix* of $\mathbf{A}$ and is denoted by $\mathbf{A}^T$, that is, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \text{ then } \mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}.$$

If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{A}^T$ is an $n \times m$ matrix. Also, both the product matrices $\mathbf{A}^T\mathbf{A}$ and $\mathbf{AA}^T$ are defined, and

$$\mathbf{A}^T\mathbf{A} = (n \times m)(m \times n) \text{ is an } n \times n \text{ square matrix}$$

and

$$\mathbf{AA}^T = (m \times n)(n \times m) \text{ is an } m \times m \text{ square matrix.}$$

A column vector $\mathbf{b}$ can also be written as $[b_1 \quad b_2 \ldots b_n]^T$.

The following results can be easily verified

1. The transpose of a row matrix is a column matrix and the transpose of a column matrix is a row matrix.
2. $(\mathbf{A}^T)^T = \mathbf{A}$.

3. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$, when the matrices **A** and **B** are conformable for addition.
4. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, when the matrices **A** and **B** are conformable for multiplication.

   If the product $\mathbf{A}_1 \mathbf{A}_2 \ldots \mathbf{A}_p$ is defined, then

$$[\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_p]^T = \mathbf{A}_p^T\, \mathbf{A}_{p-1}^T \ldots \mathbf{A}_1^T.$$

Remark 2

The product of a row vector $\mathbf{a}_i = (a_{i1} \;\; a_{i2} \;\; \ldots \;\; a_{in})$ of order $1 \times n$ and a column vector $\mathbf{b}_j = (b_{1j} \;\; b_{2j} \;\; \ldots \;\; b_{nj})^T$ of order $n \times 1$ is called the *dot product* or the *inner product* of the vectors $\mathbf{a}_i$ and $\mathbf{b}_j$, that is

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

which is a scalar. In terms of the inner products, the product matrix **C** in Eq. (3.5) can be written as

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \ldots & \mathbf{a}_1 \cdot \mathbf{b}_p \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \ldots & \mathbf{a}_2 \cdot \mathbf{b}_p \\ \ldots & \ldots & \ldots & \\ \mathbf{a}_m \cdot \mathbf{b}_1 & \mathbf{a}_m \cdot \mathbf{b}_2 & \ldots & \mathbf{b}_m \cdot \mathbf{b}_p \end{bmatrix}. \tag{3.6}$$

Symmetric and skew-symmetric matrices A real square matrix $\mathbf{A} = (a_{ij})$ is said to be

symmetric, if $a_{ij} = a_{ji}$ for all i and j, that is $\mathbf{A} = \mathbf{A}^T$

skew-symmetric, if $a_{ij} = -\,a_{ji}$ for all i and j, that is $\mathbf{A} = -\,\mathbf{A}^T$.

Remark 3

(a) In a skew-symmetric matrix $\mathbf{A} = (a_{ij})$, all its diagonal elements are zero.

(b) The matrix which is both symmetric and skew-symmetric must be a null matrix.

(c) For any real square matrix **A**, the matrix $\mathbf{A} + \mathbf{A}^T$ is always symmetric and the matrix $\mathbf{A} - \mathbf{A}^T$ is always skew-symmetric. Therefore, a real square matrix **A** can be written as the sum of a symmetric matrix and a skew-symmetric matrix. That is

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T).$$

Triangular matrices A square matrix $\mathbf{A} = (a_{ij})$ is called a *lower triangular matrix* if $a_{ij} = 0$, whenever $i < j$, that is all elements above the principal diagonal are zero and an *upper triangular matrix* if $a_{ij} = 0$, whenever $i > j$, that is all the elements below the principal diagonal are zero.

Conjugate matrix Let $\mathbf{A} = (a_{ij})$ be a complex matrix. Let $\overline{a}_{ij}$ denote the complex conjugate of a_{ij}. Then, the matrix $\overline{\mathbf{A}} = (\overline{a}_{ij})$ is called the *conjugate matrix* of *A*.

Hermitian and skew-Hermitian matrices A complex matrix **A** is called an *Hermitian matrix* if $\overline{\mathbf{A}} = \mathbf{A}^T$ or $\mathbf{A} = (\overline{\mathbf{A}})^T$ and a skew-Hermitian matrix if $\overline{\mathbf{A}} = -\mathbf{A}^T$ or $\mathbf{A} = -(\overline{\mathbf{A}})^T$. Sometimes, a Hermitian matrix is denoted by $\mathbf{A}^H$ or $\mathbf{A}^*$.

Remark 4

(a) If **A** is a real matrix, then an Hermitian matrix is same as a symmetric matrix and a skew-Hermitian matrix is same as a skew-symmetric matrix.

(b) In an Hermitian matrix, all the diagonal elements are real (let $a_{jj} = x_j + i y_j$; then $a_{jj} = \bar{a}_{jj}$ gives $x_j + i y_j = x_j - i y_j$ or $y_j = 0$ for all j).

(c) In a skew-Hermitian matrix, all the diagonal elements are either 0 or pure imaginary (let $a_{jj} = x_j + i y_j$; then $a_{jj} = -\bar{a}_{jj}$ gives $x_j + i y_j = -(x_j - i y_j)$ or $x_j = 0$ for all j).

(d) For any complex square matrix **A**, the matrix $\mathbf{A} + \overline{\mathbf{A}}^T$ is always an Hermitian matrix and the matrix $\mathbf{A} - \overline{\mathbf{A}}^T$ is always a skew-Hermitian matrix. Therefore, a complex square matrix **A** can be written as the sum of an Hermitian matrix and a skew-Hermitian matrix, that is

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \overline{\mathbf{A}}^T) + \frac{1}{2}(\mathbf{A} - \overline{\mathbf{A}}^T).$$

Example 3.1 Let **A** and **B** be two symmetric matrices of the same order. Show that the matrix **AB** is symmetric if and only if **AB** = **BA**, that is the matrices **A** and **B** commute.

Solution Since the matrices **A** and **B** are symmetric, we have

$$\mathbf{A}^T = \mathbf{A} \quad \text{and} \quad \mathbf{B}^T = \mathbf{B}.$$

Let **AB** be symmetric. Then

$$(\mathbf{AB})^T = \mathbf{AB}, \quad \text{or} \quad \mathbf{B}^T\mathbf{A}^T = \mathbf{AB}, \quad \text{or} \quad \mathbf{BA} = \mathbf{AB}.$$

Now, let **AB** = **BA**. Taking transpose on both sides, we get

$$(\mathbf{AB})^T = (\mathbf{B\,A})^T = \mathbf{A}^T\mathbf{B}^T = \mathbf{AB}.$$

Hence, the result.

3.2.3 Determinants

With every square matrix **A** of order n, we associate a determinant of order n which is denoted by *det* (**A**) or | **A** |. The determinant has a value and this value is real if the matrix **A** is real and may be real or complex, if the matrix is complex. A determinant of order n is defined as

$$det\,(\mathbf{A}) = |\,\mathbf{A}\,| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \tag{3.7}$$

We now discuss methods to find the value of a determinant. A determinant of order 2 has two rows and two columns. Its value is given by

$$|\,\mathbf{A}\,| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

We evaluate higher order determinants using minors and cofactors.

Minors and cofactors Let a_{ij} be the general element of a determinant. If we delete the ith row and the jth column from the determinant, we obtain a new determinant of order $(n-1)$ which is called the *minor* of the element a_{ij}. We denote this minor by M_{ij}. The cofactor of the element a_{ij} is defined as

$$A_{ij} = (-1)^{i+j} M_{ij}. \tag{3.8}$$

We can expand a determinant of order n through the elements of any row or any column. The value

of the determinant is the sum of the products of the elements of the ith row (or jth column) and their corresponding cofactors. Thus, we have

$$|\mathbf{A}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} a_{ij} A_{ij} \tag{3.9a}$$

when we expand through the elements of the ith row, or

$$|\mathbf{A}| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} A_{ij} \tag{3.9b}$$

when we expand through the elements of the jth column. Generally, we expand a determinant through that row or column which has a number of zeros. We can use one or more of the following properties of the determinants to simplify the evaluation of determinants.

Properties of determinants

1. If all the elements of a row (or column) are zero, then the value of the determinant is zero.
2. The value of a determinant remains unchanged if its corresponding rows and columns are interchanged, that is

$$|\mathbf{A}| = |\mathbf{A}^T|.$$

3. If any two rows (or columns) are interchanged, then the value of the determinant is multiplied by (– 1).
4. If the corresponding elements of two rows (or columns) are same, that is two rows (or columns) are identical, then the value of the determinant is zero.
5. If the corresponding elements of two rows (or columns) are proportional to each other, then the value of the determinant is zero.
6. The value of the determinant of a diagonal or a lower triangular or an upper triangular matrix is the product of its diagonal elements.
7. If each element of a row (or column) is multiplied by a scalar α, then the value of the determinant is multiplied by the scalar α. Therefore, if β is a factor of each element of a row (or column), then this factor β can be taken out of the determinant.

   Note that when we multiply a matrix by a scalar α, then every element of the matrix is multiplied by α. Therefore, $|\alpha\mathbf{A}| = \alpha^n |\mathbf{A}|$ where $\mathbf{A}$ is a matrix of order n.
8. If each element of any row (or column) can be written as the sum of two (or more) terms, then the determinant can be written as sum of two (or more) determinants.
9. If a non-zero constant multiple of the elements of some row (or column) is added to the corresponding elements of some other row (or column), then the value of the determinant remains unchanged.

Remark 5

When the elements of the jth row are multiplied by a non-zero constant k and added to the corresponding elements of the ith row, we denote this operation as $R_i \leftarrow R_i + kR_j$, where R_i is the ith row of $|\mathbf{A}|$. The elements of the jth row remain unchanged whereas the elements of the ith row get changed. This operation is called an *elementary row operation*. Similarly, the

operation $C_i \leftarrow C_i + kC_j$, where C_i is the ith column of $|\mathbf{A}|$, is called the *elementary column operation*. Therefore, under elementary row (or column) operations, the value of a determinant is unchanged.

10. The sum of the products of elements of any row (or column) with their corresponding cofactors gives the value of the determinant. However, the sum of the products of the elements of any row (or column) with the corresponding cofactors of any other row (or column) is zero. Thus, we have

$$\sum_{k=1}^{n} a_{ik}A_{jk} = \begin{cases} |\mathbf{A}|, & i = j \\ 0 & i \neq j \end{cases} \quad \text{(expansion through } i\text{th row)} \tag{3.10a}$$

or
$$\sum_{k=1}^{n} a_{ki}A_{kj} = \begin{cases} |\mathbf{A}|, & i = j \\ 0 & i \neq j \end{cases} \quad \text{(expansion through } j\text{th column).} \tag{3.10b}$$

11. $|\mathbf{A} + \mathbf{B}| \neq |\mathbf{A}| + |\mathbf{B}|$, in general.

Product of two determinants

If $\mathbf{A}$ and $\mathbf{B}$ are two square matrices of the same order, then

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|.$$

Since $|\mathbf{A}| = |\mathbf{A}^T|$, we can multiply two determinants in any one of the following ways

(i) row by row, (ii) column by column,

(iii) row by column, (iv) column by row.

The value of the determinant is same in each case.

Rank of a matrix

The *rank* of a matrix $\mathbf{A}$, denoted by r or $r(\mathbf{A})$ is the order of the largest non-zero minor of $|\mathbf{A}|$. Therefore, the rank of a matrix is the largest value of r, for which there exists at least one $r \times r$ submatrix of $\mathbf{A}$ whose determinant is not zero. Thus, for an $m \times n$ matrix $r \leq \min(m, n)$. For a square matrix $\mathbf{A}$ of order n, the rank $r = n$ if $|\mathbf{A}| \neq 0$, otherwise $r < n$. The rank of a null matrix is zero and if the rank of a matrix is 0, then it must be a null matrix.

Example 3.2 Find the value of the determinant

$$|\mathbf{A}| = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 2 & 4 \\ -1 & 3 & 2 \end{vmatrix}$$

(i) using elementary row operations, (ii) using elementary column operations.

Solution

(i) Applying the operations $R_2 \leftarrow R_2 - (3/2)R_1$ and $R_3 \leftarrow R_3 + (1/2)R_1$, we obtain

$$|\mathbf{A}| = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 11/2 \\ 0 & 7/2 & 3/2 \end{vmatrix}$$

Applying the operation $R_3 \leftarrow R_3 - 7R_2$, we obtain

$$|\mathbf{A}| = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 11/2 \\ 0 & 0 & -37 \end{vmatrix} = 2(1/2)(-37) = -37$$

since the value of the determinant of an upper triangular matrix is the product of diagonal elements.

(ii) Applying the operations $C_2 \leftarrow C_2 - (1/2)C_1$ and $C_3 \leftarrow C_3 + (1/2)C_1$, we obtain

$$|\mathbf{A}| = \begin{vmatrix} 2 & 0 & 0 \\ 3 & 1/2 & 11/2 \\ -1 & 7/2 & 3/2 \end{vmatrix}$$

Applying the operation $C_3 \leftarrow C_3 - 11C_2$, we obtain

$$|\mathbf{A}| = \begin{vmatrix} 2 & 0 & 0 \\ 3 & 1/2 & 0 \\ -1 & 7/2 & -37 \end{vmatrix} = 2(1/2)(-37) = -37$$

since the value of the determinant of a lower triangular matrix is the product of diagonal elements.

Example 3.3 Show that

$$|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha).$$

Solution Applying the operations $C_1 \leftarrow C_1 - C_3$ and $C_2 \leftarrow C_2 - C_3$, we get

$$|\mathbf{A}| = \begin{vmatrix} 0 & 0 & 1 \\ \alpha - \gamma & \beta - \gamma & \gamma \\ \alpha^2 - \gamma^2 & \beta^2 - \gamma^2 & \gamma^2 \end{vmatrix} = (\alpha - \gamma)(\beta - \gamma) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & \gamma \\ \alpha + \gamma & \beta + \gamma & \gamma^2 \end{vmatrix}$$

$$= (\alpha - \gamma)(\beta - \gamma) \begin{vmatrix} 1 & 1 \\ \alpha + \gamma & \beta + \gamma \end{vmatrix} = (\alpha - \gamma)(\beta - \gamma)(\beta + \gamma - \alpha - \gamma)$$

$$= (\alpha - \gamma)(\beta - \gamma)(\beta - \alpha) = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha).$$

Example 3.4 Given the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -5 & 2 \\ 1 & 7 & 1 \end{bmatrix}$$

verify that $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.

Solution We have

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 1 & -5 & 2 \\ 1 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 13 & 8 \\ 23 & 25 & 20 \\ 22 & 117 & 0 \end{bmatrix}$$

Therefore, $|\mathbf{AB}| = \begin{vmatrix} 8 & 13 & 8 \\ 23 & 25 & 20 \\ 22 & 117 & 0 \end{vmatrix} = 8(0 - 2340) - 23(0 - 936) + 22(260 - 200)$

$$= -18720 + 21528 + 1320 = 4128.$$

We can find the value of the product $|\mathbf{A}||\mathbf{B}|$ directly as

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix}\begin{vmatrix} 3 & 2 & 1 \\ 1 & -5 & 2 \\ 1 & 7 & 1 \end{vmatrix} = \begin{vmatrix} 10 & -3 & 18 \\ 28 & -9 & 45 \\ 14 & 65 & -40 \end{vmatrix} \quad \text{(multiplying determinants row by row)}$$

$$= 10(360 - 2925) - 28(120 - 1170) + 14(-135 + 162)$$

$$= -25650 + 29400 + 378 = 4128.$$

We can also find $|\mathbf{A}|$ and $|\mathbf{B}|$ and then multiply.

Example 3.5 Without evaluating the determinant, show that

$$D = \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0.$$

Solution Expanding all the terms, we have

$$D = \begin{vmatrix} \cos A\cos P + \sin A\sin P & \cos A\cos Q + \sin A\sin Q & \cos A\cos R + \sin A\sin R \\ \cos B\cos P + \sin B\sin P & \cos B\cos Q + \sin B\sin Q & \cos B\cos R + \sin B\sin R \\ \cos C\cos P + \sin C\sin P & \cos C\cos Q + \sin C\sin Q & \cos C\cos R + \sin C\sin R \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix}\begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix} = 0 \times 0 = 0.$$

Example 3.6 Find all values of μ for which rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$$

is equal to 3.

Solution Since the matrix $\mathbf{A}$ is of order 4, $r(\mathbf{A}) \leq 4$. Now, $r(\mathbf{A}) = 3$, if $|\mathbf{A}| = 0$ and there is at least one submatrix of order 3 whose determinant is not zero. Expanding the determinant through the elements of first row, we get

$$|\mathbf{A}| = \mu \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 11 & -6 & 1 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 \\ 0 & \mu & -1 \\ -6 & -6 & 1 \end{vmatrix} = \mu[\mu(\mu-6)+11]-6$$

$$= \mu^3 - 6\mu^2 + 11\mu - 6 = (\mu-1)(\mu-2)(\mu-3).$$

Setting $|\mathbf{A}| = 0$, we obtain $\mu = 1, 2, 3$. For $\mu = 1, 2, 3$, the determinant of the leading third order submatrix

$$|\mathbf{A}_1| = \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 0 & 0 & \mu \end{vmatrix} = \mu^3 \neq 0.$$

Hence $r(\mathbf{A}) = 3$, when $\mu = 1$ or 2 or 3. For other values of μ, $r(\mathbf{A}) = 4$.

3.2.4 Inverse of a Square Matrix

Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n. Then, $\mathbf{A}$ is called a

(i) *singular matrix* if $|\mathbf{A}| = 0$,
(ii) *non-singular matrix* if $|\mathbf{A}| \neq 0$.

In other words, a square matrix of order n is singular if its rank $r(\mathbf{A}) < n$ and non-singular if its rank $r(\mathbf{A}) = n$. A square non-singular matrix $\mathbf{A}$ of order n is said to be *invertible*, if there exists a non-singular square matrix $\mathbf{B}$ of order n such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \tag{3.11}$$

where $\mathbf{I}$ is an identity matrix of order n. The matrix $\mathbf{B}$ is called the *inverse matrix* of $\mathbf{A}$ and we write $\mathbf{B} = \mathbf{A}^{-1}$ or $\mathbf{A} = \mathbf{B}^{-1}$. Hence, we say that $\mathbf{A}^{-1}$ is the inverse of the matrix $\mathbf{A}$, if

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}. \tag{3.12}$$

The inverse $\mathbf{A}^{-1}$ of the matrix $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\, adj(\mathbf{A}) \tag{3.13}$$

where $adj(\mathbf{A})$ = adjoint matrix of $\mathbf{A}$
= transpose of the matrix of cofactors of $\mathbf{A}$.

Remark 6

(a)
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

This result can be easily proved. We have

$$(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{I}.$$

Premultiplying both sides first by $\mathbf{A}^{-1}$ and then by $\mathbf{B}^{-1}$ we obtain

$$\mathbf{B}^{-1}\mathbf{A}^{-1}(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B}(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \quad \text{or} \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

In general, we have $(\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_p)^{-1} = \mathbf{A}_p^{-1}\mathbf{A}_{p-1}^{-1} \ldots \mathbf{A}_1^{-1}$.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are non-singular matrices, then $\mathbf{AB}$ is also a non-singular matrix.

(c) If $\mathbf{AB} = \mathbf{0}$ and $\mathbf{A}$ is a non-singular matrix, then $\mathbf{B}$ must be a null matrix, since $\mathbf{AB} = \mathbf{0}$ can be premultiplied by $\mathbf{A}^{-1}$. If $\mathbf{B}$ is a non-singular matrix, then $\mathbf{A}$ must be a null matrix, since $\mathbf{AB} = \mathbf{0}$ can be post multiplied by $\mathbf{B}^{-1}$.

(d) If $\mathbf{AB} = \mathbf{AC}$ and $\mathbf{A}$ is a non-singular matrix, then $\mathbf{B} = \mathbf{C}$ (see Remark 1(d)).

(e) $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$, in general.

Properties of inverse matrices

1. If $\mathbf{A}^{-1}$ exists, then it is unique.
2. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
3. $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$, (From $(\mathbf{AA}^{-1})^T = \mathbf{I}^T = \mathbf{I}$, we get $(\mathbf{A}^{-1})^T\mathbf{A}^T = \mathbf{I}$. Hence, the result).
4. Let $\mathbf{D} = \text{diag}\ (d_{11}, d_{22}, \ldots, d_{nn})$, $d_{ii} \neq 0$. Then $\mathbf{D}^{-1} = \text{diag}\ (1/d_{11}, 1/d_{22}, \ldots, 1/d_{nn})$.
5. The inverse of a non-singular upper or lower triangular matrix is respectively an upper or a lower triangular matrix.
6. The inverse of a non-singular symmetric matrix is a symmetric matrix.
7. $(\mathbf{A}^{-1})^n = \mathbf{A}^{-n}$ for any positive integer n.

Example 3.7 Show that the matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I} = \mathbf{0}$ where $\mathbf{I}$ is an identity matrix of order 3. Hence, find the matrix (i) $\mathbf{A}^{-1}$ and (ii) $\mathbf{A}^{-2}$.

Solution We have

$$\mathbf{A}^2 = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\,\mathbf{A} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix}.$$

Substituting in $\mathbf{B} = \mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I}$, we get

$$\mathbf{B} = \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix} - \begin{bmatrix} 24 & -6 & -30 \\ 90 & 6 & -30 \\ 30 & 24 & 54 \end{bmatrix} + \begin{bmatrix} 22 & 0 & -11 \\ 55 & 11 & 0 \\ 0 & 11 & 33 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$$

(i) Premultiplying $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I} = \mathbf{0}$ by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{A}^3 - 6\mathbf{A}^{-1}\mathbf{A}^2 + 11\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1} = \mathbf{0}$$

or $\quad \mathbf{A}^{-1} = \mathbf{A}^2 - 6\mathbf{A} + 11\,\mathbf{I}$

$$= \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix} - \begin{bmatrix} 12 & 0 & -6 \\ 30 & 6 & 0 \\ 0 & 6 & 18 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}.$$

(ii) $$\mathbf{A}^{-2} = (\mathbf{A}^{-1})^2 = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -11 & 10 \\ -160 & 61 & -55 \\ 55 & -21 & 19 \end{bmatrix}.$$

3.2.5 Solution of $n \times n$ Linear System of Equations

Consider the system of n equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n. \end{aligned} \tag{3.14}$$

In matrix form, we can write the system of equations (3.14) as

$$\mathbf{Ax} = \mathbf{b} \tag{3.15}$$

where $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and **A, b, x** are respectively called the *coefficient matrix*, the right hand side column vector and the solution vector. If $\mathbf{b} \neq \mathbf{0}$, that is, at least one of the elements $b_1, b_2, \ldots, b_n$ is not zero, then the system of equations is called *non-homogeneous*. If $\mathbf{b} = \mathbf{0}$, then the system of equations is called *homogeneous*. The system of equations is called *consistent* if it has at least one solution and *inconsistent* if it has no solution.

Non-homogeneous system of equations

The non-homogeneous system of equations $\mathbf{Ax} = \mathbf{b}$ can be solved by the following methods.

Matrix method

Let $\mathbf{A}$ be non-singular. Premultiplying $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^{-1}$, we obtain

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{3.16}$$

The system of equations is consistent and has a unique solution. If $\mathbf{b} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$ (trivial solution) is the only solution.

Cramer's rule

Let $\mathbf{A}$ be non-singular. The Cramer's rule for the solution of $\mathbf{Ax} = \mathbf{b}$ is given by

$$x_i = \frac{|\mathbf{A}_i|}{|\mathbf{A}|}, \quad i = 1, 2, \ldots, n \tag{3.17}$$

where $|\mathbf{A}_i|$ is the determinant of the matrix $\mathbf{A}_i$ obtained by replacing the ith column of $\mathbf{A}$ by the right hand side column vector $\mathbf{b}$.

We discuss the following cases.

Case 1 When $|\mathbf{A}| \neq 0$, the system of equations is consistent and the unique solution is obtained by using Eq. (3.17).

Case 2 When $|\mathbf{A}| = 0$ and one or more of $|\mathbf{A}_i|$, $i = 1, 2, \ldots, n$ are not zero, then the system of equations has no solution, that is the system is inconsistent.

Case 3 When $|\mathbf{A}| = 0$ and all $|\mathbf{A}_i| = 0$, $i = 1, 2, \ldots, n$, then the system of equations is consistent and has infinite number of solutions. The system of equations has at least a one-parameter family of solutions.

Homogeneous system of equations

Consider the homogeneous system of equations

$$\mathbf{Ax} = \mathbf{0}. \tag{3.18}$$

Trivial solution $\mathbf{x} = \mathbf{0}$ is always a solution of this system.

If $\mathbf{A}$ is non-singular, then again $\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$ is the solution.

Therefore, a homogeneous system of equations is always consistent. We conclude that non-trivial solutions for $\mathbf{Ax} = \mathbf{0}$ exist if and only if $\mathbf{A}$ is singular. In this case, the homogeneous system of equations has infinite number of solutions.

Example 3.8 Show that the system of equations

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

has a unique solution. Solve this system using (i) matrix method, (ii) Cramer's rule.

Solution We find that

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1+3) - 2(-1-1) + 1(3-1) = 10 \neq 0.$$

Therefore, the coefficient matrix $\mathbf{A}$ is non-singular and the given system of equations has a unique solution. Let $\mathbf{x} = [x, y, z]^T$.

(i) We obtain

$$\mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

Therefore, $$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence, $x = 2$, $y = -1$ and $z = 1$.

(ii) We have

$$|\mathbf{A}_1| = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 4(1+3) - 0 + 2(3-1) = 20 \cdot$$

$$|\mathbf{A}_2| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 1(0+6) - 2(4-2) + 1(-12-0) = -10 \cdot$$

$$|\mathbf{A}_3| = \begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-0) - 2(-2-4) + 1(0-4) = 10 \cdot$$

Therefore, $$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = 2, \quad y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = -1, \quad z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = 1.$$

Example 3.9 Show that the system of equations

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

has infinite number of solutions. Hence, find the solutions.

Solutions We find that

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix} = 0, \quad |\mathbf{A}_1| = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{vmatrix} = 0,$$

$$|\mathbf{A}_2| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 0, \quad |\mathbf{A}_3| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix} = 0.$$

Therefore, the system of equations has infinite number of solutions. Using the first two equations

$$x_1 - x_2 = 3 - 3x_3$$

$$2x_1 + 3x_2 = 2 - x_3$$

and solving, we obtain $x_1 = (11 - 10\,x_3)/5$ and $x_2 = (5x_3 - 4)/5$ where x_3 is arbitrary. This solution satisfies the third equation.

Example 3.10 Show that the system of equations

$$\begin{bmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix}$$

is inconsistent.

Solution We find that

$$|\,\mathbf{A}\,| = \begin{vmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{vmatrix} = 0, \quad |\,\mathbf{A}_1\,| = \begin{vmatrix} 6 & 9 & 3 \\ 2 & 3 & 1 \\ 7 & 6 & 2 \end{vmatrix} = 0,$$

$$|\,\mathbf{A}_2\,| = \begin{vmatrix} 4 & 6 & 3 \\ 2 & 2 & 1 \\ 2 & 7 & 2 \end{vmatrix} = 6.$$

Since $|\,\mathbf{A}\,| = 0$ and $|\,\mathbf{A}_2\,| \neq 0$, the system of equations is inconsistent.

Example 3.11 Solve the homogeneous system of equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -2 \\ 4 & 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution We find that $|\,\mathbf{A}\,| = 0$. Hence, the given system has infinite number of solutions. Solving the first two equations

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3z \\ 2z \end{bmatrix}$$

we obtain $x = 13z$, $y = -\,8z$ where z is arbitrary. This solution satisfies the third equation.

Exercise 3.1

1. Given the matrices $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, verify that

   (i) $|\,\mathbf{AB}\,| = |\,\mathbf{A}\,|\,|\,\mathbf{B}\,|$, (ii) $|\,\mathbf{A} + \mathbf{B}\,| \neq |\,\mathbf{A}\,| + |\,\mathbf{B}\,|$.

2. If $\mathbf{A}^T = [1, -5, 7]$, $\mathbf{B} = [3, 1, 2]$, verify that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

3. Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I} = \mathbf{0}$. Hence, find $\mathbf{A}^{-1}$.

4. Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^3 - 6\mathbf{A}^2 + 5\mathbf{A} + 11\mathbf{I} = \mathbf{0}$.

   Hence, find $\mathbf{A}^{-1}$.

5. For the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$, verify that

   (i) $[adj\ (\mathbf{A})]^T = adj\ (\mathbf{A}^T)$, (ii) $[adj\ (\mathbf{A})]^{-1} = adj\ (\mathbf{A}^{-1})$.

6. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that

   (i) $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$, (ii) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

7. For the matrices $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 0 & 9 \end{bmatrix}$, verify that

   (i) $adj\ (\mathbf{AB}) = adj\ (\mathbf{A})\ adj\ (\mathbf{B})$, (ii) $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$.

8. For any non-singular matrix $\mathbf{A} = (a_{ij})$ of order n, show that

   (i) $|\ adj\ (\mathbf{A})\ | = |\ \mathbf{A}\ |^{n-1}$, (ii) $adj\ (adj\ (\mathbf{A})) = |\ \mathbf{A}\ |^{n-2}\mathbf{A}$.

9. For any non-singular matrix $\mathbf{A}$, show that $|\ \mathbf{A}^{-1}\ | = 1/|\ \mathbf{A}\ |$.

10. For any symmetric matrix $\mathbf{A}$, show that $\mathbf{BAB}^T$ is symmetric, where $\mathbf{B}$ is any matrix for which the product matrix $\mathbf{BAB}^T$ is defined.

11. If $\mathbf{A}$ is a symmetric matrix, prove that $(\mathbf{BA}^{-1})^T(\mathbf{A}^{-1}\mathbf{B}^T)^{-1} = \mathbf{I}$ where $\mathbf{B}$ is any matrix for which the product matrices are defined.

12. If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices, then prove that

    (i) $\mathbf{A} + \mathbf{B}$ is symmetric, (ii) $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ are both symmetric,

    (iii) $\mathbf{AB} - \mathbf{BA}$ is skew-symmetric.

13. If $\mathbf{A}$ and $\mathbf{B}$ are non-singular, commutative and symmetric matrices, then prove that

    (i) $\mathbf{AB}^{-1}$, (ii) $\mathbf{A}^{-1}\mathbf{B}$, (iii) $\mathbf{A}^{-1}\mathbf{B}^{-1}$

    are symmetric.

14. Let $\mathbf{A}$ be a non-singular matrix. Show that

    (i) if $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots + \mathbf{A}^n = \mathbf{0}$, then $\mathbf{A}^{-1} = \mathbf{A}^n$,

    (ii) if $\mathbf{I} - \mathbf{A} + \mathbf{A}^2 - \ldots + (-1)^n\mathbf{A}^n = \mathbf{0}$, then $\mathbf{A}^{-1} = (-1)^{n-1}\mathbf{A}^n$.

15. Let $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{A}$ be non-singular square matrices of order n and $\mathbf{PAQ} = \mathbf{I}$, then show that $\mathbf{A}^{-1} = \mathbf{QP}$.

16. If $\mathbf{I} - \mathbf{A}$ is a non-singular matrix, then show that

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots$$

    assuming that the series on the right hand side converges.

17. For any three non-singular matrices **A**, **B**, **C**, each of order n, show that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

Establish the following identities:

18. $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$, where w is a cube root of unity.

19. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ac & b(a+c) \\ 1 & ab & c(a+b) \end{vmatrix} = 0.$

20. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0.$

21. $\begin{vmatrix} bc & a^2 & a^2 \\ b^2 & ac & b^2 \\ c^2 & c^2 & ab \end{vmatrix} = \begin{vmatrix} bc & ab & ac \\ ab & ac & bc \\ ac & bc & ab \end{vmatrix}.$

22. $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2.$

23. $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (a+c)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$

24. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$

25. $\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2 = (x^3+y^3+z^3-3xyz)^2.$

26. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta).$

27. $\begin{vmatrix} \sin(a+\alpha) & \sin(b+\alpha) & \sin(c+\alpha) \\ \sin(a+\beta) & \sin(b+\beta) & \sin(c+\beta) \\ \sin(a+\gamma) & \sin(b+\gamma) & \sin(c+\gamma) \end{vmatrix} = 0$ for all a, b, c, α, β and γ.

Solve the following system of equations:

28. $x - y + z = 2,\ 2x + 3y - z = 5,\ x + y - z = 0.$
29. $x + 2y + 3z = 6,\ 2x + 4y + z = 7,\ 3x + 2y + 9z = 14.$
30. $-x + y + 2z = 2,\ 3x - y + z = 3,\ -x + 3y + 4z = 6.$
31. $2x - z = 1,\ 5x + y = 7,\ y + 3z = 5.$
32. Determine the values of k for which the system of equations

$$x - ky + z = 0, \quad kx + 3y - kz = 0, \quad 3x + y - z = 0$$

has (i) only trivial solution, (ii) non-trivial solution.

33. Find the value of θ for which the system of equations

$$2(\sin\theta)x + y - 2z = 0, \quad 3x + 2(\cos 2\theta)y + 3z = 0, \quad 5x + 3y - z = 0$$

has a non-trivial solution.

34. If the system of equations $x + ay + az = 0$, $bx + y + bz = 0$, $cx + cy + z = 0$, where a, b, c are non-zero and non-unity, has a non-trivial solution, then show that

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} = -1.$$

35. Find the values of λ and μ for which the system of equations

$$x + 2y + z = 6, \quad x + 4y + 3z = 10, \quad x + 4y + \lambda z = \mu$$

has a (i) unique solution, (ii) infinite number of solutions, (iii) no solution.

Find the rank of the matrix **A**, where **A** is given by

36. $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$.

37. $\begin{bmatrix} 1 & 3 & -4 \\ -1 & -3 & 4 \\ 2 & 6 & -8 \end{bmatrix}$.

38. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$.

39. $\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ p^3 & q^3 & r^3 \end{bmatrix}$.

40. (a) $\begin{bmatrix} 2 & 1 & 5 & -1 \\ -1 & 2 & 5 & 3 \\ 3 & 2 & 9 & -1 \end{bmatrix}$,

(b) $\begin{bmatrix} 0 & c_1 & -b_1 & a_2 \\ -c_1 & 0 & a_1 & b_2 \\ b_1 & -a_1 & 0 & c_2 \\ -a_2 & -b_2 & -c_2 & 0 \end{bmatrix}$, $a_i, b_i, c_i \neq 0, i = 1, 2$.

41. Prove that if **A** is an Hermitian matrix, then i**A** is a skew-Hermitian matrix and if **A** is a skew-Hermitian matrix, then i**A** is an Hermitian matrix.

42. Prove that if **A** is a real matrix and $\mathbf{A}^n \to \mathbf{0}$ as $n \to \infty$, then $\mathbf{I} + \mathbf{A}$ is invertible.

43. Let **A**, **B** be $n \times n$ real matrices. Then, show that

(i) Trace $(\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha$ Trace $(\mathbf{A}) + \beta$ Trace $(\mathbf{B})$ for any scalars α and β,

(ii) Trace $(\mathbf{AB})$ = Trace $(\mathbf{BA})$,

(iii) $\mathbf{AB} - \mathbf{BA} = \mathbf{I}$ is never true.

44. If **B**, **C** are $n \times n$ matrices, $\mathbf{A} = \mathbf{B} + \mathbf{C}$, $\mathbf{BC} = \mathbf{CB}$ and $\mathbf{C}^2 = \mathbf{0}$, then show that $\mathbf{A}^{p+1} = \mathbf{B}^p[\mathbf{B} + (p+1)\mathbf{C}]$ for any positive integer p.

45. Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n, such that $a_{ij} = d$, $i \neq j$ and $a_{ij} = c$, $i = j$. Then show that $|\mathbf{A}| = (c-d)^{n-1}[c + (n-1)d]$.

Identify the following matrices as symmetric, skew-symmetric, Hermitian, skew-Hermitian or none of these:

46. $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & 4 \\ -3 & -4 & 6 \end{bmatrix}$.

47. $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.

48. $\begin{bmatrix} 0 & b & c \\ -b & 0 & e \\ -c & -e & 0 \end{bmatrix}$.

49. $\begin{bmatrix} 1 & 2+4i & 1-i \\ 2-4i & -5 & 3-5i \\ 1+i & 3+5i & 6 \end{bmatrix}$.

50. $\begin{bmatrix} 1 & 2+4i & 1-i \\ -2+4i & -5 & 3-5i \\ -1-i & -3-5i & 6 \end{bmatrix}$.

51. $\begin{bmatrix} 0 & 2+4i & 1-i \\ -2+4i & 0 & 3-5i \\ -1-i & -3-5i & 0 \end{bmatrix}$.

52. $\begin{bmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{bmatrix}$.

53. $\begin{bmatrix} 0 & -i & 1+i \\ -i & -2i & 0 \\ -1+i & 0 & i \end{bmatrix}$.

54. $\begin{bmatrix} 1 & -1 & i \\ -1 & 0 & 1-i \\ -i & 1+i & 2 \end{bmatrix}$.

55. $\begin{bmatrix} 1 & 2i & -i \\ -2i & i & 1 \\ i & 1 & 2 \end{bmatrix}$.

3.3 Vector Spaces

Let V be a non-empty set of certain objects, which may be vectors, matrices, functions or some other objects. Each object is an element of V and is called a vector. The elements of V are denoted by **a**, **b**, **c**, **u**, **v**, etc. Assume that the two algebraic operations

(i) vector addition and (ii) scalar multiplication

are defined on elements of V.

If the vector addition is defined as the usual addition of vectors, then

$$\mathbf{a} + \mathbf{b} = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

If the scalar multiplication is defined as the usual scalar multiplication of a vector by the scaler α, then

$$\alpha\mathbf{a} = \alpha(a_1, a_2, \ldots, a_n) = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n).$$

The set V defines a vector space if for any elements **a**, **b**, **c** in V and any scalars α, β the following properties (axioms) are satisfied.

Properties (axioms) with respect to vector addition

1. $\mathbf{a} + \mathbf{b}$ is in V.
2. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. (commutative law)
3. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$. (associative law)
4. $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$. (existence of a unique zero element in V)
5. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$. (existence of additive inverse or negative vector in V)

Properties (axioms) with respect to scalar multiplication

6. $\alpha\mathbf{a}$ is in V.
7. $(\alpha + \beta)\,\mathbf{a} = \alpha\mathbf{a} + \beta\,\mathbf{a}$. (left distributive law)
8. $(\alpha\beta)\,\mathbf{a} = \alpha\,(\beta\,\mathbf{a})$.
9. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$. (right distributive law)
10. $1\mathbf{a} = \mathbf{a}$. (existence of multiplicative inverse)

The properties defined in **1** and **6** are called the *closure* properties. When these two properties are satisfied, we say that the vector space is closed under the vector addition and scalar multiplication. The vector addition and scalar multiplication defined above need not always be the usual addition and multiplication operators. Thus, *the vector space depends not only on the set V of vectors, but also on the definition of vector addition and scalar multiplication on V.*

If the elements of V are real, then it is called a *real vector space* when the scalars α, β are real numbers, whereas V is called a *complex vector space*, if the elements of V are complex and the scalars α, β may be real or complex numbers or if the elements of V are real and the scalars α, β are complex numbers.

Remark 7

(a) If even one of the above properties is not satisfied, then V is not a vector space. We usually check the closure properties first before checking the other properties.

(b) The concepts of length, dot product, vector product etc. are not part of the properties to be satisfied.

(c) The set of real numbers and complex numbers are called *fields* of scalars. We shall consider vector spaces only on the fields of scalars. In an advanced course on linear algebra, vector spaces over arbitrary fields are considered.

(d) The vector space $V = \{\mathbf{0}\}$ is called a trivial vector space.

The following are some examples of vector spaces under the usual operations of vector addition and scalar multiplication.

1. The set V of real or complex numbers.
2. The set of real valued continuous functions f on any closed interval $[a, b]$. The $\mathbf{0}$ vector defined in property **4** is the zero function.
3. The set of polynomials P_n of degree less than or equal to n.
4. The set V of n-tuples in $\mathbb{R}^n$ or $\mathbb{C}^n$.
5. The set V of all $m \times n$ matrices. The element $\mathbf{0}$ defined in property **4** is the null matrix of order $m \times n$.

The following are some examples which are not vector spaces. Assume that usual operations of vector addition and scalar multiplication are being used.

1. The set V of all polynomials of degree n. Let P_n and Q_n be two polynomials of degree n in V. Then, $\alpha P_n + \beta Q_n$ need not be a polynomial of degree n and thus may not be in V. For example, if $P_n = x^n + a_1 x^{n-1} + \ldots + a_n$ and $Q_n = -x^n + b_1 x^{n-1} + \ldots + b_n$, then $P_n + Q_n$ is a polynomial of degree $(n - 1)$.
2. The set V of all real-valued functions of one variable x, defined and continuous on the closed interval $[a, b]$ such that the value of the function at b is some non-zero constant p. For example, let $f(x)$ and $g(x)$ be two elements in V. Now, $f(b) = g(b) = p$. Since $f(b) + g(b) = 2p$, $f(x) + g(x)$ is not in V. Note that if $p = 0$, then V forms a vector space.

Example 3.12 Let V be the set of all polynomials, with real coefficients, of degree n, where addition is defined by $\mathbf{a} + \mathbf{b} = \mathbf{ab}$ and under usual scalar multiplication. Show that V is not a vector space.

Solution Let P_n and Q_n be two elements in V. Now, $P_n + Q_n = (P_n)(Q_n)$ is a polynomial of degree $2n$, which is not in V. Therefore, V does not define a vector space.

Example 3.13 Let V be the set of all ordered pairs (x, y), where x, y are real numbers. Let $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ be two elements in V. Define the addition as

$$\mathbf{a} + \mathbf{b} = (x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$$

and the scalar multiplication as

$$\alpha(x_1, y_1) = (\alpha x_1/3, \alpha y_1/3).$$

Show that V is not a vector space. Which of the properties are not satisfied?

Solution We illustrate the properties that are not satisfied.

(i) $(x_2, y_2) + (x_1, y_1) = (2x_2 - 3x_1, y_2 - y_1) \neq (x_1, y_1) + (x_2, y_2)$.

Therefore, property **2** (commutative law) does not hold.

(ii)
$$\begin{aligned}((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (2x_1 - 3x_2, y_1 - y_2) + (x_3, y_3)\\ &= (4x_1 - 6x_2 - 3x_3, y_1 - y_2 - y_3)\\ (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (2x_2 - 3x_3, y_2 - y_3)\\ &= (2x_1 - 6x_2 + 9x_3, y_1 - y_2 + y_3).\end{aligned}$$

Therefore, property **3** (associative law) is not satisfied.

(iii) $1(x_1, y_1) = (x_1/3, y_1/3) \neq (x_1, y_1)$.

Therefore, property **10** (existence of multiplicative inverse) is not satisfied.

Hence, V is not a vector space.

Example 3.14 Let V be the set of all ordered pairs (x, y), where x, y are real numbers. Let $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ be two elements in V. Define the addition as

$$\mathbf{a} + \mathbf{b} = (x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

and the scalar multiplication as

$$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1).$$

Show that V is not a vector space. Which of the properties are not satisfied?

Solution Note that $(1, 1)$ is an element of V. From the given definition of vector addition, we find that

$$(x_1, y_1) + (1, 1) = (x_1, y_1).$$

and this is true only for the element $(1, 1)$. Therefore, the element $(1, 1)$ plays the role of $\mathbf{0}$ element as defined in property **4**. Now, there is no element in V for which $(\mathbf{a}) + (-\mathbf{a}) = \mathbf{0} = (1, 1)$, since

$$(x_1, y_1) + (-x_1, -y_1) = (-x_1^2, -y_1^2) \neq (1, 1).$$

Therefore, property **5** is not satisfied.

Now, let $\alpha = 1$, $\beta = 2$ be any two scalars. We have

$$(\alpha + \beta)(x_1, y_1) = 3(x_1, y_1) = (3x_1, 3y_1)$$

and $$\alpha(x_1, y_1) + \beta(x_1, y_1) = 1(x_1, y_1) + 2(x_1, y_1) = (x_1, y_1) + (2x_1, 2y_1) = (2x_1^2, 2y_1^2)$$

Therefore, $(\alpha + \beta)(x_1, y_1) \neq \alpha(x_1, y_1) + \beta(x_1, y_1)$ and property **7** is not satisfied.

Similarly, it can be shown that property **9** is not satisfied. Hence, V is not a vector space.

3.3.1 Subspaces

Let V be an arbitrary vector space defined under a given vector addition and scalar multiplication. A non-empty subset W of V, such that W is also a vector space under the same two operations of vector addition and scalar multiplication, is called a *subspace* of V. Thus, W is also closed under the two given algebraic operations on V. As a convention, the vector space V is also taken as a subspace of V.

Remark 8

To show that W is a subspace of a vector space V, it is not necessary to verify all the 10 properties as given in section 3.3. If it is shown that W is closed under the given definition of vector addition and scalar multiplication, then the properties **2, 3, 7, 8, 9** and **10** are automatically satisfied because these properties are valid for all elements in V and hence are also valid for all elements in W. Thus, we need to verify the remaining properties, that is, the existence of the zero element and the additive inverse in W.

Consider the following examples:

1. Let V be the set of n-tuples $(x_1\ x_2 \ldots x_n)$ in $\mathbb{R}^n$ with usual addition and scalar multiplication. Then
   - **(i)** W consisting of n-tuples $(x_1\ \ x_2\ \ \ldots\ \ x_n)$ with $x_1 = 0$ is a subspace of V.
   - **(ii)** W consisting of n-tuples $(x_1\ \ x_2\ \ \ldots\ \ x_n)$ with $x_1 \geq 0$ is not a subspace of V, since W is not closed under scalar multiplication ($\alpha\mathbf{x}$, when α is a negative real number, is not in W).
   - **(iii)** W consisting of n-tuples $(x_1\ \ x_2\ \ \ldots\ \ x_n)$ with $x_2 = x_1 + 1$ is not a subspace of V, since W is not closed under addition.
     (Let $\mathbf{x} = (x_1\ \ x_2\ \ \ldots\ \ x_n)$ with $x_2 = x_1 + 1$ and $\mathbf{y} = (y_1\ \ y_2\ \ \ldots\ \ y_n)$ with $y_2 = y_1 + 1$ be two elements in W. Then
     $$\mathbf{x} + \mathbf{y} = (x_1 + y_1\ .x_2 + y_2\ \ \ldots\ \ x_n + y_n)$$
     is not in W as $x_2 + y_2 = x_1 + y_1 + 2 \neq x_1 + y_1 + 1$).
2. Let V be the set of all real polynomials P of degree $\leq m$ with usual addition and scalar multiplication. Then
   - **(i)** W consisting of all real polynomials of degree $\leq m$ with $P(0) = 0$ is a subspace of V.
   - **(ii)** W consisting of all real polynomials of degree $\leq m$ with $P(0) = 1$ is not a subspace of V. since W is not closed under addition (If P and $Q \in W$, then $P + Q \notin W$).
   - **(iii)** W consisting of all polynomials of degree $\leq m$ with real positive coefficients is not a subspace of V since W is not closed under scalar multiplication (If P is an element of W, then $-P \notin W$).
3. Let V be the set of all $n \times n$ real square matrices with usual matrix addition and scalar multiplication. Then

(i) W consisting of all symmetric/skew-symmetric matrices of order n is a subspace of V.

(ii) W consisting of all upper/lower triangular matrices of order n is a subspace of V.

(iii) W consisting of all $n \times n$ matrices having real positive elements is not a subspace of V since W is not closed under scalar multiplication (if $\mathbf{A}$ is an element of W, then $-\mathbf{A} \notin W$).

4. Let V be the set of all $n \times n$ complex matrices with usual matrix addition and scalar multiplication. Then

(i) W consisting of all Hermitian matrices of order n forms a vector space when scalars are real numbers and does not form a vector space when scalars are complex numbers (W is not closed under scalar multiplication).

Let
$$\mathbf{A} = \begin{pmatrix} a & x+iy \\ x-iy & b \end{pmatrix} \in W.$$

Let $\alpha = i$. We get $\alpha\mathbf{A} = i\mathbf{A} = \begin{pmatrix} ai & xi-y \\ xi+y & bi \end{pmatrix} \notin W.$

(ii) W consisting of all skew-Hermitian matrices of order n forms a vector space when scalars are real numbers and does not form a vector space when scalars are complex numbers.

Let
$$\mathbf{A} = \begin{pmatrix} i & x+iy \\ -x+iy & 2i \end{pmatrix} \in W.$$

Let $\alpha = i$. We get $i\mathbf{A} - \begin{pmatrix} -1 & ix-y \\ -ix-y & -2 \end{pmatrix} \notin W.$

Example 3.15 Let F and G be subspaces of a vector space V such that $F \cap G = \{\mathbf{0}\}$. The *sum* of F and G is written as $F + G$ and is defined by

$$F + G = \{\mathbf{f} + \mathbf{g} : \mathbf{f} \in F, \mathbf{g} \in G\}.$$

Show that $F + G$ is a subspace of V assuming the usual definition of vector addition and scalar multiplication.

Solution Let $W = F + G$ and $\mathbf{f} \in F$, $\mathbf{g} \in G$. Since $\mathbf{0} \in F$ and $\mathbf{0} \in G$, we have $\mathbf{0} + \mathbf{0} = \mathbf{0} \in W$. Let $\mathbf{f}_1 + \mathbf{g}_1$ and $\mathbf{f}_2 + \mathbf{g}_2$ belong to W where $\mathbf{f}_1, \mathbf{f}_2 \in F$ and $\mathbf{g}_1, \mathbf{g}_2 \in G$. Then

$$(\mathbf{f}_1 + \mathbf{g}_1) + (\mathbf{f}_2 + \mathbf{g}_2) = (\mathbf{f}_1 + \mathbf{f}_2) + (\mathbf{g}_1 + \mathbf{g}_2) \in F + G = W.$$

Also for any scalar α,
$$\alpha(\mathbf{f} + \mathbf{g}) = \alpha\mathbf{f} + \alpha\mathbf{g} \in F + G = W.$$

Therefore, $W = F + G$ is a subspace of V.

We now present an important result on subspaces.

Theorem 3.1 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be any r elements of a vector space V under usual vector addition and scalar multiplication. Then, the set of all linear combinations of these elements, that is the set of all elements of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_r \mathbf{v}_r \tag{3.19}$$

is a subspace of V, where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are scalars.

Proof Let W be the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Let

$$\mathbf{w}_1 = \sum_{i=1}^{r} a_i \mathbf{v}_i \quad \text{and} \quad \mathbf{w}_2 = \sum_{i=1}^{r} b_i \mathbf{v}_i$$

be any two linear combinations (any two elements of W). Then,

$$\mathbf{w}_1 + \mathbf{w}_2 = (a_1 + b_1)\mathbf{v}_1 + (a_2 + b_2)\mathbf{v}_2 + \ldots + (a_r + b_r)\mathbf{v}_r \in W$$

$$\alpha \mathbf{w}_1 = (\alpha a_1)\mathbf{v}_1 + (\alpha a_2)\mathbf{v}_2 + \ldots + (\alpha a_r)\mathbf{v}_r \in W$$

$$\alpha \mathbf{w}_2 = (\alpha b_1)\mathbf{v}_1 + (\alpha b_2)\mathbf{v}_2 + \ldots + (\alpha b_r)\mathbf{v}_r \in W$$

and $\quad \alpha(\mathbf{w}_1 + \mathbf{w}_2) = \alpha \mathbf{w}_1 + \alpha \mathbf{w}_2$.

Taking $\alpha = 0$, we find that $0\mathbf{w}_1 = \mathbf{0} \in W$. This implies that $\mathbf{w}_1 + \mathbf{0} = \mathbf{0} + \mathbf{w}_1 = \mathbf{w}_1$

Taking $\alpha = -1$, we find that $(-1)\mathbf{w}_1 = (-\mathbf{w}_1) \in W$. This implies that $\mathbf{w}_1 + (-\mathbf{w}_1) = \mathbf{0}$

Therefore, W is a subspace of V.

The elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are in the subspace W as

$$\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_r,\ \mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2 + \ldots + 0\mathbf{v}_r, \ldots$$

We say that the subspace W is *spanned* by the r elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Also, any subspace that contains the elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ must contain every linear combination of these elements.

Spanning set Let S be a subset of a vector space V and suppose that every element in V can be obtained as a linear combination of the elements taken from S. Then S is said to be the *spanning set* for V. We also say that S spans V.

Example 3.16 Let V be the vector space of all 2×2 real matrices. Show that the sets

(i) $$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

(ii) $$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

span V.

Solution Let $\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary element of V.

(i) We write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since every element of V can be written as a linear combination of the elements of S, the set S spans the vector space V.

(ii) We need to determine the scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Equating the corresponding elements, we obtain the system of equations

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = a, \quad \alpha_2 + \alpha_3 + \alpha_4 = b,$$

$$\alpha_3 + \alpha_4 = c, \qquad \alpha_4 = d.$$

The solution of this system of equations is

$$\alpha_4 = d,\ \alpha_3 = c - d,\ \alpha_2 = b - c,\ \alpha_1 = a - b.$$

Therefore, we can write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a-b) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (b-c) \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (c-d) \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since every element of V can be written as a linear combination of the elements of S, the set S spans the vector space V.

Example 3.17 Let V be the vector space of all polynomials of degree ≤ 3. Determine whether or not the set

$$S = \{t^3, t^2 + t, t^3 + t + 1\}$$

spans V?

Solution Let $P(t) = \alpha t^3 + \beta t^2 + \gamma t + \delta$ be an arbitrary element in V. We need to find whether or not there exist scalars a_1, a_2, a_3 such that

$$\alpha t^3 + \beta t^2 + \gamma t + \delta = a_1 t^3 + a_2 (t^2 + t) + a_3 (t^3 + t + 1)$$

or

$$\alpha t^3 + \beta t^2 + \gamma t + \delta = (a_1 + a_3)\, t^3 + a_2 t^2 + (a_2 + a_3)\, t + a_3.$$

Comparing the coefficients of various powers of t, we get

$$a_1 + a_3 = \alpha,\ a_2 = \beta,\ a_2 + a_3 = \gamma,\ a_3 = \delta.$$

The solution of the first three equations is given by

$$a_1 = \alpha + \beta - \gamma,\ a_2 = \beta,\ a_3 = \gamma - \beta.$$

Substituting in the last equation, we obtain $\gamma - \beta = \delta$, which may not be true for all elements in V. For example, the polynomial $t^3 + 2t^2 + t + 3$ does not satisfy this condition and therefore, it cannot be written as a linear combination of the elements of S. Therefore, S does not span the vector space V.

3.3.2 Linear Independence of Vectors

Let V be a vector space. A finite set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ of the elements of V is said to be *linearly dependent* if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{0}. \tag{3.20}$$

If Eq. (3.20) is satisfied only for $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, then the set of vectors is said to be *linearly independent.*

The above definition of linear dependence of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ can be written alternately as follows.

Theorem 3.2 The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent if and only if at least one element of the set is a linear combination of the remaining elements.

Proof Let the elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly dependent. Then, there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_{i-1}\mathbf{v}_{i-1} + \alpha_i\mathbf{v}_i + \alpha_{i+1}\mathbf{v}_{i+1} + \ldots + \alpha_n\mathbf{v}_n = \mathbf{0}.$$

Let $\alpha_i \neq 0$. Then, we can write

$$\mathbf{v}_i = -\left(\frac{\alpha_1}{\alpha_i}\right)\mathbf{v}_1 - \left(\frac{\alpha_2}{\alpha_i}\right)\mathbf{v}_2 - \ldots - \left(\frac{\alpha_{i-1}}{\alpha_i}\right)\mathbf{v}_{i-1} - \left(\frac{\alpha_{i+1}}{\alpha_i}\right)\mathbf{v}_{i+1} - \ldots - \left(\frac{\alpha_n}{\alpha_i}\right)\mathbf{v}_n$$

$$= \alpha_1^*\mathbf{v}_1 + \alpha_2^*\mathbf{v}_2 + \ldots + \alpha_{i-1}^*\mathbf{v}_{i-1} + \alpha_{i+1}^*\mathbf{v}_{i+1} + \ldots + \alpha_n^*\mathbf{v}_n$$

where $\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*$ are some scalars. Hence, the vector $\mathbf{v}_i$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n$.

Now let $\mathbf{v}_i$ be a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n$. Therefore, we have

$$\mathbf{v}_i = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_{i-1}\mathbf{v}_{i-1} + a_{i+1}\mathbf{v}_{i+1} + \ldots + a_n\mathbf{v}_n$$

where a_i's are scalars. Then

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_{i-1}\mathbf{v}_{i-1} + (-1)\,\mathbf{v}_i + a_{i+1}\mathbf{v}_{i+1} + \ldots + a_n\mathbf{v}_n = \mathbf{0}.$$

Since the coefficient of $\mathbf{v}_i$ is not zero, the elements are linearly dependent.

Remark 9

Eq. (3.20) gives a homogeneous system of algebraic equations. Non-trivial solutions exist if *det* (coefficient matrix) = 0, that is the vectors are linearly dependent in this case. If the *det* (coefficient matrix) $\neq 0$, then by Cramer's rule, $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ and the vectors are linearly independent.

Example 3.18 Let $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (0, 1, -1)$ and $\mathbf{v}_3 = (0, 0, 1)$ be elements of $\mathbb{R}^3$. Show that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Solution We consider the vector equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}.$$

Substituting for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we obtain

$$\alpha_1(1, -1, 0) + \alpha_2(0, 1, -1) + \alpha_3(0, 0, 1) = \mathbf{0}$$

or

$$(\alpha_1, -\alpha_1 + \alpha_2, -\alpha_2 + \alpha_3) = \mathbf{0}.$$

Comparing, we obtain $\alpha_1 = 0$, $-\alpha_1 + \alpha_2 = 0$ and $-\alpha_2 + \alpha_3 = 0$. The solution of these equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, the given set of vectors is linearly independent.

Alternative

$$det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1 \neq 0.$$

Therefore, the given vectors are linearly independent.

Example 3.19 Let $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (0, 1, -1)$, $\mathbf{v}_3 = (0, 2, 1)$ and $\mathbf{v}_4 = (1, 0, 3)$ be elements of $\mathbb{R}^3$. Show that the set of vectors $\{\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4\}$ is linearly dependent.

Solution The given set of elements will be linearly dependent if there exists scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4 = \mathbf{0}. \tag{3.21}$$

Substituting for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and comparing, we obtain

$$\alpha_1 + \alpha_4 = 0, -\alpha_1 + \alpha_2 + 2\alpha_3 = 0, -\alpha_2 + \alpha_3 + 3\alpha_4 = 0.$$

The solution of this system of equations is

$$\alpha_1 = -\alpha_4,\ \alpha_2 = 5\alpha_4/3,\ \alpha_3 = -4\alpha_4/3,\ \alpha_4 \text{ arbitrary}.$$

Substituting in Eq. (3.21) and cancelling α_4, we obtain

$$-\mathbf{v}_1 + \frac{5}{3}\mathbf{v}_2 - \frac{4}{3}\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}.$$

Hence, there exist scalars not all zero, such that Eq. (3.21) is satisfied. Therefore, the set of vectors is linearly dependent.

3.3.3 Dimension and Basis

Let V be a vector space. If for some positive integer n, there exists a set S of n linearly independent elements of V and if every set of $n + 1$ or more elements in V is linearly dependent, then V is said to have *dimension n*. Then, we write $dim(V) = n$. Thus, the maximum number of linearly independent elements of V is the dimension of V. The set S of n linearly independent vectors is called the *basis* of V. Note that a vector space whose only element is zero has dimension zero.

Theorem 3.3 Let V be a vector space of dimension n. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the linearly independent elements of V. Then, every other element of V can be written as a linear combination of these elements. Further, this representation is unique.

Proof Let $\mathbf{v}$ be an element of V. Then, the set $\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent as it has $n + 1$ elements. Therefore, there exist scalars $\alpha_0, \alpha_1, \ldots, \alpha_n$, not all zero, such that

$$\alpha_0 \mathbf{v} + \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{0}. \tag{3.22}$$

Now, $\alpha_0 \neq 0$. Because, if $\alpha_0 = 0$, we get $\alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{0}$ and since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, we get $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$. This implies that the set of $n + 1$ elements $\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly independent, which is not possible as the dimension of V is n.

Therefore, we obtain from Eq. (3.22)

$$\mathbf{v} = \sum_{i=1}^{n} (-\alpha_i/\alpha_0)\mathbf{v}_i. \tag{3.23}$$

Hence, $\mathbf{v}$ is a linear combination of n linearly independent vectors of V.

Now, let there be two representations of $\mathbf{v}$ given by

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n \quad \text{and} \quad \mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_n\mathbf{v}_n$$

where $b_i \neq a_i$ for at least one i. Subtracting these two equations, we get

$$\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \ldots + (a_n - b_n)\mathbf{v}_n.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, we get

$$a_i - b_i = 0 \quad \text{or} \quad a_i = b_i,\ i = 1, 2, \ldots, n.$$

Therefore, both the representations of $\mathbf{v}$ are same and the representation of $\mathbf{v}$ given by Eq. (3.23) is unique.

Remark 10

(a) A set of $(n + 1)$ vectors in $\mathbb{R}^n$ is linearly dependent.

(b) A set of vectors containing $\mathbf{0}$ as one of its elements is linearly dependent as $\mathbf{0}$ is the linear combination of any set of vectors.

Theorem 3.4 Let V be an n-dimensional vector space. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, $k < n$ are linearly independent elements of V, then there exist elements $\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \ldots, \mathbf{v}_n$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V.

Proof There exists an element $\mathbf{v}_{k+1}$ such that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly independent. Otherwise, every element of V can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and therefore V has dimension $k < n$. This argument can be continued. If $n > k + 1$, we keep adding elements $\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \ldots, \mathbf{v}_n$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V.

Since all the elements of a vector space V of dimension n can be represented as linear combinations of the n elements in the basis of V, the basis of V spans V. However, there can be many basis for the same vector space. For example, consider the vector space $\mathbb{R}^3$. Each of the following set of vectors

(i) [1, – 1, 0], [0, 1, – 1], [0, 0, 1]

(ii) [1, – 1, 0], [0, 0, 1], [1, 2, 3]

(iii) [1, 0, 0], [0, 1, 0], [0, 0, 1]

are linearly independent and therefore forms a basis in $\mathbb{R}^3$. Some of the standard basis are the following.

1. If V consists of n-tuples in $\mathbb{R}^n$, then

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \mathbf{e}_n = (0, 0, \ldots, 0, 1)$$

is called a standard basis in $\mathbb{R}^n$.

2. If V consists of all $m \times n$ matrices, then

$$\mathbf{E}_{rs} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}, r = 1, 2, \ldots, m \ \text{ and } \ s = 1, 2, \ldots, n$$

where 1 is located in the (r, s) location, that is in the rth row and the sth column, is called its standard basis.

For example, if V consists of all 2×3 matrices, then any matrix $\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$ in V can be written as

$$\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = a\,\mathbf{E}_{11} + b\,\mathbf{E}_{12} + c\,\mathbf{E}_{13} + x\,\mathbf{E}_{21} + y\,\mathbf{E}_{22} + z\,\mathbf{E}_{23}$$

where $\mathbf{E}_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{E}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ etc.

3. If V consists of all polynomials $P(t)$ of degree $\leq n$, then $\{1, t, t^2, \ldots, t^n\}$ is taken as its standard basis.

Example 3.20 Determine whether the following set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis in $\mathbb{R}^3$, where

(i) $\mathbf{u} = (2, 2, 0)$, $\mathbf{v} = (3, 0, 2)$, $\mathbf{w} = (2, -2, 2)$

(ii) $\mathbf{u} = (0, 1, -1)$, $\mathbf{v} = (-1, 0, -1)$, $\mathbf{w} = (3, 1, 3)$.

Solution If the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis in $\mathbb{R}^3$, then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ must be linearly independent. Let $\alpha_1, \alpha_2, \alpha_3$ be scalars. Then, the only solution of the equation

$$\alpha_1\mathbf{u} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w} = \mathbf{0} \tag{3.24}$$

must be $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

(i) Using Eq. (3.24), we obtain the system of equations

$$2\alpha_1 + 3\alpha_2 + 2\alpha_3 = 0,\ 2\alpha_1 - 2\alpha_3 = 0 \quad \text{and} \quad 2\alpha_2 + 2\alpha_3 = 0.$$

The solution of this system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent and they form a basis in $\mathbb{R}^3$.

(ii) Using Eq. (3.24), we obtain the system of equations

$$-\alpha_2 + 3\alpha_3 = 0,\ \alpha_1 + \alpha_3 = 0,\ -\alpha_1 - \alpha_2 + 3\alpha_3 = 0.$$

The solution of this system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent and they form a basis in $\mathbb{R}^3$.

Example 3.21 Find the dimension of the subspace of $\mathbb{R}^4$ spanned by the set $\{(1\ \ 0\ \ 0\ \ 0), (0\ \ 1\ \ 0\ \ 0), (1\ \ 2\ \ 0\ \ 1), (0\ \ 0\ \ 0\ \ 1)\}$. Hence find its basis.

Solution The dimension of the set is ≤ 4. If it is 4, then the only solution of the vector equation

$$\alpha_1(1\ \ 0\ \ 0\ \ 0) + \alpha_2(0\ \ 1\ \ 0\ \ 0) + \alpha_3(1\ \ 2\ \ 0\ \ 1) + \alpha_4(0\ \ 0\ \ 0\ \ 1) = \mathbf{0} \tag{3.25a}$$

should be $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Comparing, we obtain the system of equations

$$\alpha_1 + \alpha_3 = 0,\ \alpha_2 + 2\alpha_3 = 0,\ \alpha_3 + \alpha_4 = 0.$$

The solution of this system of equations is given by

$$\alpha_1 = \alpha_4,\ \alpha_2 = 2\alpha_4,\ \alpha_3 = -\alpha_4, \text{ where } \alpha_4 \text{ is arbitrary.}$$

Hence, the vector equation (3.25a) is satisfied for non-zero values of $\alpha_1, \alpha_2, \alpha_3$ and α_4. Therefore, the dimension of the set is less than 4.

Now, consider any three elements of the set, say (1 0 0 0), (0 1 0 0) and (1 2 0 1). Consider the vector equation

$$\alpha_1(1 \quad 0 \quad 0 \quad 0) + \alpha_2(0 \quad 1 \quad 0 \quad 0) + \alpha_3(1 \quad 2 \quad 0 \quad 1) = \mathbf{0}. \tag{3.25b}$$

Comparing, we obtain the system of equations

$$\alpha_1 + \alpha_3 = 0, \ \alpha_2 + 2\alpha_3 = 0 \quad \text{and} \quad \alpha_3 = 0.$$

The solution of this system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Hence, these three elements are linearly independent. Therefore, the dimension of the given set is 3 and the basis is the set of vectors {(1 0 0 0), (0 1 0 0), (1 2 0 1)}. We find that the fourth vector can be written as

$$(0 \quad 0 \quad 0 \quad 1) = -(1 \quad 0 \quad 0 \quad 0) - 2\,(0 \quad 1 \quad 0 \quad 0) + 1(1 \quad 2 \quad 0 \quad 1).$$

Example 3.22 Let $\mathbf{u} = \{(a, b, c, d)$, such that $a + c + d = 0, \ b + d = 0\}$ be a subspace of $\mathbb{R}^4$. Find the dimension and the basis of the subspace.

Solution $\mathbf{u}$ satisfies the closure properties. From the given equations, we have

$$a + c + d = 0 \quad \text{and} \quad b + d = 0 \quad \text{or} \quad a = -c - d \quad \text{and} \quad b = -d.$$

We have two free parameters, say, c and d. Therefore, the dimension of the given subspace is 2. Choosing $c = 0, d = 1$ and $c = 1, d = 0$, we may write a basis as {(– 1 –1 0 1), (– 1 0 1 0)}.

3.3.4 Lnear Transformations

Let A and B be two arbitrary sets. A rule that assigns to elements of A exactly one element of B is called a *function* or a *mapping* or a *transformation*. Thus, a transformation maps the elements of A into the elements of B. The set A is called the *domain* of the transformation. We use capital letters T, S etc. to denote a transformation. If T is a transformation from A into B, we write

$$T : A \to B. \tag{3.26}$$

For each element $\mathbf{a} \in A$, we get a unique element $\mathbf{b} \in B$. We write $\mathbf{b} = T(\mathbf{a})$ or $\mathbf{b} = T\mathbf{a}$ and $\mathbf{b}$ is called the image of $\mathbf{a}$ under the mapping T. The collection of all such images in B is called the *range* or the image set of the transformation T.

In this section, we shall discuss mappings from a vector space into a vector space. let V and W be two vector spaces, both real or complex, over the same field F of scalars. Let T be a mapping from V into W. The mapping T is said to be a *linear transformation* or a *linear mapping*, if it satisfies the following two properties:

(i) For every scalar α and every element $\mathbf{v}$ in V

$$T(\alpha\mathbf{v}) = \alpha T(\mathbf{v}). \tag{3.27}$$

(ii) For any two elements $\mathbf{v}_1$, $\mathbf{v}_2$ in V

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2). \tag{3.28}$$

Since V is a vector space, the product $\alpha\mathbf{v}$ and the sum $\mathbf{v}_1 + \mathbf{v}_2$ are defined and are elements in V. Then, T defines a mapping from V into W. Since $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are in W, the product $\alpha T(\mathbf{v})$ and the sum $T(\mathbf{v}_1) + T(\mathbf{v}_2)$ are in W. The conditions given in Eqs. (3.27) and (3.28) are equivalent to

$$T(\alpha\,\mathbf{v}_1 + \beta\,\mathbf{v}_2) = T(\alpha\,\mathbf{v}_1) + T(\alpha\,\mathbf{v}_2) = \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2)$$

for $\mathbf{v}_1$ and $\mathbf{v}_2$ in V and any scalars α, β.

Let V be a vector space of dimension n and let the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be its basis. Then, any element $\mathbf{v}$ in V can be written as a linear combination of the elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Therefore,
$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$$
where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars, not all zero. If T is a linear transformation defined in V, then
$$\begin{aligned} T(\mathbf{v}) &= T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n) \\ &= T(\alpha_1\mathbf{v}_1) + T(\alpha_2\mathbf{v}_2) + \ldots + T(\alpha_n\mathbf{v}_n) \\ &= \alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \ldots + \alpha_n T(\mathbf{v}_n). \end{aligned}$$
Thus, a linear transformation is completely determind by its action on the basis vectors of a vector space.

Letting $\alpha = 0$ in Eq. (3.27), we find that for every element $\mathbf{v}$ in V
$$T(0\,\mathbf{v}) = T(\mathbf{0}) = 0T(\mathbf{v}) = \mathbf{0}.$$
Therefore, the zero element in V is mapped into zero element in W by the linear transformation T.

The collection of all elements $\mathbf{w} = T(\mathbf{v})$ is called the *range* of T and is written as *ran* (T). The set of all elements of V that are mapped into the zero element by the linear transformation T is called the *kernel* or the *null-space* of T and is denoted by *ker* (T). Therefore, we have
$$ker(T) = \{\mathbf{v} \mid T(\mathbf{v}) = \mathbf{0}\} \quad \text{and} \quad ran(T) = \{T(\mathbf{v}) \mid \mathbf{v} \in V\}.$$
Thus, the null space of T is a subspace of V and the range of T is a subspace of W.

The dimension of *ran* (T) is called the rank (T) and the dimension of *ker* (T) is called the *nullity* of T. We have the following result.

Theorem 3.5 If T has rank r and the dimension of V is n, then the nullity of T is $n - r$, that is,
$$\text{rank}(T) + \text{nullity} = n = dim(V).$$
We shall discuss the linear transformation only in the context of matrices.

Let $\mathbf{A}$ be an $m \times n$ real (or complex) matrix. Let the rows of $\mathbf{A}$ represent the elements in $\mathbb{R}^n$ (or $\mathbb{C}^n$) and the columns of $\mathbf{A}$ represent the elements in $\mathbb{R}^m$ (or $\mathbb{C}^m$). If $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{Ax}$ is in $\mathbb{R}^m$. Thus, an $m \times n$ matrix maps the elements in $\mathbb{R}^n$ into the elements in $\mathbb{R}^m$. We write
$$T = \mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m, \quad \text{and} \quad T\mathbf{x} = \mathbf{Ax}.$$
We now prove that the mapping $\mathbf{A}$ is a linear transformation. Let $\mathbf{v}_1$, $\mathbf{v}_2$ be two elements in $\mathbb{R}^n$ and α, β be scalars. Then
$$T(\alpha\,\mathbf{v}_1) = \mathbf{A}(\alpha\,\mathbf{v}_1) = \alpha\mathbf{A}\mathbf{v}_1$$
$$T(\beta\,\mathbf{v}_2) = \mathbf{A}(\beta\,\mathbf{v}_2) = \beta\,\mathbf{A}\mathbf{v}_2$$
and
$$T(\alpha\,\mathbf{v}_1 + \beta\,\mathbf{v}_2) = \mathbf{A}(\alpha\,\mathbf{v}_1 + \beta\,\mathbf{v}_2) = \alpha\,\mathbf{A}\mathbf{v}_1 + \beta\mathbf{A}\mathbf{v}_2.$$
The range of T is a linear subspace of $\mathbb{R}^m$ and the kernel of T is a linear subspace of $\mathbb{R}^n$.

Sum and product of linear transformations

Let T_1 and T_2 be two linear transformations from V into W. We define the sum $T_1 + T_2$ to be the transformation S such that
$$S\mathbf{v} = T_1\mathbf{v} + T_2\mathbf{v},\ \mathbf{v} \in V.$$

It can be easily verified that $T_1 + T_2$ is a linear transformation and $T_1 + T_2 = T_2 + T_1$.

Now, let U, V, W be three vector spaces, all real or all complex, on the same field of scalars. Let T_1 and T_2 be linear transformations such that

$$T_1 : U \to V \quad \text{and} \quad T_2 : V \to W.$$

The product $T_2 T_1$ is defined to be the transformation S from U into W such that

$$\mathbf{w} = S\mathbf{u} = T_2(T_1\mathbf{u}),\ \mathbf{u} \in U.$$

The transformation T_2T_1 is also called a *composite* transformation (Fig. 3.1). The transformation T_2T_1 means applying first the transformation T_1 and then applying the transformation T_2. It can be easily verified that T_2T_1 is a linear transformation.

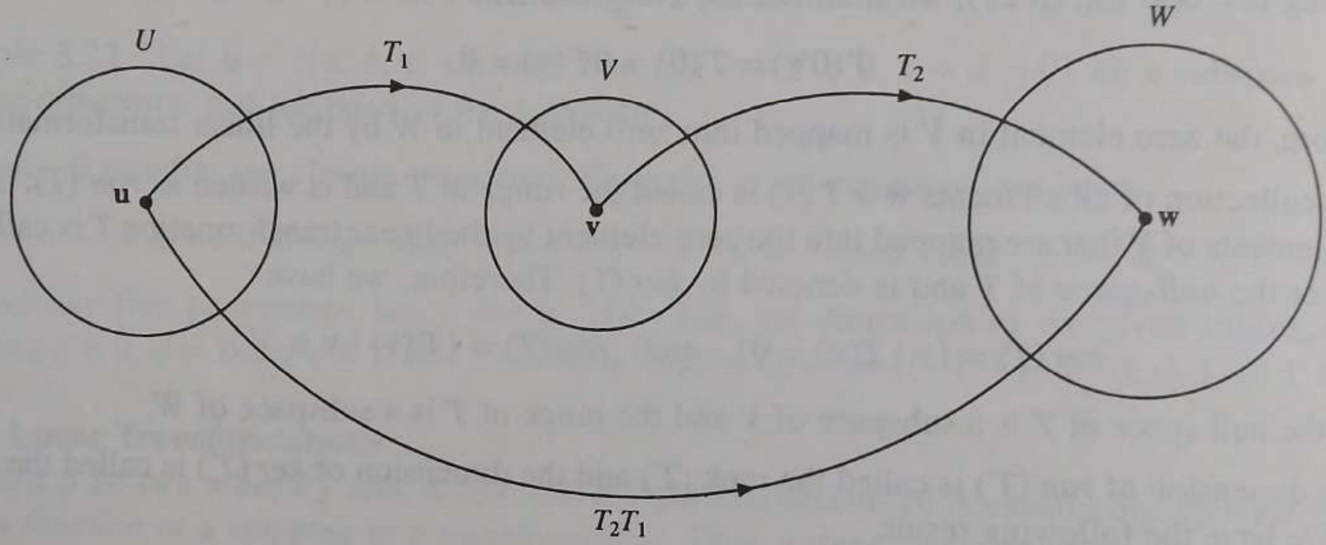


Fig. 3.1. Composite transformation.

If $T_1 : V \to V$ and $T_2 : V \to V$ are linear transformations, then both T_2T_1 and T_1T_2 are defined and map V into V. In general, $T_2T_1 \neq T_1T_2$. For example, let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices and $\mathbf{x}$ be any element in $\mathbb{R}^n$. Let T_1 and T_2 be the transformations

$$T_1(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad \text{and} \quad T_2(\mathbf{x}) = \mathbf{B}\mathbf{x}$$

from $\mathbb{R}^n$ into $\mathbb{R}^n$. Then

$$T_2(T_1(\mathbf{x})) = \mathbf{BA}\mathbf{x} \quad \text{and} \quad T_1(T_2(\mathbf{x})) = \mathbf{AB}\mathbf{x}.$$

Therefore, $T_2T_1 \neq T_1T_2$ unless the matrices $\mathbf{A}$ and $\mathbf{B}$ commute.

Example 3.23 Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by the relations

$$T\mathbf{x} = \mathbf{A}\mathbf{x},\ \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Find $T\mathbf{x}$ when $\mathbf{x}$ is given by $[3 \quad 4 \quad 5]^T$.

Solution We have

$$T\mathbf{x} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ 62 \end{bmatrix}.$$

Example 3.24 Let T be a linear transformation defined by

$$T\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

Find $T\left[\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix}\right]$.

Solution The matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are linearly independent and hence form a basis in the space of 2×2 matrices. We write for any scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, not all zero

$$\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \end{bmatrix}.$$

Comparing the elements and solving the resulting system of equations, we get $\alpha_1 = 4$, $\alpha_2 = 1$, $\alpha_3 = -2$, $\alpha_4 = 5$. Since T is a linear transformation, we get

$$T\left[\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix}\right] = \alpha_1 T\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] + \alpha_2 T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] + \alpha_3 T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] + \alpha_4 T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= 4\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 20 \\ 36 \end{pmatrix}.$$

Example 3.25 Let T be a linear transformation defined by

$$T\mathbf{x} = \mathbf{A}\mathbf{x}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{x} = (x_1, x_2)^T.$$

Find all points, if any, that are mapped into the point (3, 2).

Solution Let $(y_1, y_2)^T$ be the point that is mapped into (3, 2). Therefore, we have

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Multiplying and comparing we obtain the system of equations $y_1 + 2y_2 = 3$, $3y_1 + 4y_2 = 2$. The solution of this system of equations is $y_1 = -4$, $y_2 = 7/2$.

Example 3.26 For the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = (1, 3)^T$, $\mathbf{x}_2 = (4, 6)^T$, are in $\mathbb{R}^2$, find the matrix of linear transformation T: $\mathbb{R}^2 \to \mathbb{R}^3$, such that

$$T\mathbf{x}_1 = (-2 \quad 2 \quad -7)^T \quad \text{and} \quad T\mathbf{x}_2 = (-2 \quad -4 \quad -10)^T.$$

Solution The transformation T maps column vectors in $\mathbb{R}^2$ into column vectors in $\mathbb{R}^3$. Therefore, T must be a matrix $\mathbf{A}$ of order 3×2. Let

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

Therefore, we have

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -7 \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -10 \end{bmatrix}.$$

Multiplying and comparing the corresponding elements, we get

$$\begin{aligned} a_1 + 3b_1 &= -2, & 4a_1 + 6b_1 &= -2, \\ a_2 + 3b_2 &= 2, & 4a_2 + 6b_2 &= -4, \\ a_3 + 3b_3 &= -7, & 4a_3 + 6b_3 &= -10. \end{aligned}$$

Solving these equations, we obtain

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -4 & 2 \\ 2 & -3 \end{bmatrix}.$$

Example 3.27 Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, where $T\mathbf{x} = \mathbf{A}\mathbf{x}$, $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = (x \quad y \quad z)^T$. Find *ker*$(T)$, *ran*$(T)$ and their dimensions.

Solution To find *ker*(T), we need to determine all $\mathbf{v} = (v_1 \; v_2 \; v_3)^T$ such that $T\mathbf{v} = \mathbf{0}$. Now, $T\mathbf{v} = \mathbf{A}\mathbf{v} = \mathbf{0}$ gives the equations

$$v_1 + v_2 = 0, \; -v_1 + v_3 = 0$$

whose solution is $v_1 = -v_2 = v_3$. Therefore $\mathbf{v} = v_1[1 \; -1 \; 1]^T$.

Therefore, dimension of *ker*(T) is 1.

Now, *ran*(T) is defined as $\{T(\mathbf{v}) \mid \mathbf{v} \in V\}$. We have

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ -v_1 + v_3 \end{bmatrix}$$

$$= v_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + v_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the dimension of *ran* (T) is 2.

Example 3.28 Let T be a linear transformation $T\mathbf{x} = \mathbf{A}\mathbf{x}$ from $\mathbb{R}^2$ into $\mathbb{R}^3$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Find *ker* (T), *ran* (T) and their dimensions.

Solution To find *ker* (T), we need to determine all $\mathbf{v} = (v_1 \;\; v_2)^T$ such that $T\mathbf{v} = \mathbf{0}$. Now, $T\mathbf{v} = A\mathbf{v} = \mathbf{0}$ gives the equations

$$2v_1 + v_2 = 0, \;\; v_1 - v_2 = 0 \;\; \text{and} \;\; 3v_1 + 2v_2 = 0$$

whose solution is $v_1 = v_2 = 0$. Therefore $\mathbf{v} = (0 \;\; 0)^T$ and the dimension of *ker* (T) is zero.

Now,
$$ran(T) = T(\mathbf{v}) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + v_2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Since $(2 \;\; 1 \;\; 3)^T$, $(1 \;\; -1 \;\; 2)^T$ are linearly independent, the dimension of *ran* (T) is 2.

Example 3.29 Find the matrix of a linear transformation T from $\mathbb{R}^3$ into $\mathbb{R}^3$ such that

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}, \;\; T\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}, \;\; T\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 5 \end{pmatrix}.$$

Solution The transformation T maps elements in $\mathbb{R}^3$ into $\mathbb{R}^3$. Therefore, the transformation is a matrix of order 3×3. Let this matrix be written as

$$T = \mathbf{A} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

We determine the elements of the matrix $\mathbf{A}$ such that

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}.$$

Equating the elements and solving the resulting equations, we obtain

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ -15/2 & 3 & 13/2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Example 3.30 Let T be a transformation from $\mathbb{R}^3$ into $\mathbb{R}^1$ defined by

$$T(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2.$$

Show that T is not a linear transformation.

Solution Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be any two elements in $\mathbb{R}^3$. Then

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3).$$

We have

$$T(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2,\ T(\mathbf{y}) = y_1^2 + y_2^2 + y_3^2$$

$$T(\mathbf{x} + \mathbf{y}) = (x_1 + y_1)^2 + (x_2 + y_2)^2 + (x_3 + y_3)^2 \neq T(\mathbf{x}) + T(\mathbf{y}).$$

Therefore, T is not a linear transformation.

Matrix representation of a linear transformation

We observe from the earlier discussion that a matrix $\mathbf{A}$ of order $m \times n$ is a linear transformation whic maps the elements in $\mathbb{R}^n$ into the elements in $\mathbb{R}^m$. Now, let T be a linear transformation from finite dimensional vector space into another finite dimensional vector space over the same field F. W shall now show that with this linear transformation, we may associate a matrix $\mathbf{A}$.

Let V and W be respectively, n-dimensional and m-dimensional vector spaces over the same fiel F. Let T be a linear transformation such that $T: V \to W$. Let

$$X = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\},\ Y = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$$

be the ordered basis of V and W respectively. Let $\mathbf{v}$ be an arbitrary element in V and $\mathbf{w}$ be an arbitrar element in W. Then, there exist scalars, $\alpha_1, \alpha_2, \ldots \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_m$, not all zero, such that

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n \quad (3.29\text{ i}$$

$$\mathbf{w} = \beta_1 \mathbf{w}_1 + \beta_2 \mathbf{w}_2 + \ldots + \beta_m \mathbf{w}_m \quad (3.29\text{ ii}$$

and

$$\mathbf{w} = T\mathbf{v} = T(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n)$$
$$= \alpha_1 T\mathbf{v}_1 + \alpha_2 T\mathbf{v}_2 + \ldots + \alpha_n T\mathbf{v}_n \quad (3.29\text{ iii)}$$

Since every element $T\mathbf{v}_i$, $i = 1, 2, \ldots, n$ is in W, it can be written as a linear combination of the basis vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ in W. That is, there exist scalars a_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ not all zero, such that

$$T\mathbf{v}_i = a_{1i}\mathbf{w}_1 + a_{2i}\mathbf{w}_2 + \ldots + a_{mi}\mathbf{w}_m$$
$$= [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m]\,[a_{1i}, a_{2i}, \ldots, a_{mi}],\ i = 1, 2, \ldots, n \quad (3.29\text{ iv)}$$

Hence, we can write

$$T[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n] = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m]\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (3.29\text{ v)}$$

or $T\mathbf{X} = \mathbf{Y}\mathbf{A}$

where $\mathbf{A}$ is the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \tag{3.29 vi}$$

The $m \times n$ matrix $\mathbf{A}$ is called the matrix representation of T or the matrix of T with respect to the ordered basis $\mathbf{X}$ and $\mathbf{Y}$. It may be observed that $\mathbf{X}$ is a basis of the vector space V, on which T acts and $\mathbf{Y}$ is the basis of the vector space W that contains the range of T. Therefore, the matrix representation of T depends not only on T but also on the basis $\mathbf{X}$ and $\mathbf{Y}$. For a given linear transformation T, the elements a_{ij} of the matrix $\mathbf{A} = (a_{ij})$ are determined from (3.29 v), using the given basis vectors in $\mathbf{X}$ and $\mathbf{Y}$. From (3.29 iii), we have (using (3.29 iv))

$$\begin{aligned}\mathbf{w} &= \alpha_1(a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \dots + a_{m1}\mathbf{w}_m) + \alpha_2(a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \dots + a_{m2}\mathbf{w}_m) \\ &\quad + \dots + \alpha_n(a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \dots + a_{mn}\mathbf{w}_m) \\ &= (\alpha_1 a_{11} + \alpha_2 a_{12} + \dots + \alpha_n a_{1n})\,\mathbf{w}_1 + (\alpha_1 a_{21} + \alpha_2 a_{22} + \dots + \alpha_n a_{2n})\mathbf{w}_2 \\ &\quad + \dots + (\alpha_1 a_{m1} + \alpha_2 a_{m2} + \dots + \alpha_n a_{mn})\,\mathbf{w}_m \\ &= \beta_1\mathbf{w}_1 + \beta_2\mathbf{w}_2 + \dots + \beta_m\mathbf{w}_m\end{aligned}$$

where $\beta_i = \alpha_1 a_{i1} + \alpha_2 a_{i2} + \dots + \alpha_n a_{in}$, $i = 1, 2, \dots, m$.

Hence,

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

or

$$\boldsymbol{\beta} = \mathbf{A}\,\boldsymbol{\alpha} \tag{3.29 vii}$$

where the matrix $\mathbf{A}$ is as defined in (3.29 vi) and

$$\boldsymbol{\beta} = [\beta_1, \beta_2, \dots, \beta_m]^{\mathrm{T}}, \boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_n]^{\mathrm{T}}.$$

For a given ordered basis vectors $\mathbf{X}$ and $\mathbf{Y}$ of vector spaces V and W respectively, and a linear transformation T: $V \to W$, the matrix $\mathbf{A}$ obtained from (3.29 v) is unique. We prove this result as follows:

Let $\mathbf{A} = (a_{ij})$ and $B = (b_{ij})$ be two matries each of order $m \times n$ such that

$$TX = Y\mathbf{A} \quad \text{and} \quad TX = YB.$$

Therefore, we have

$$Y\mathbf{A} = YB$$

or

$$\sum_{i=1}^{m} \mathbf{w}_i a_{ij} = \sum_{i=1}^{m} \mathbf{w}_i b_{ij}, j = 1, 2, \dots, n.$$

Since $Y = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ is a given basis, we obtain $a_{ij} = b_{ij}$ for all i and j and hence $\mathbf{A} \equiv B$.

Example 3.31 Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ y-z \end{pmatrix}$$

Determine the matrix of the linear transformation T, with respect to the ordered basis

(i) $X = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$ in $\mathbb{R}^3$ and $Y = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ in $\mathbb{R}^2$

(standard basis e_1, e_2, e_3 in $\mathbb{R}^3$ and e_1, e_2, in $\mathbb{R}^2$).

(ii) $X = \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$ in $\mathbb{R}^3$ and $Y = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ in $\mathbb{R}^2$.

(iii) $X = \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$ in $\mathbb{R}^3$ and $Y = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ in $\mathbb{R}^2$.

Solution Let $V = \mathbb{R}^3$, $W = \mathbb{R}^2$. Let $X = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $Y = \{\mathbf{w}_1, \mathbf{w}_2\}$.

(i) We have $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We obtain

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0),$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(1),$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(-1).$$

Using the notation given in (3.29 v), that is $TX = YA$, we write

$$T[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = [\mathbf{w}_1, \mathbf{w}_2]\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

or
$$T\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Therefore, the matrix of the linear transformation T with respect to the given basis vectors is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

(ii) We have
$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We obtain

$$T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(2) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0),$$

$$T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(-1),$$

$$T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(1).$$

Using (3.29 v), that is $TX = Y\mathbf{A}$, we write

$$T\left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Therefore, the matrix of the linear transformation T with respect to the given basis vectors is given by

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(iii) We have $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

We obtain

$$T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}(1)$$

$$T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(0) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}(1)$$

$$T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}(0)$$

Using (3.29 v), that is $TX = Y\mathbf{A}$, we write

$$T\left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Therefore, the matrix of the linear transformation T with respect to the given basis vectors is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Exercise 3.2

Discuss whether V defined in Problems **1** to **10** is a vector space or not. If V is not a vector space, state which of the properties are not satisfied.

1. Let V be the set of the real polynomials of degree $\leq m$ and having 2 as a root with the usual addition and scalar multiplication.
2. Let V be the set of all real polynomials of degree 4 or 6 with the usual addition and scalar multiplication.
3. Let V be the set of all real polynomials of degree ≥ 4 with the usual addition and scalar multiplication.
4. Let V be the set of all rational numbers with the usual addition and scalar multiplication.
5. Let V be the set of all positive real numbers with addition defined as $x + y = xy$ and usual scalar multiplication.
6. Let V be the set of all ordered pairs (x, y) in $\mathbb{R}^2$ with vector addition defined as $(x, y) + (u, v) = (x + u, y + v)$ and scalar multiplication defined as $\alpha(x, y) = (3\alpha x, y)$.

7. Let V be the set of all ordered triplets (x, y, z), $x, y, z \in \mathbb{R}$, with vector addition defined as
$$(x, y, z) + (u, v, w) = (3x + 4u, y - 2v, z + w)$$
and scalar multiplication defined as
$$\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z/3).$$

8. Let V be the set of all positive real numbers with addition defined as $x + y = xy$ and scalar multiplication defined as $\alpha x = x^{\alpha}$.

9. Let V be the set of all real valued continuous functions f on $[a, b]$ such that (i) $\int_a^b f(x)\,dx = 0$ and (ii) $\int_a^b f(x)\,dx = 2$ with usual addition and scalar multiplication.

10. Let V be the set of all solutions of the
    (i) homogeneous linear differential equation $y'' - 3y' + 2y = 0$.
    (ii) non-homogeneous linear differential equation $y'' - 3y' + 2y = x$.
    under the usual addition and scalar multiplication.

Is W a subspace of V in Problems **11** to **15**? If not, state why?

11. Let V be the set of all 3×1 real matrices with usual matrix addition and scalar multiplication and W consisting of all 3×1 real matrices of the form

    (i) $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$, (ii) $\begin{bmatrix} a \\ a \\ a^2 \end{bmatrix}$, (iii) $\begin{bmatrix} a \\ b \\ 2 \end{bmatrix}$, (iv) $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$.

12. Let V be the set of all 3×3 real matrices with the usual matrix addition and scalar multiplication and W consisting of all 3×3 matrices $\mathbf{A}$ which
    (i) have positive elements, (ii) are non-singular,
    (iii) are symmetric, (iv) $\mathbf{A}^2 = \mathbf{A}$.

13. Let V be the set of all 2×2 complex matrices with the usual matrix addition and scalar multiplication and W consisting of all matrices of the form $\begin{bmatrix} z & x+iy \\ x-iy & u \end{bmatrix}$, where x, y, z, u are real numbers and (i) scalars are real numbers, (ii) scalars are complex numbers.

14. Let V consist of all real polynomials of degree ≤ 4 with the usual polynomial addition and scalar multiplication and W consisting of polynomials of degree ≤ 4 having
    (i) constant term 1, (ii) coefficient of t^2 as 0,
    (iii) coefficient of t^3 as 1, (iv) only real roots.

15. Let V be the vector space of all triplets of the form (x_1, x_2, x_3) in $\mathbb{R}^3$ with the usual addition and scalar multiplication and W be the set of triplets of the form (x_1, x_2, x_3) such that
    (i) $x_1 = 2x_2 = 3x_3$, (ii) $x_1 = x_2 = x_3 + 1$,
    (iii) $x_1 \ge 0$, x_2, x_3 arbitrary, (iv) $x_1^2 + x_2^2 + x_3^2 \le 4$.
    (v) x_3 is an integer.

16. Let $\mathbf{u} = (1, 2, -1)$, $\mathbf{v} = (2, 3, 4)$ and $\mathbf{w} = (1, 5, -3)$. Determine whether or not $\mathbf{x}$ is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$, where $\mathbf{x}$ is given by

(i) (4, 3, 10), (ii) (3, 2, 5), (iii) (– 2, 1, – 5).

17. Let $\mathbf{u} = (1, -2, 1, 3)$, $\mathbf{v} = (1, 2, -1, 1)$ and $\mathbf{w} = (2, 3, 1, -1)$. Determine whether or not $\mathbf{x}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, where $\mathbf{x}$ is given by

(i) (3, 0, 5, – 1), (ii) (2, – 7, 1, 11), (iii) (4, 3, 0, 3).

18. Let $P_1(t) = t^2 - 4t - 6$, $P_2(t) = 2t^2 - 7t - 8$, $P_3(t) = 2t - 3$. Write $P(t)$ as a linear combination of $P_1(t)$, $P_2(t)$, $P_3(t)$, when

(i) $P(t) = -t^2 + 1$, (ii) $P(t) = 2t^2 - 3t - 25$.

19. Let V be the set of all 3×1 real matrices. Show that the set

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ spans } V.$$

20. Let V be the set of all 2×2 real matrices. Show that the set

$$S = \left\{ \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} \right\} \text{ spans } V.$$

21. Examine whether the following vectors in $\mathbb{R}^3/\mathbb{C}^3$ are linearly independent.

(i) (2, 2, 1), (1, – 1, 1), (1, 0, 1), (ii) (1, 2, 3), (3, 4, 5), (6, 7, 8),

(iii) (0, 0, 0), (1, 2, 3), (3, 4, 5), (iv) $(2, i, -1), (1, -3, i), (2i, -1, 5)$,

(v) (1, 3, 4), (1, 1, 0), (1, 4, 2), (1, – 2, 1).

22. Examine whether the following vectors in $\mathbb{R}^4$ are linearly independent.

(i) (4, 1, 2, – 6), (1, 1, 0, 3), (1, – 1, 0, 2), (– 2, 1, 0, 3),

(ii) (1, 2, 3, 1), (2, 1, – 1, 1), (4, 5, 5, 3), (5, 4, 1, 3),

(iii) (1, 2, 3, 4), (2, 0, 1, – 2), (3, 2, 4, 2),

(iv) (1, 1, 0, 1), (1, 1, 1, 1), (– 1, – 1, 1, 1), (1, 0, 0, 1),

(v) (1, 2, 3, – 1), (0, 1, – 1, 2), (1, 5, 1, 8), (– 1, 7, 8, 3).

23. If $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are linearly independent vectors in $\mathbb{R}^3$, then show that

(i) $\mathbf{x} + \mathbf{y}$, $\mathbf{y} + \mathbf{z}$, $\mathbf{z} + \mathbf{x}$; (ii) $\mathbf{x}$, $\mathbf{x} + \mathbf{y}$, $\mathbf{x} + \mathbf{y} + \mathbf{z}$

are also linearly independent in $\mathbb{R}^3$.

24. Write (– 4, 7, 9) as a linear combination of the elements of the set S : {(1, 2, 3), (– 1, 3, 4), (3, 1, 2)}. Show that S is not a spanning set in $\mathbb{R}^3$.

25. Write $t^2 + t + 1$ as a linear combination of the elements of the set $S : \{3t, t^2 - 1, t^2 + 2t + 2\}$. Show that S is the spanning set for all polynomials of degree 2 and can be taken as its basis.

26. Let V be the set of all vectors in $\mathbb{R}^4$ and S be a subset of V consisting of all vectors of the form

(i) $(x, y, -y, -x)$, (ii) (x, y, z, w) such that $x + y + z - w = 0$,

(iii) $(x, 0, z, w)$, (iv) (x, x, x, x).

Find the dimension and the basis of S.

27. For what values of k do the following set of vectors form a basis in $\mathbb{R}^3$?

(i) $\{(k, 1-k, k), (0, 3k-1, 2), (-k, 1, 0)\}$,

(ii) $\{(k, 1, 1), (0, 1, 1), (k, 0, k)\}$,

(iii) $\{(k, k, k), (0, k, k), (k, 0, k)\}$,

(iv) $\{(1, k, 5), (1, -3, 2), (2, -1, 1)\}$.

28. Find the dimension and the basis for the vector space V, when V is the set of all 2×2 (i) real matrices, (ii) symmetric matrices, (iii) skew-symmetric matrices, (iv) skew-Hermitian matrices, (v) real matrices $\mathbf{A} = (a_{ij})$ with $a_{11} + a_{22} = 0$, (vi) real matrices $\mathbf{A} = (a_{ij})$ with $a_{11} + a_{12} = 0$.

29. Find the dimension and the basis for the vector space V, when V is the set of all 3×3 (i) diagonal matrices, (ii) upper triangular matrices, (iii) lower triangular matrices.

30. Find the dimension of the vector space V, when V is the set of all $n \times n$ (i) real matrices, (ii) diagonal matrices, (iii) symmetric matrices, (iv) skew-symmetric matrices.

Examine whether the transformation T given in problems **31** to **35** is linear or not. If not linear, state why?

31. $T: \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \end{pmatrix} = x + y + a$, $a \neq 0$, a real constant.

32. $T: \mathbb{R}^3 \to \mathbb{R}^2$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x + z \end{pmatrix}$.

33. $T: \mathbb{R}^1 \to \mathbb{R}^2$; $T(x) = \begin{pmatrix} x^2 \\ 3x \end{pmatrix}$.

34. $T: \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} 0 & x \neq 0, y \neq 0 \\ 2y, & x = 0 \\ 3x, & y = 0. \end{cases}$

35. $T: \mathbb{R}^3 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + x + z$.

Find $ker(T)$ and $ran(T)$ and their dimensions in problems **36** to **42**.

36. $T: \mathbb{R}^3 \to \mathbb{R}^3$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ z \\ x - y \end{pmatrix}$.

37. $T: \mathbb{R}^2 \to \mathbb{R}^3$; $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ y - x \\ 3x + 4y \end{pmatrix}$.

38. $T: \mathbb{R}^4 \to \mathbb{R}^3$; $T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x + y + w \\ z \\ y + 2w \end{pmatrix}$.

39. $T: \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \end{pmatrix} = x + 3y$.

40. $T: \mathbb{R}^3 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + 3y$.

41. $T: \mathbb{R}^2 \to \mathbb{R}^2$; $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x - y \end{pmatrix}$.

42. $T: \mathbb{R}^3 \to \mathbb{R}^2$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y \\ 3x + z \end{pmatrix}$.

43. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation difined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x-z \end{pmatrix}.$$

Find the matrix representation of T with respect to the ordered basis

(i) $$X = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } \mathbb{R}^3 \text{ and } Y = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \text{ in } \mathbb{R}^2.$$

(ii) $$X = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } \mathbb{R}^3 \text{ and } Y = \left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right\} \text{ in } \mathbb{R}^2.$$

44. Let V and W be two vector spaces in $\mathbb{R}^3$. Let $T: V \to W$ be a linear transformation difined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ x+y \\ x+y+z \end{pmatrix}.$$

Find the matrix representation of T with respect to the ordered basis

(i) $$X = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } V \text{ and } Y = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} \text{ in } W$$

(ii) $$X = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} \text{ in } V \text{ and } Y = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } W.$$

45. Let V and W be two vector spaces in $\mathbb{R}^3$. Let $T : V \to W$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} x+z \\ x+y \\ x+y+z \end{pmatrix}.$$

Find the matrix representation of T with respect to the ordered basis

$$X = \left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\} \text{ in } V \text{ and } Y = \left\{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\} \text{ in } W$$

46. Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ x+z \\ x+y+z \end{pmatrix}.$$

Find the matrix representation of T with respect to the ordered basis

$$X = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } \mathbb{R}^3 \quad \text{and} \quad Y = \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\} \text{ in } \mathbb{R}^4$$

47. Let T: $\mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation.

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$ be the matrix representation of the linear transformation T with respect to the ordered basis vectors $\mathbf{v}_1 = [1, 2]^T$, $\mathbf{v}_2 = [3, 4]^T$ in $\mathbb{R}^2$ and $\mathbf{w}_1 = [-1, 1, 1]^T$, $\mathbf{w}_2 = [1, -1, 1]^T$, $\mathbf{w}_3 = [1, 1, -1]^T$ in $\mathbb{R}^3$. Then, determine the linear transformation T.

48. Let T: $\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & -4 \end{bmatrix}$ be the matrix representation of the linear transformation with respect to the ordered basis vectors $\mathbf{v}_1 = [1, -1, 1]^T$, $\mathbf{v}_2 = [2, 3, -1]^T$, $\mathbf{v}_3 = [1, 1, -1]^T$ in $\mathbb{R}^3$ and $\mathbf{w}_1 = [1, 1]^T$, $\mathbf{w}_2 = [2, 3]^T$ in $\mathbb{R}^2$. Then, determine the linear transformation T.

49. Let T: $P_1(t) \to P_2(t)$ be a linear tansformation. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 1 \end{bmatrix}$ be the matrix representation of the linear transformation with respect to the ordered basis $[1 + t, t]$ in $P_1(t)$ and $[1 - t, 2t, 2 + 3t - t^2]$ in $P_2(t)$. Then, determine the linear transformation T.

50. Let V be the set of all vectors of the form (x_1, x_2, x_3) in $\mathbb{R}^3$ satisfying (i) $x_1 - 3x_2 + 2x_3 = 0$; (ii) $3x_1 - 2x_2 + x_3 = 0$ and $4x_1 + 5x_2 = 0$. Find the dimension and basis for V.

3.4 Solution of General Linear System of Equations

In section 3.2.5, we have discussed the matrix method and the Cramer's rule for solving a system of n equations in n unknowns, $\mathbf{Ax} = \mathbf{b}$. We assumed that the coefficient matrix $\mathbf{A}$ is non-singular, that is $|\mathbf{A}| \neq 0$, or the rank of the matrix $\mathbf{A}$ is n. The matrix method requires evaluation of n^2 determinants each of order $(n - 1)$, to generate the cofactor matrix, and one determinant of order n, whereas the Cramer's rule requires evaluation of $(n + 1)$ determinants each of order n. Since the evaluation of high order determinants is very time consuming, these methods are not used for large values of n, say

$n > 4$. In this section, we discuss a method for solving a general system of m equations in n unknow given by

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.3}$$

where $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

are respectively called the *coefficient matrix, right hand side column vector* and the *solution vect* The orders of the matrices **A**, **b**, **x** are respectively $m \times n$, $m \times 1$ and $n \times 1$. The matrix

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right] \tag{3.3}$$

is called the *augmented matrix* and has m rows and $(n + 1)$ columns. The augmented matrix descrit completely the system of equations. The solution vector of the system of equations (3.30) is n-tuple $(x_1, x_2, \dots, x_n)$ that satisfies all the equations. There are three possibilities:

(i) the system has a unique solution,

(ii) the system has no solution,

(iii) the system has infinite number of solutions.

The system of equations is said to be *consistent*, if it has atleast one solution and *inconsistent*, if it h no solution. Using the concepts of ranks and vector spaces, we now obtain the necessary and sufficie conditions for the existence and uniqueness of the solution of the linear system of equations.

3.4.1 Existence and Uniqueness of the Solution

Let V_n be a vector space consisting of n-tuples in $\mathbb{R}^n$ (or $\mathbb{C}^n$). The row vectors $R_1, R_2, \dots, R_m$ of t $m \times n$ matrix **A** are n-tuples which belong to V_n. Let S be the subspace of V_n generated by the rov of **A**. Then, S is called the *row-space* of the matrix **A** and its dimension is the *row-rank* of **A** and denoted by $rr(\mathbf{A})$. Therefore,

$$\text{row-rank of } \mathbf{A} = rr(\mathbf{A}) = dim(S). \tag{3.3}$$

Similarly, we define the column-space of **A** and the column-rank of **A** denoted by $cr(\mathbf{A})$.

Since the row-space of $m \times n$ matrix **A** is generated by m row vectors of **A**, we have $\dim(S) \leq$ and since S is a subspace of V_n, we have $dim(S) \leq n$. Therefore, we have

$$rr(\mathbf{A}) \leq min(m, n) \text{ and similarly } cr(\mathbf{A}) \leq min(m, n). \tag{3.3}$$

Theorem 3.6 Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix. Then the row-rank and the column-rank of **A** are same

Proof Let S be the row-space of **A**. The dimension of S is the number of linearly independent row of **A**. Let the dimension of S be r. Therefore, r rows of the matrix A are linearly independent and th

remaining $m - r$ rows can be written as a linear combination of these r rows. Let $R_1, R_2, \ldots, R_r$ be the linearly independent rows of **A**. Then, we can write

$$\begin{aligned}
R_{r+1} &= \alpha_{r+1,1} R_1 + \alpha_{r+1,2} R_2 + \ldots + \alpha_{r+1,r} R_r \\
R_{r+2} &= \alpha_{r+2,1} R_1 + \alpha_{r+2,2} R_2 + \ldots + \alpha_{r+2,r} R_r \\
&\ldots \\
R_m &= \alpha_{m,1} R_1 + \alpha_{m,2} R_2 + \ldots + \alpha_{m,r} R_r
\end{aligned}$$

where $\alpha_{i,j}$ are scalars.

Therefore, the jth element of the row R_{r+1} is given by

$$a_{r+1,j} = \alpha_{r+1,1}\, a_{1j} + \alpha_{r+1,2}\, a_{2j} + \ldots + \alpha_{r+1,r}\, a_{rj}.$$

Similarly, the jth elements of the rows $R_{r+2}, \ldots, R_m$ can be written.

Hence, the jth column of the matrix **A** can be written as

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \ldots \\ a_{rj} \\ a_{r+1,j} \\ \ldots \\ a_{m,j} \end{bmatrix} = a_{1j} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \ldots \\ 0 \\ \alpha_{r+1,1} \\ \ldots \\ \alpha_{m,1} \end{bmatrix} + a_{2j} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \ldots \\ 0 \\ \alpha_{r+1,2} \\ \ldots \\ \alpha_{m,2} \end{bmatrix} + \ldots + a_{rj} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ 1 \\ \alpha_{r+1,r} \\ \ldots \\ \alpha_{m,r} \end{bmatrix}.$$

Thus, every column of the matrix **A** can be written as a linear combination of r linearly independent rows of **A**. Therefore, the dimension of the column-space cannot exceed r, which is the maximum number of linearly independent rows of **A**, that is

$$cr\,(\mathbf{A}) \le r = \text{row-rank of } \mathbf{A}.$$

Similarly, by reversing the roles of rows and columns in the above discussion, we obtain

$$rr\,(\mathbf{A}) \le r = \text{column-rank of } \mathbf{A}.$$

Combining the above results, we have

$$\text{rank}\,(\mathbf{A}) = rr\,(\mathbf{A}) = cr\,(\mathbf{A}) = r.$$

Now, we prove the important result which is known as the *fundamental theorem of linear algebra*.

Theorem 3.7 The non-homogeneous system of equations $\mathbf{Ax} = \mathbf{b}$, where **A** is an $m \times n$ matrix, has a solution if and only if the matrix **A** and the augmented matrix $(\mathbf{A} \mid \mathbf{b})$ have the same rank.

Proof We can write the given system of equations $\mathbf{Ax} = \mathbf{b}$ as

$$x_1 C_1 + x_2 C_2 + \ldots + x_n C_n = \mathbf{b} \tag{3.34}$$

where C_i is the ith column of **A**. Thus, finding solution of the system $\mathbf{Ax} = \mathbf{b}$ is equivalent to finding scalars $x_1, x_2, \ldots, x_n$ which satisfy the equation (3.34).

Let rank (**A**) = rank (**A** | **b**) = r. Then, the column-rank of the matrix (**A** | **b**) is r and $r \le n$. Therefore, there are r linearly independent column vectors. Suppose these are the first r columns,. Then, the remaining columns $C_{r+1}, C_{r+2}, \ldots, C_{n+1}$ can be written as a linear combination of these r linearly independent column vectors. Thus, the $(n + 1)$th column of (**A** | **b**) is a linear combination of its first n columns, that is

$$\mathbf{b} = \alpha_1 C_1 + \alpha_2 C_2 + \ldots + \alpha_n C_n$$

which means that $\mathbf{A}\boldsymbol{\alpha} = \mathbf{b}$, or $\mathbf{Ax} = \mathbf{b}$ has a solution.

Conversely, let $\mathbf{Ax} = \mathbf{b}$ have a solution, say $\mathbf{x} = \boldsymbol{\alpha}$. Then, we can write

$$\mathbf{b} = \alpha_1 C_1 + \alpha_2 C_2 + \ldots + \alpha_n C_n.$$

Thus, the column-spaces of **A** and (**A** | **b**) are the same and have the same dimension. Since, these dimensions are rank (**A**) and rank (**A** | **b**) respectively, we obtain rank (**A**) = rank (**A** | **b**).

Remark 11

A system of linear equations $\mathbf{Ax} = \mathbf{b}$ is consistent, if the vector **b** can be written as a linear combination of the columns $C_1, C_2, \ldots, C_n$ of **A**. If **b** is not a linear combination of the columns of **A**, that is, **b** is linearly independent of the columns $C_1, C_2, \ldots, C_n$, then no scalars can be determined that satisfy Eq. (3.34) and the system is inconsistent in this case.

In section 3.2.3, we defined the rank of an $m \times n$ matrix **A** in terms of the determinants of the submatrices of **A**. An $m \times n$ matrix has rank r if it has at least one square submatrix of order r which is non-singular and all square submatrices of order greater than r are singular. This approach is very time consuming when n is large. Now, we discuss an alternative procedure to obtain the rank of a matrix.

3.4.2 Elementary Row and Column Operations

The following three operations on a matrix **A** are called the *elementary row operations*:

(i) Interchange of any two rows (written as $R_i \sim R_j$).

(ii) Multiplication/division of any row by a non-zero scalar (written as αR_i).

(iii) Adding/subtracting a scalar multiple of any row to another row (written as $R_i \leftarrow R_i + \alpha R_j$, that is α multiples of the elements of the jth row are added to the corresponding elements of the ith row. The elements of the jth row remain unchanged, whereas, the elements of the ith row get changed).

These operations change the form of **A** but do not change the row-rank of **A** as they do not change the row-space of **A**. A matrix **B** is said to be *row equivalent* to a matrix **A**, if the matrix **B** can be obtained from the matrix **A** by a finite sequence of elementary row operations. Then, we usually write $\mathbf{B} \approx \mathbf{A}$. We observe that

(i) every matrix is row equivalent to itself.

(ii) if **A** is row equivalent to **B**, then **B** is row equivalent to **A**.

(iii) if **A** is row equivalent to **B** and **B** is row equivalent to **C**, then **A** is row equivalent to **C**.

The above operations performed on columns (that is column in place of row) are called *elementary column operations*.

3.4.3 Echelon Form of a Matrix

An $m \times n$ matrix is called a *row echelon* matrix or in *row echelon* form if the number of zeros preceeding the first non-zero entry of a row increases row by row until a row having all zero entries (or no other elimination is possible) is obtained. Therefore, a matrix is in row echelon form if the following are satisfied.

not even constant

(i) If the ith row contains all zeros, it is true for all subsequent rows.

(ii) If a column contains a non-zero entry of any row, then every subsequent entry in this column is zero, that is, if the ith and $(i + 1)$th rows are both non-zero rows, then the initial non-zero entry of the $(i + 1)$th row appears in a later column than that of the ith row.

(iii) Rows containing all zeros occur only after all non-zero rows.

For example, the following matrices are in row echelon form.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathbf{A} = (a_{ij})$ be a given $m \times n$ matrix. Assume that $a_{11} \neq 0$. If $a_{11} = 0$, we interchange the first row with some other row to make the element in the (1, 1) position as non-zero. Using elementary row operations, we reduce the matrix $\mathbf{A}$ to its row echelon form (elements of first column below a_{11} are made zero, then elements in the second column below a_{22} are made zero and so on).

Similarly, we define the column echelon form of a matrix.

Rank of A The number of non-zero rows in the row echelon form of a matrix $\mathbf{A}$ gives the rank of the matrix $\mathbf{A}$ (that is, the dimension of the row-space of the matrix $\mathbf{A}$) and the set of the non-zero rows in the row echelon form gives the basis of the row-space.

Similar results hold for column echelon matrices.

Remark 12

(i) If $\mathbf{A}$ is a square matrix, then the row-echelon form is an upper triangular matrix and the column echelon form is a lower triangular matrix.

(ii) This approach can also be used to examine whether a given set of vectors are linearly independent or not. We form the matrix with each vector as its row (or column) and reduce it to the row (column) echelon form. The given vectors are linearly independent, if the row echelon form has no row with all its elements as zeros. The number of non-zero rows is the dimension of the given set of vectors and the set of vectors consisting of the non-zero rows is the basis.

Example 3.32 Reduce the following matrices to row echelon form and find their ranks.

(i) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$.

Solution Let the given matrix be denoted by **A**. We have

(i) $$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 + 2R_1 \end{matrix} \approx \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix} R_3 + 2R_2 \approx \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is the row echelon form of **A**. Since the number of non-zero rows in the row echelon form is 2, we get rank (**A**) = 2.

(ii) $$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - 8R_1 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \begin{matrix} R_3 + R_2 \\ R_4 - 5R_2 \end{matrix}$$

$$\approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the number of non-zero rows in the row echelon form of **A** is 2, we get rank (**A**) = 2.

Example 3.33 Reduce the following matrices to column echelon form and find their ranks.

(i) $$\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix},$$ (ii) $$\begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix}.$$

Solution Let the given matrix be denoted by **A**. We have

(i) $$\mathbf{A} = \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix} \begin{matrix} C_2 - C_1/3 \\ C_3 - 7C_1/3 \end{matrix} \approx \begin{bmatrix} 3 & 0 & 0 \\ 1 & 5/3 & 5/3 \\ 4 & -7/3 & -7/3 \\ 2 & 1/3 & 1/3 \end{bmatrix} C_3 - C_2 \approx \begin{bmatrix} 3 & 0 & 0 \\ 1 & 5/3 & 0 \\ 4 & -7/3 & 0 \\ 2 & 1/3 & 0 \end{bmatrix}.$$

Since the column echelon form of **A** has two non-zero columns, rank (**A**) = 2.

(ii) $$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix} \begin{matrix} C_2 - C_1 \\ C_3 + C_1 \\ C_4 - C_1 \end{matrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & -2 \\ 1 & -1 & 2 & 1 \\ 1 & -2 & 4 & 2 \end{bmatrix} \begin{matrix} C_3 + 2C_2 \\ C_4 + C_2 \end{matrix}$$

$$\approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}.$$

Since the column echelon form of **A** has 2 non-zero columns, rank $(\mathbf{A}) = 2$.

Example 3.34 Examine whether the following set of vectors is linearly independent. Find the dimension and the basis of the given set of vectors.

(i) $(1, 2, 3, 4), (2, 0, 1, -2), (3, 2, 4, 2)$,

(ii) $(1, 1, 0, 1), (1, 1, 1, 1), (-1, 1, 1, 1), (1, 0, 0, 1)$,

(iii) $(2, 3, 6, -3, 4), (4, 2, 12, -3, 6), (4, 10, 12, -9, 10)$.

Solution Let each given vector represent a row of a matrix **A**. We reduce **A** to row echelon form. If all the rows of the row echelon form have some non-zero elements, then the given set of vectors are linearly independent.

(i)
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & 2 & 4 & 2 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & -4 & -5 & -10 \end{bmatrix} R_3 - R_2$$

$$\approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since all the rows in the row echelon form of **A** are not non-zero, the given set of vectors are linearly dependent. Since the number of non-zero rows is 2, the dimension of the given set of vectors is 2. The basis can be taken as the set of vectors $\{(1\ \ 2\ \ 3\ \ 4), (0, -4, -5, -10)\}$.

(ii)
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 + R_1 \\ R_4 - R_1 \end{matrix} \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix} R_2 \sim R_3 \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} R_4 + R_2/2$$

$$\approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{bmatrix} R_4 - R_3/2 \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since all the rows in the row echelon form of **A** are non-zero, the given set of vectors are linearly independent and the dimension of the given set of vectors is 4. The set of vectors $\{(1, 1, 0, 1), (0, 2, 1, 2), (0, 0, 1, 0), (0, 0, 0, 1)\}$ or the given set itself forms the basis.

(iii) $$\mathbf{A} = \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 4 & 2 & 12 & -3 & 6 \\ 4 & 10 & 12 & -9 & 10 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix} \approx \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 4 & 0 & -3 & 2 \end{bmatrix} R_3 + R_4$$

$$\approx \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since all the rows in the row echelon form of **A** are not non-zero, the given set of vectors are linearly dependent. Since the number of non-zero rows is 2, the dimension of the given set of vectors is 2 and its basis can be taken as the set $\{(2, 3, 6, -3, 4), (0, -4, 0, 3, -2)\}$.

3.4.4 Gauss Elimination Method for Non-homogeneous Systems

Consider a non-homogeneous system of m equations in n unknowns

$$\mathbf{Ax} = \mathbf{b} \tag{3.35}$$

where $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

We assume that at least one element of **b** is not zero. We write the augmented matrix of order $m \times (n + 1)$ as

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

and reduce it to the row echelon form by using elementary row operations. We need a maximum of $(m - 1)$ stages of eliminations to reduce the given augmented matrix to the equivalent row echelon form. This process may terminate at an earlier stage. We then have an equivalent system of the form

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccccccc|c} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1n} & b_1 \\ 0 & \bar{a}_{22} & \cdots & \bar{a}_{2r} & \cdots & \bar{a}_{2n} & \bar{b}_2 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & a^*_{rr} & \cdots & a^*_{rn} & b^*_r \\ 0 & 0 & \cdots & 0 & \cdots & 0 & b^*_{r+1} \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & b^*_m \end{array}\right]$$

where $r \leq m$ and $a_{11} \neq 0, \bar{a}_{22} \neq 0, \ldots, a^*_{rr} \neq 0$ are called *pivots*. We have the following cases:

(a) Let $r < m$ and one or more of the elements $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are not zero. Then rank $(\mathbf{A}) \neq$ rank $(\mathbf{A} \mid \mathbf{b})$ and the system of equations has no solution.

(b) Let $m \geq n$ and $r = n$ (the number of columns in $\mathbf{A}$) and $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are all zeros. In this case, rank $(\mathbf{A})$ = rank $(\mathbf{A} \mid \mathbf{b}) = n$ and the system of equations has a unique solution. We solve the nth equations for x_n, the $(n-1)$th equation for x_{n-1} and so on. This procedure is called the *back substitution method*.

For example, if we have 10 equations in 5 variables, then the augmented matrix is of order 10×6. When rank $(\mathbf{A})$ = rank $(\mathbf{A} \mid \mathbf{b}) = 5$, the system has a unique solution.

(c) Let $r < n$ and $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are all zeros. In this case, r unknowns, $x_1, x_2, \ldots, x_r$ can be determined in terms of the remaining $(n - r)$ unknowns $x_{r+1}, x_{r+2}, \ldots, x_n$ by solving the rth equation for x_r, $(r-1)$th equation for x_{r-1} and so on. In this case, we obtain an $(n-r)$ parameter family of solutions, that is infinitely many solutions.

Remark 13

(a) We do not, normally use column elementary operations in solving the linear system of equations. When we interchange two columns, the order of the unknowns in the given system of equations is also changed. Keeping track of the order of unknowns is quite difficult.

(b) Gauss elimination method may be written as

$$(\mathbf{A} \mid \mathbf{b}) \xrightarrow[\text{row operations}]{\text{Elementary}} (\mathbf{B} \mid \mathbf{c}).$$

The matrix $\mathbf{B}$ is the row echelon form of the matrix $\mathbf{A}$ and $\mathbf{c}$ is the new right hand side column vector. We obtain the solution vector (if it exists) using the back substitution method.

(c) If $\mathbf{A}$ is a square matrix of order n, then $\mathbf{B}$ is an upper triangular matrix of order n.

(d) Gauss elimination method can be used to solve p systems of the form $\mathbf{Ax} = \mathbf{b}_1$, $\mathbf{Ax} = \mathbf{b}_2, \ldots,$ $\mathbf{Ax} = \mathbf{b}_p$ which have the same coefficient matrix but different right hand side column vectors. We form the augmented matrix as $(\mathbf{A} \mid \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p)$, which has m rows and $(n + p)$ columns. Using the elementary row operations, we obtain the row equivalent system $(\mathbf{B} \mid \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p)$, where $\mathbf{B}$ is the row echelon form of A. Now, we solve the systems $\mathbf{Bx} = \mathbf{c}_1$, $\mathbf{Bx} = \mathbf{c}_2, \ldots,$ $\mathbf{Bx} = \mathbf{c}_p$, using the back substitution method.

Example 3.35 Solve the following systems of equations (if possible) using Gauss elimination method.

(i) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$,

(iii) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

Solution We write the augmented matrix and reduce it to row echelon form by applying elementary row operations.

(i) $$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{rrr|r} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{array}\right] \begin{matrix} R_2 - R_1/2 \\ R_3 + R_1/2 \end{matrix} \approx \left[\begin{array}{rrr|r} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 5/2 & -3/2 & 4 \end{array}\right] R_3 + 5R_2/3$$

$$\approx \left[\begin{array}{rrr|r} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 0 & 8/3 & -8/3 \end{array}\right].$$

Using the back substitution method, we obtain the solution as

$$\frac{8}{3} z = -\frac{8}{3}, \qquad \text{or} \quad z = -1,$$

$$-\frac{3}{2} y + \frac{5}{2} z = -4, \qquad \text{or} \quad y = 1,$$

$$2x + y - z = 4, \qquad \text{or} \quad x = 1.$$

Therefore, the system of equations has the unique solution $x = 1$, $y = 1$, $z = -1$.

(ii) $$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{rrr|r} 2 & 0 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 4 & -2 & 3 & 3 \end{array}\right] \begin{matrix} R_2 - R_1/2 \\ R_3 - 2R_1 \end{matrix} \approx \left[\begin{array}{rrr|r} 2 & 0 & 1 & 3 \\ 0 & -1 & 1/2 & -1/2 \\ 0 & -2 & 1 & -3 \end{array}\right] R_3 - 2R_2$$

$$\approx \left[\begin{array}{rrr|r} 2 & 0 & 1 & 3 \\ 0 & -1 & 1/2 & -1/2 \\ 0 & 0 & 0 & -2 \end{array}\right].$$

We find that rank $(\mathbf{A}) = 2$ and rank $(\mathbf{A} \mid \mathbf{b}) = 3$. Therefore, the system of equations has no solution.

(iii) $$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 5 & -2 & 2 & 5 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix} \approx \left[\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] R_3 - R_2$$

$$\approx \left[\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system is consistent and has infinite number of solutions. We find that the last equation is satisfied for all values of x, y, z. From the second equation we get $3y - 3z = 0$, or $y = z$. From the first equation, we get $x - y + z = 1$, or $x = 1$. Therefore, we obtain the solution $x = 1$, $y = z$ and z is arbitrary.

Example 3.36 Solve the following systems of equations using Gauss elimination method.

(i)
$$\begin{aligned} 4x - 3y - 9z + 6w &= 0 \\ 2x + 3y + 3z + 6w &= 6 \\ 4x - 21y - 39z - 6w &= -24, \end{aligned}$$

(ii)
$$\begin{aligned} x + 2y - 2z &= 1 \\ 2x - 3y + z &= 0 \\ 5x + y - 5z &= 1 \\ 3x + 14y - 12z &= 5. \end{aligned}$$

Solution We have

(i) $(\mathbf{A} \mid \mathbf{b})$

$$= \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 2 & 3 & 3 & 6 & 6 \\ 4 & -21 & -39 & -6 & -24 \end{array}\right] \begin{matrix} \\ R_2 - R_1/2 \\ R_3 - R_1 \end{matrix} \approx \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 0 & 9/2 & 15/2 & 3 & 6 \\ 0 & -18 & -30 & -12 & -24 \end{array}\right] R_3 + 4R_2$$

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$$\approx \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 0 & 9/2 & 15/2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system of equations is consistent and has infinite number of solutions. Choose w as arbitrary. From the second equation, we obtain

$$\frac{9}{2}y + \frac{15}{2}z = 6 - 3w, \quad \text{or} \quad y = \frac{2}{9}\left(6 - 3w - \frac{15}{2}z\right) = \frac{1}{3}(4 - 5z - 2w).$$

From the first equation, we obtain

$$4x = 3y + 9z - 6w = 4 - 5z - 2w + 9z - 6w = 4 + 4z - 8w$$

or $\qquad x = 1 + z - 2w.$

Thus, we obtain a two parameter family of solutions

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$$x = 1 + z - 2w \quad \text{and} \quad y = (4 - 5z - 2w)/3$$

where z and w are arbitrary.

(ii)
$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & -3 & 1 & 0 \\ 5 & 1 & -5 & 1 \\ 3 & 14 & -12 & 5 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \\ R_4 - 3R_1 \end{matrix} \approx \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & -9 & 5 & -4 \\ 0 & 8 & -6 & 2 \end{array}\right] \begin{matrix} R_3 - 9R_2/7 \\ R_4 + 8R_2/7 \end{matrix}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & 0 & -10/7 & -10/7 \\ 0 & 0 & -2/7 & -2/7 \end{array}\right] R_4 - 5R_3 \approx \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & 0 & -10/7 & -10/7 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The last equation is satisfied for all values of x, y, z. From the third equation, we obtain $z = 1$. Back substitution gives $y = 1$, $x = 1$. Hence, the system of equations has a unique solution $x = 1$, $y = 1$ and $z = 1$. Since $R_4 = (24R_1 - 7R_2 + R_3)/5$, the last equation is redundent.

3.4.5 Homogeneous System of Linear Equations

Consider the homogeneous system of equations

$$\mathbf{Ax} = \mathbf{0} \tag{3.36}$$

where $\mathbf{A}$ is an $m \times n$ matrix. The homogeneous system is always consistent since $\mathbf{x} = \mathbf{0}$ (trivial solution) is always a solution. In this case, rank $(\mathbf{A})$ = rank $(\mathbf{A} \mid \mathbf{0})$. Therefore, for the homogeneous system to have a non-trivial solution, we require that rank $(\mathbf{A}) < n$. If rank $(\mathbf{A}) = r < n$ we obtain an $(n - r)$ parameter family of solutions which form a vector space of dimension $(n - r)$ as $(n - r)$ parameters can be chosen arbitrarily.

The solution space of the homogeneous system is called the *null space* and its dimension is called the *nullity* of $\mathbf{A}$. Therefore, we obtain the result

$$\text{rank }(\mathbf{A}) + \text{nullity }(\mathbf{A}) = n \quad \text{(see Theorem 3.5).}$$

Remark 14

(a) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of a linear homogeneous system, then $\alpha\,\mathbf{x}_1 + \beta\mathbf{x}_2$ is also a solution of the homogeneous system for any scalars α, β. This result does not hold for non-homogeneous systems.

(b) A homogeneous system of m equations in n unknowns and $m < n$, always possesses a non-trivial solution.

Theorem 3.8 If a non-homogeneous system of linear equations $\mathbf{Ax} = \mathbf{b}$ has solutions, then all these solutions are of the form $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$, where $\mathbf{x}_0$ is any fixed solution of $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x}_h$ is any solution of the corresponding homogeneous system.

Proof Let $\mathbf{x}$ be any solution and $\mathbf{x}_0$ be any fixed solution of $\mathbf{Ax} = \mathbf{b}$. Therefore, we have

$$\mathbf{Ax} = \mathbf{b} \quad \text{and} \quad \mathbf{Ax}_0 = \mathbf{b}.$$

Subtracting, we get

$$\mathbf{Ax} - \mathbf{Ax}_0 = \mathbf{0}, \quad \text{or} \quad \mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{0}.$$

Thus, the difference $\mathbf{x} - \mathbf{x}_0$ between any solution $\mathbf{x}$ of $\mathbf{Ax} = \mathbf{b}$ and any fixed solution $\mathbf{x}_0$ of $\mathbf{Ax} = \mathbf{b}$ is a solution of the homogeneous system $\mathbf{Ax} = \mathbf{0}$, say $\mathbf{x}_h$. Hence, the result.

Remark 15

If the non-homogeneous system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is an $m \times n$ matrix $(m \geq n)$ has a unique solution, that is, rank $(\mathbf{A}) = n$, then the corresponding homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution, that is $\mathbf{x}_h = \mathbf{0}$.

Example 3.37 Solve the following homogeneous system of equations $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{A}$ is given by

(i) $\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, (iii) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & -6 & 1 \end{bmatrix}$.

Find the rank $(\mathbf{A})$ and nullity $(\mathbf{A})$.

Solution We write the augmented matrix $(\mathbf{A} \mid \mathbf{0})$ and reduce it to row echelon form.

(i) $(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 0 \end{array}\right] \begin{matrix} R_2 - R_1/2 \\ R_3 - 3R_1/2 \end{matrix} \approx \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -3/2 & 0 \\ 0 & 1/2 & 0 \end{array}\right] R_3 + R_2/3 \approx \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right].$

Since, rank $(\mathbf{A}) = 2$ = number of unknowns, the system has only a trivial solution. Hence, nullity $(\mathbf{A}) = 0$.

(ii) $(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \approx \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -3 & 4 & 0 \end{array}\right] R_3 - 3R_2 \approx \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right].$

Since rank $(\mathbf{A}) = 3$ = number of unknowns, the homogeneous system has only a trivial solution. Therefore, nullity $(\mathbf{A}) = 0$.

(iii) $(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 2 & 3 & 1 & 4 & 0 \\ 3 & 2 & -6 & 1 & 0 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \approx \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & -1 & -3 & -2 & 0 \end{array}\right] R_3 + R_2 \approx \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$

Therefore, rank $(\mathbf{A}) = 2$ and the number of unknowns is 4. Hence, we obtain a two parameter family of solutions as $x_2 = -3x_3 - 2x_4$, $x_1 = -x_2 + x_3 - x_4 = 4x_3 + x_4$, where x_3 and x_4 are arbitrary. Therefore, nullity $(\mathbf{A}) = 2$.

3.4.6 Gauss-Jordan Method to Find the Inverse of a Matrix

Let $\mathbf{A}$ be a non-singular matrix of order n. Therefore, its inverse $\mathbf{B} = \mathbf{A}^{-1}$ exists and $\mathbf{AB} = \mathbf{I}$. Let the matrix $\mathbf{B}$ be written as $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]$, where $\mathbf{b}_i$ is the ith column of $\mathbf{B}$.

From $\mathbf{AB} = \mathbf{I}$, we obtain

$$\mathbf{AB} = \mathbf{A}[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n] = \mathbf{I} = [\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n]. \tag{3.37}$$

where $\mathbf{I}_i$ is the column vector with 1 in the ith position and zeros elsewhere. Using Gauss elimination method for solving n systems with the same coefficient matrix (see Remark 13(d)), we form the augmented matrix

$$(\mathbf{A} \mid \mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n) \text{ which is same as } (\mathbf{A} \mid \mathbf{I}),$$

where $\mathbf{I}$ is the identity matrix of order n. Using elementary row operations, we obtain

$$(\mathbf{A} \mid \mathbf{I}) \xrightarrow[\text{row operations}]{\text{Elementary}} (\mathbf{I} \mid \mathbf{B}). \tag{3.38}$$

Hence, $\mathbf{B} = \mathbf{A}^{-1}$. This method is called the *Gauss-Jordan* method. In the first step, all the elements below the pivot a_{11} are made zero. In the second step, all the elements above and below the second pivot $\bar{a}_{22}$ are made zero. At the kth step, all the elements above and below the pivot a_{kk}^* are made zero. The pivot in the (i, i) position can be made 1 at every step or when the elimination is completed.

Example 3.38 Using Gauss-Jordan method, find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Solution We have

$$(\mathbf{A} \mid \mathbf{I}) = \left[\begin{array}{rrr|rrr} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right].$$

The pivot element a_{11} is -1. We make it 1 by multiplying the first row by -1. Therefore,

$$(\mathbf{A} \mid \mathbf{I}) \approx \left[\begin{array}{rrr|rrr} 1 & -1 & -2 & -1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_2 - 3R_1 \\ R_3 + R_1 \end{matrix} \approx \left[\begin{array}{rrr|rrr} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] R_2/2$$

$$\approx \left[\begin{array}{rrr|rrr} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{matrix} R_1 + R_2 \\ R_3 - 2R_2 \end{matrix} \approx \left[\begin{array}{rrr|rrr} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] (-R_3)/5$$

$$\approx \left[\begin{array}{rrr|rrr} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right] \begin{matrix} R_1 - 3R_3/2 \\ R_2 - 7R_3/2 \end{matrix} \approx \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -7/10 & 2/10 & 3/10 \\ 0 & 1 & 0 & -13/10 & -2/10 & 7/10 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right].$$

Hence,

$$\mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}.$$

Exercise 3.3

Using the elementary row operations, determine the ranks of the following matrices.

1. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$

2. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 2 \\ 5 & -5 & 11 \end{bmatrix}.$

3. $\begin{bmatrix} 2 & 1 & -2 \\ -1 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix}.$

4. $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & 4 & -13 & -5 \end{bmatrix}.$

5. $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \end{bmatrix}.$

6. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 3 & -1 \\ 8 & 13 & 14 \end{bmatrix}.$

7. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 7 & 11 & 15 & 19 \\ 9 & 15 & 21 & 27 \end{bmatrix}.$

8. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$

9. $\begin{bmatrix} 2 & 0 & -1 & 0 \\ 4 & 1 & 0 & 5 \\ 0 & 1 & 3 & 6 \\ 6 & 1 & -2 & 6 \end{bmatrix}.$

10. $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}.$

Using the elementary column operations, determine the ranks of the following matrices.

11. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix}$.

12. $\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$.

13. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ -1 & 1 & 3 & -5 \\ 2 & 3 & 4 & 5 \end{bmatrix}$.

14. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$.

15. $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$.

Determine whether the following set of vectors is linearly independent. Find also its dimension.

16. $\{(3, 2, 4), (1, 0, 2), (1, -1, -1)\}$.

17. $\{(2, 2, 1), (1, -1, 1), (1, 0, 1)\}$.

18. $\{(2, 1, 0), (1, -1, 1), (4, 1, 2), (2, -3, 3)\}$.

19. $\{(2, 2, 1), (2, i, -1), (1 + i, -i, 1)\}$.

20. $\{(1, 1, 1), (i, i, i), (1 + i, -1 - i, i)\}$.

21. $\{(1, 1, 1, 1), (-1, 1, 1, -1), (1, 0, 1, 1), (1, 1, 0, 1)\}$.

22. $\{(1, 2, 3, 1), (2, 1, -1, 1), (4, 5, 5, 3), (5, 4, 1, 3)\}$.

23. $\{(1, 2, 3, 4), (0, 1, -1, 2), (1, 4, 1, 8), (3, 7, 8, 14)\}$.

24. $\{(1, 1, 0, 1), (1, 1, 1, 1), (4, 4, 1, 1), (1, 0, 0, 1)\}$.

25. $\{(2, 2, 0, 2), (4, 1, 4, 1), (3, 0, 4, 0)\}$.

Determine which of the following systems are consistent and find all the solutions for the consistent systems.

26. $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$.

27. $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

28. $\begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 3 \end{bmatrix}$.

29. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & -9 & 2 \\ 5 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$.

30. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \\ 22 \end{bmatrix}$.

31. $\begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & -3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

32. $\begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$.

33. $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

34. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$.

35. $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Find all the solutions of the following homogeneous systems $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{A}$ is given as the following.

36. $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}$. **37.** $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & -7 \\ -1 & -2 & 11 \end{bmatrix}$. **38.** $\begin{bmatrix} 3 & -11 & 5 \\ 4 & 1 & -10 \\ 4 & 9 & -6 \end{bmatrix}$.

39. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 6 & 12 \end{bmatrix}$. **40.** $\begin{bmatrix} 2 & -1 & -3 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & -7 & -13 & -1 \\ -1 & 5 & 9 & 1 \end{bmatrix}$. **41.** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$.

42. $\begin{bmatrix} 3 & 1 & 1 & 4 \\ 0 & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{bmatrix}$. **43.** $\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & -2 & -3 \\ 3 & 0 & -5 & -1 \\ 5 & -1 & -7 & -4 \end{bmatrix}$. **44.** $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$.

45. $\begin{bmatrix} 1 & 1 & -2 & -1 \\ 2 & 1 & 1 & -2 \\ 3 & 2 & -1 & -3 \\ 4 & 2 & 2 & -4 \end{bmatrix}$.

Using the Gauss-Jordan method find the inverses of the following matrices.

46. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. **47.** $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix}$. **48.** $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

49. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$. **50.** $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 4 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

3.5 Eigenvalue Problems

Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n. The matrix $\mathbf{A}$ may be singular or non-singular. Consider the homogeneous system of equations

$$\mathbf{Ax} = \lambda \mathbf{x} \quad \text{or} \quad (\mathbf{A} - \lambda \mathbf{I})\, \mathbf{x} = \mathbf{0} \tag{3.39}$$

where λ is a scalar and $\mathbf{I}$ is an identity matrix of order n. The homogeneous system of equations (3.39) always has a trivial solution. We need to find values of λ for which the homogeneous system (3.39) has non-trivial solutions. The values of λ, for which non-trivial solutions of the homogeneous system (3.39) exist, are called the *eigenvalues* or the *characteristic values* of $\mathbf{A}$ and the corresponding non-trivial solution vectors $\mathbf{x}$ are called the *eigenvectors* or the *characteristic vectors* of $\mathbf{A}$. If $\mathbf{x}$ is a non-trivial solution of the homogeneous system (3.39), then $\alpha \mathbf{x}$, where α is any constant is also a solution of the homogeneous system. Hence, an eigenvector is unique only upto a constant multiple. The

problem of determining the eigenvalues and the corresponding eigenvectors of a square matrix **A** is called an *eigenvalue problem*.

3.5.1 Eigenvalues and Eigenvectors

If the homogeneous system (3.39) has a non-trivial solution, then the rank of the coefficient matrix $(\mathbf{A} - \lambda\,\mathbf{I})$ is less than n, that is, the coefficient matrix must be singular. Therefore,

$$det\,(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0. \tag{3.40}$$

Expanding the determinant given in Eq. (3.40), we obtain a polynomial of degree n in λ, which is of the form

$$P_n(\lambda) = |\,\mathbf{A} - \lambda\,\mathbf{I}\,| = (-1)^n[\lambda^n - c_1\,\lambda^{n-1} + c_2\,\lambda^{n-2} - \dots + (-1)^n c_n] = 0$$

or

$$\lambda^n - c_1\,\lambda^{n-1} + c_2\,\lambda^{n-2} - \dots + (-1)^n c_n = 0. \tag{3.41}$$

where $c_1, c_2, \dots, c_n$ can be expressed in terms of the elements a_{ij} of the matrix **A**. This equation is called the *characteristic* equation of the matrix **A**. The polynomial equation $P_n(\lambda) = 0$ has n roots which can be real or complex, simple or repeated. The roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of the polynomial equation $P_n(\lambda) = 0$ are called the *eigenvalues*. By using the relation between the roots and the coefficients, we can write

$$\begin{aligned} \lambda_1 + \lambda_2 + \dots + \lambda_n &= c_1 = a_{11} + a_{22} + \dots + a_{nn} \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{n-1}\lambda_n &= c_2 \\ &\vdots \\ \lambda_1\lambda_2\dots\lambda_n &= c_n. \end{aligned}$$

If we set $\lambda = 0$ in Eq. (3.40), we get

$$|\,\mathbf{A}\,| = (-1)^{2n} c_n = c_n = \lambda_1\,\lambda_2\,\dots\,\lambda_n. \tag{3.42}$$

Therefore, we get

$$\text{sum of eigenvalues} = trace\,(\mathbf{A}) \quad \text{and} \quad \text{product of eigenvalues} = |\,\mathbf{A}\,|.$$

The set of eigenvalues is called the *spectrum* of **A** and the largest eigenvalue in magnitude is called the *spectral radius* of **A** and is denoted by ρ (**A**). If $|\,\mathbf{A}\,| = 0$, that is the matrix is singular, then from Eq. (3.42), we find that one of the eigenvalues must be zero. Conversely, if one of the eigenvalues is zero, then $|\,\mathbf{A}\,| = 0$. Note that if **A** is a diagonal or an upper triangular or a lower triangular matrix, then the diagonal elements of the matrix **A** are the eigenvalues of **A**.

After determining the eigenvalues λ_i's, we solve the homogeneous system $(\mathbf{A} - \lambda_i\,\mathbf{I})\mathbf{x} = \mathbf{0}$ for each $\lambda_i, i = 1, 2, \dots, n$ to obtain the corresponding eigenvectors.

Properties of eigenvalues and eigenvectors

Let λ be an eigenvalue of **A** and **x** be its corresponding eigenvector. Then we have the following results.

1. $\alpha\,\mathbf{A}$ has eigenvalue $\alpha\lambda$ and the corresponding eigenvector is **x**.

$$\mathbf{A}\mathbf{x} = \lambda\,\mathbf{x} \Rightarrow \alpha\mathbf{A}\mathbf{x} = (\alpha\lambda)\mathbf{x}.$$

2. $\mathbf{A}^m$ has eigenvalue λ^m and the corresponding eigenvector is $\mathbf{x}$ for any positive integer m. Premultiplying both sides of $\mathbf{Ax} = \lambda\mathbf{x}$ by $\mathbf{A}$, we get

$$\mathbf{AAx} = \mathbf{A}\lambda\mathbf{x} = \lambda\mathbf{Ax} = \lambda(\lambda\mathbf{x}) \quad \text{or} \quad \mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}.$$

Therefore, $\mathbf{A}^2$ has the eigenvalue λ^2 and the corresponding eigenvector is $\mathbf{x}$. Premultiplying successively m times, we obtain the result.

3. $\mathbf{A} - k\mathbf{I}$ has the eigenvalue $\lambda - k$, for any scalar k and the corresponding eigenvector is $\mathbf{x}$.

$$\mathbf{Ax} = \lambda\mathbf{x} \Rightarrow \mathbf{Ax} - k\,\mathbf{Ix} = \lambda\mathbf{x} - k\mathbf{x}$$

or
$$(\mathbf{A} - k\mathbf{I})\mathbf{x} = (\lambda - k)\,\mathbf{x}.$$

4. $\mathbf{A}^{-1}$ (if it exists) has the eigenvalue $1/\lambda$ and the corresponding eigenvector is $\mathbf{x}$. Premultiplying both sides of $\mathbf{Ax} = \lambda\mathbf{x}$ by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{Ax} = \lambda\mathbf{A}^{-1}\mathbf{x} \quad \text{or} \quad \mathbf{A}^{-1}\mathbf{x} = (1/\lambda)\,\mathbf{x}.$$

5. $(\mathbf{A} - k\mathbf{I})^{-1}$ has the eigenvalue $1/(\lambda - k)$ and the corresponding eigenvector is $\mathbf{x}$ for ar scalar k.

6. $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues, since a determinant can be expanded by rows columns.

7. For a real matrix $\mathbf{A}$, if $\alpha + i\beta$ is an eigenvalue, then its conjugate $\alpha - i\beta$ is also an eigenval (since the characteristic equation has real coefficients). When the matrix $\mathbf{A}$ is complex, th property does not hold.

We now present an important result which gives the relationship of a matrix $\mathbf{A}$ and its characteris equation.

Theorem 3.9 **(Cayley-Hamilton theorem)** Every square matrix $\mathbf{A}$ satisfies its own characteris equation, that is

$$\mathbf{A}^n - c_1\mathbf{A}^{n-1} + \ldots + (-1)^{n-1}c_{n-1}\mathbf{A} + (-1)^n c_n\mathbf{I} = \mathbf{0}. \tag{3.4}$$

Proof The cofactors of the elements of the determinant $|\,\mathbf{A} - \lambda\mathbf{I}\,|$ are polynomials in λ of degr $(n-1)$ or less. Therefore, the elements of the adjoint matrix (transpose of the cofactor matrix) are al polynomials in λ of degree $(n-1)$ or less. Hence, we can express the adjoint matrix as a polynom in λ whose coefficients $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$ are square matrices of order n having elements as functio of the elements of the matrix $\mathbf{A}$. Thus, we can write

$$adj\,(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \ldots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n.$$

We also have

$$(\mathbf{A} - \lambda\mathbf{I})\,adj\,(\mathbf{A} - \lambda\mathbf{I}) = |\,\mathbf{A} - \lambda\mathbf{I}\,|\,\mathbf{I}.$$

Therefore, we can write for any λ

$$(\mathbf{A} - \lambda\mathbf{I})(\mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \ldots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n) = \lambda^n\mathbf{I} - c_1\lambda^{n-1}\mathbf{I} + \ldots + (-1)^{n-1}c_{n-1}\lambda\mathbf{I} + (-1)^n c$$

Comparing the coefficients of various powers of λ, we obtain

$$-\mathbf{B}_1 = \mathbf{I}$$
$$\mathbf{AB}_1 - \mathbf{B}_2 = -c_1\mathbf{I}$$
$$\mathbf{AB}_2 - \mathbf{B}_3 = c_2\mathbf{I}$$

$$\dots$$

$$\mathbf{AB}_{n-1} - \mathbf{B}_n = (-1)^{n-1} c_{n-1} \mathbf{I}$$

$$\mathbf{AB}_n = (-1)^n c_n \mathbf{I}.$$

Premultiplying these equations by $\mathbf{A}^n, \mathbf{A}^{n-1}, \dots, \mathbf{A}, \mathbf{I}$ respectively and adding, we get

$$\mathbf{A}^n - c_1 \mathbf{A}^{n-1} + \dots + (-1)^{n-1} c_{n-1} \mathbf{A} + (-1)^n c_n \mathbf{I} = \mathbf{0}$$

which proves the theorem.

Remark 16a We have for any non-zero vector $\mathbf{x}$

$$\mathbf{Ix} = 1\mathbf{x}$$

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{A}^2\mathbf{x} = \lambda^2 \mathbf{x}$$

$$\dots$$

$$\mathbf{A}^n \mathbf{x} = \lambda^n \mathbf{x}.$$

Multiplying these equations by $(-1)^n c_n, (-1)^{n-1} c_{n-1}, \dots, (-1) c_1$, 1 respectively and adding, we get

$$(-1)^n c_n \mathbf{Ix} + (-1)^{n-1} c_{n-1} \mathbf{Ax} + \dots + (-1)^1 c_1 \mathbf{A}^{n-1}\mathbf{x} + \mathbf{A}^n \mathbf{x}$$

$$= (-1)^n c_n \mathbf{x} + (-1)^{n-1} c_{n-1} \lambda \mathbf{x} + \dots + (-1)^1 c_1 \lambda^{n-1} \mathbf{x} + \lambda^n \mathbf{x}$$

or

$$[\mathbf{A}^n - c_1 \mathbf{A}^{n-1} + \dots + (-1)^{n-1} c_{n-1} \mathbf{A} + (-1)^n c_n \mathbf{I}]\mathbf{x}$$

$$= [\lambda^n - c_1 \lambda^{n-1} + \dots + (-1)^{n-1} c_{n-1} \lambda + (-1)^n c_n]\, \mathbf{x} = 0\,\mathbf{x} = \mathbf{0}.$$

Since $\mathbf{x} \neq \mathbf{0}$, there is a possibility that

$$\mathbf{A}^n - c_1 \mathbf{A}^{n-1} + \dots + (-1)^{n-1} c_{n-1} \mathbf{A} + (-1)^n c_n \mathbf{I} = \mathbf{0}.$$

Remark 16b

(a) We can use Eq. (3.43) to find $\mathbf{A}^{-1}$ (if it exists) in terms of the powers of the matrix $\mathbf{A}$.

Premultiplying both sides in Eq. (3.43) by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{A}^n - c_1 \mathbf{A}^{-1}\mathbf{A}^{n-1} + \dots + (-1)^{n-1} c_{n-1} \mathbf{A}^{-1}\mathbf{A} + (-1)^n c_n \mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$$

or

$$\mathbf{A}^{-1} = -\frac{(-1)^n}{c_n}[\mathbf{A}^{n-1} - c_1 \mathbf{A}^{n-2} + \dots + (-1)^{n-1} c_{n-1} \mathbf{I}] \tag{3.44}$$

(b) We can use Eq. (3.43) to obtain $\mathbf{A}^n$ in terms of lower powers of $\mathbf{A}$ as

$$\mathbf{A}^n = c_1 \mathbf{A}^{n-1} - c_2 \mathbf{A}^{n-2} + \dots + (-1)^{n-1} c_n \mathbf{I}. \tag{3.45}$$

Example 3.39 Verify Cayley-Hamilton theorem for the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Also (i) obtain $\mathbf{A}^{-1}$ and $\mathbf{A}^3$, (ii) find eigenvalues of $\mathbf{A}, \mathbf{A}^2$ and verify that eigenvalues of $\mathbf{A}^2$ are squares

of those of **A**, (iii) find the spectral radius of **A**.

Solution The characteristic equation of **A** is given by

$$|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\{(1-\lambda)^2-4\} - 2\{-(1-\lambda)-2\}$$

$$= (1-\lambda)(\lambda^2-2\lambda-3) - 2(\lambda-3) = -\lambda^3+3\lambda^2-\lambda+3 = 0.$$

Now,
$$\mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}.$$

We have

$$-\mathbf{A}^3+3\mathbf{A}^2-\mathbf{A}+3\mathbf{I} = -\begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix} + 3\begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}. \tag{3.46}$$

Hence, **A** satisfies the characteristic equation $-\lambda^3+3\lambda^2-\lambda+3=0$.

(i) From Eq. (3.46), we get

$$\mathbf{A}^{-1} = \frac{1}{3}[\mathbf{A}^2-3\mathbf{A}+\mathbf{I}] = \frac{1}{3}\left[\begin{pmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 6 & 0 \\ -3 & 3 & 6 \\ 3 & 6 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}.$$

From Eq. (3.46), we get

$$\mathbf{A}^3 = 3\mathbf{A}^2-\mathbf{A}+3\mathbf{I} = \begin{bmatrix} -3 & 12 & 12 \\ 0 & 9 & 12 \\ 0 & 18 & 15 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}.$$

(ii) Eigenvalues of **A** are the roots of

$$\lambda^3 - 3\lambda^2 + \lambda - 3 = 0 \quad \text{or} \quad (\lambda - 3)(\lambda^2 + 1) = 0 \quad \text{or} \quad \lambda = 3, i, -i.$$

The characteristic equation of $\mathbf{A}^2$ is given by

$$\begin{vmatrix} -1-\lambda & 4 & 4 \\ 0 & 3-\lambda & 4 \\ 0 & 6 & 5-\lambda \end{vmatrix} = (-1-\lambda)[(3-\lambda)(5-\lambda) - 24] = 0$$

or $$(\lambda + 1)(\lambda^2 - 8\lambda - 9) = 0 \quad \text{or} \quad (\lambda + 1)(\lambda - 9)(\lambda + 1) = 0.$$

The eigenvalues of $\mathbf{A}^2$ are $9, -1, -1$ which are the squares of the eigenvalues of $\mathbf{A}$.

(iii) The spectral radius of $\mathbf{A}$ is given by

$$\rho(\mathbf{A}) = \text{largest eigenvalue in magnitude} = \max_i |\lambda_i| = 3.$$

Example 3.40 If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then show that $\mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}$ for $n \geq 3$. Hence, find $\mathbf{A}^{50}$.

Solution The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1) = 0, \quad \text{or} \quad \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

Using Cayley-Hamilton theorem, we get

$$\mathbf{A}^3 - \mathbf{A}^2 - \mathbf{A} + \mathbf{I} = \mathbf{0}, \quad \text{or} \quad \mathbf{A}^3 - \mathbf{A}^2 = \mathbf{A} - \mathbf{I}.$$

Premultiplying both sides successively by $\mathbf{A}$, we obtain

$$\begin{aligned} \mathbf{A}^3 - \mathbf{A}^2 &= \mathbf{A} - \mathbf{I} \\ \mathbf{A}^4 - \mathbf{A}^3 &= \mathbf{A}^2 - \mathbf{A} \\ &\dots \\ \mathbf{A}^{n-1} - \mathbf{A}^{n-2} &= \mathbf{A}^{n-3} - \mathbf{A}^{n-4} \\ \mathbf{A}^n - \mathbf{A}^{n-1} &= \mathbf{A}^{n-2} - \mathbf{A}^{n-3}. \end{aligned}$$

Adding these equations, we get

$$\mathbf{A}^n - \mathbf{A}^2 = \mathbf{A}^{n-2} - \mathbf{I}, \quad \text{or} \quad \mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}, \; n \geq 3.$$

Using this equation recursively, we obtain

$$\begin{aligned} \mathbf{A}^n &= (\mathbf{A}^{n-4} + \mathbf{A}^2 - \mathbf{I}) + \mathbf{A}^2 - \mathbf{I} = \mathbf{A}^{n-4} + 2(\mathbf{A}^2 - \mathbf{I}) \\ &= (\mathbf{A}^{n-6} + \mathbf{A}^2 - \mathbf{I}) + 2(\mathbf{A}^2 - \mathbf{I}) = \mathbf{A}^{n-6} + 3(\mathbf{A}^2 - \mathbf{I}) \\ &\dots \\ &= \mathbf{A}^{n-(n-2)} + \frac{1}{2}(n-2)(\mathbf{A}^2 - \mathbf{I}) = \frac{n}{2}\mathbf{A}^2 - \frac{1}{2}(n-2)\mathbf{I}. \end{aligned}$$

Substituting $n = 50$, we get

$$\mathbf{A}^{50} = 25\mathbf{A}^2 - 24\mathbf{I} = 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}.$$

Example 3.41 Find the eigenvalues and the corresponding eigenvectors of the following matrices

(i) $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, (ii) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, (iii) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$.

Solution

(i) The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 3\lambda - 10 = 0, \quad \text{or} \quad \lambda = -2, 5.$$

Corresponding to the eigenvalue $\lambda = -2$, we have

$$(\mathbf{A} + 2\mathbf{I})\mathbf{x} = \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad 3x_1 + 4x_2 = 0 \quad \text{or} \quad x_1 = -\frac{4}{3}x_2.$$

Hence, the eigenvector $\mathbf{x}$ is given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_2/3 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}.$$

Since an eigenvector is unique upto a constant multiple, we can take the eigenvector as $[-4, 3]^T$.

Corresponding to the eigenvalue $\lambda = 5$, we have

$$(\mathbf{A} - 5\mathbf{I})\mathbf{x} = \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad x_1 - x_2 = 0, \quad \text{or} \quad x_1 = x_2.$$

Therefore, the eigenvector is given by $\mathbf{x} = (x_1, x_2)^T = x_1(1, 1)^T$ or simply $(1, 1)^T$.

(ii) The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 - 2\lambda + 2 = 0, \quad \text{or} \quad \lambda = 1 \pm i.$$

Corresponding to the eigenvalue $\lambda = 1 + i$, we have

$$[\mathbf{A} - (1+i)\mathbf{I}]\mathbf{x} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$-ix_1 + x_2 = 0 \quad \text{and} \quad -x_1 - ix_2 = 0.$$

Both the equations reduce to $-x_1 - ix_2 = 0$. Choosing $x_2 = 1$, we get $x_1 = -i$. Therefore, the eigenvector is $\mathbf{x} = [-i, 1]^T$.

Corresponding to the eigenvalue $\lambda = 1 - i$, we have

$$[\mathbf{A} - (1 - i)\mathbf{I}]\mathbf{x} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or
$$ix_1 + x_2 = 0 \quad \text{and} \quad -x_1 + ix_2 = 0.$$

Both the equations reduce to $-x_1 + ix_2 = 0$. Choosing $x_2 = 1$, we get $x_1 = i$. Therefore, the eigenvector is $\mathbf{x} = [i, 1]^T$.

Remark 17

For a real matrix **A**, the eigenvalues and the corresponding eigenvectors can be complex.

(iii) The characteristic equation of **A** is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

or $\lambda = 1, 2, 3$.

Corresponding to the eigenvalue $\lambda = 1$, we have

$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} x_2 + x_3 = 0 \\ x_1 + x_3 = 0. \end{cases}$$

We obtain two equations in three unknowns. One of the variables x_1, x_2, x_3 can be chosen arbitrarily. Taking $x_3 = 1$, we obtain the eigenvector as $[-1, -1, 1]^T$.

Corresponding to the eigenvalue $\lambda = 2$, we have

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $x_1 = 0$, $x_3 = 0$ and x_2 arbitrary. Taking $x_2 = 1$, we obtain the eigenvector as $[0, 1, 0]^T$.

Corresponding to the eigenvalue $\lambda = 3$, we have

$$(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} x_1 = 0 \\ x_2 + x_3 = 0. \end{cases}$$

Choosing $x_3 = 1$, we obtain the eigenvector as $[0, -1, 1]^T$.

Example 3.42 Find the eigenvalues and the corresponding eigenvectors of the following matrices

(i) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, (ii) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (iii) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution In each of the above problems, we obtain the characteristic equation as $(1-\lambda)^3 = 0$. Therefore, the eigenvalues are $\lambda = 1, 1, 1$, a repeated value. Since a 3×3 matrix has 3 eigenvalues, it is important to know, whether the given matrix has 3 linearly independent eigenvectors or it has lesser number of linearly independent eigenvectors.

Corresponding to the eigenvalue $\lambda = 1$, we obtain the following eigenvectors.

(i) $$(\mathbf{A}-\mathbf{I})\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{cases} x_2 = 0, \\ x_3 = 0 \\ x_1 \text{ arbitrary.} \end{cases}$$

Choosing $x_1 = 1$, we obtain the solution as $[1, 0, 0]^T$.
Hence, **A** has only one independent eigenvector.

(ii) $$(\mathbf{A}-\mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{cases} x_2 = 0 \\ x_1, x_3 \text{ arbitrary.} \end{cases}$$

Taking $x_1 = 0, x_3 = 1$ and $x_1 = 1, x_3 = 0$, we obtain two linearly independent solutions

$$\mathbf{x}_1 = [0, 0, 1]^T, \mathbf{x}_2 = [1, 0, 0]^T.$$

In this case **A** has two linearly independent eigenvectors.

(iii) $$(\mathbf{A}-\mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system is satisfied for arbitrary values of all the three variables. Hence, we obtain three linearly independent eigenvectors, which can be written as

$$\mathbf{x}_1 = [1, 0, 0]^T, \mathbf{x}_2 = [0, 1, 0]^T, \mathbf{x}_3 = [0, 0, 1]^T.$$

We now state some important results regarding the relationship between the eigenvalues of a matrix and the corresponding linearly independent eigenvectors.

1. Eigenvectors corresponding to distinct eigenvalues are linearly independent.
2. If λ is an eigenvalue of multiplicity m of a square matrix **A** of order n, then the number of linearly independent eigenvectors associated with λ is given by

$$p = n - r, \quad \text{where} \quad r = \text{rank}\,(\mathbf{A} - \lambda\,\mathbf{I}),\ 1 \le p \le m.$$

Remark 18

In Example 3.41, all the eigenvalues are distinct and therefore, the corresponding eigenvectors are linearly independent. In Example 3.42 the eigenvalue $\lambda = 1$ is of multiplicity 3. We find that in

(i) Example 3.42(i), the rank of the matrix **A** – **I** is 2 and we obtain one linearly independent eigenvector.

(ii) Example 3.42(ii), the rank of the matrix **A** – **I** is 1 and we obtain two linearly independent eigenvectors.

(iii) Example 3.42(iii), the rank of the matrix $\mathbf{A} - \mathbf{I}$ is 0 and we obtain three linearly independent eigenvectors.

3.5.2 Similar and Diagonalizable Matrices

Similar matrices

Let **A** and **B** be square matrices of the same order. The matrix **A** is said to be similar to the matrix **B** if there exists an invertible matrix **P** such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP} \quad \text{or} \quad \mathbf{PA} = \mathbf{BP}. \tag{3.47}$$

Postmultiplying both sides in Eq. (3.47) by $\mathbf{P}^{-1}$, we get

$$\mathbf{PAP}^{-1} = \mathbf{B}.$$

Therefore, **A** is similar to **B** if and only if **B** is similar to **A**. The matrix **P** is called the *similarity matrix*. We now prove a result regarding eigenvalues of similar matrices.

Theorem 3.10 Similar matrices have the same characteristic equation (and hence the same eigenvalues). Further, if $\mathbf{x}$ is an eigenvector of **A** corresponding to the eigenvalue λ, then $\mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of **B** corresponding to the eigenvalue λ, where **P** is the similarity matrix.

Proof Let λ be an eigenvalue and $\mathbf{x}$ be the corresponding eigenvector of **A**. That is

$$\mathbf{Ax} = \lambda\mathbf{x}.$$

Premultiplying both sides by an invertible matrix $\mathbf{P}^{-1}$, we obtain

$$\mathbf{P}^{-1}\mathbf{Ax} = \lambda\mathbf{P}^{-1}\mathbf{x}.$$

Set $\mathbf{x} = \mathbf{Py}$. We get

$$\mathbf{P}^{-1}\mathbf{APy} = \lambda\mathbf{P}^{-1}\mathbf{Py}, \quad \text{or} \quad (\mathbf{P}^{-1}\mathbf{AP})\mathbf{y} = \lambda\,\mathbf{y} \quad \text{or} \quad \mathbf{By} = \lambda\mathbf{y}.$$

where $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$. Therefore, **B** has the same eigenvalues as **A**, that is the characteristic equation of **B** is same as the characteristic equation of **A**. Now, **A** and **B** are similar matrices. Therefore, similar matrices have the same characteristic equation (and hence the same eigenvalues). Also $\mathbf{x} = \mathbf{Py}$, that is eigenvectors of **A** and **B** are related by $\mathbf{x} = \mathbf{Py}$ or $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$.

Remark 19

(a) Theorem 3.10 states that if two matrices are similar, then they have the same characteristic equation and hence the same eigenvalues. However, the converse of this theorem is not true. Two matrices which have the same characteristic equation need not always be similar.

(b) If **A** is similar to **B** and **B** is similar to **C**, then **A** is similar to **C**.
Let there be two invertible matrices **P** and **Q** such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP} \quad \text{and} \quad \mathbf{B} = \mathbf{Q}^{-1}\mathbf{CQ}.$$

Then $$\mathbf{A} = \mathbf{P}^{-1}\mathbf{Q}^{-1}\mathbf{CQP} = \mathbf{R}^{-1}\mathbf{CR}, \quad \text{where} \quad \mathbf{R} = \mathbf{QP}.$$

Example 3.43 Examine whether **A** is similar to **B**, where

(i) $\mathbf{A} = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$.

(ii) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution The given matrices are similar if there exists an invertible matrix **P** such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP} \quad \text{or} \quad \mathbf{PA} = \mathbf{BP}.$$

Let $\mathbf{P} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We shall determine a, b, c, d such that $\mathbf{PA} = \mathbf{BP}$ and then check whether **P** is non-singular.

(i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, or $\begin{bmatrix} 5a-2b & 5a \\ 5c-2d & 5c \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -3a+4c & -3b+4d \end{bmatrix}$.

Equating the corresponding elements, we obtain the system of equations

$$\begin{aligned} 5a - 2b &= a + 2c, &\quad \text{or} \quad && 4a - 2b - 2c = 0 \\ 5a &= b + 2d, &\quad \text{or} \quad && 5a - b - 2d = 0 \\ 5c - 2d &= -3a + 4c, &\quad \text{or} \quad && 3a + c - 2d = 0 \\ 5c &= -3b + 4d, &\quad \text{or} \quad && 3b + 5c - 4d = 0. \end{aligned}$$

A solution to this system of equations is $a = 1, b = 1, c = 1, d = 2$.

Therefore, we get $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, which is a non-singular matrix. Hence the matrices **A** and **B** are similar.

(ii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$.

Equating the corresponding elements, we get

$$a = a + c, b = b + d \quad \text{or} \quad c = d = 0.$$

Therefore, $\mathbf{P} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, which is a singular matrix.

Since an invertible matrix **P** does not exist, the matrices **A** and **B** are not similar.

It can be verified that the eigenvalues of **A** are 1, 1 whereas the eigenvalues of **B** are 0, 2.

In practice, it is usually difficult to obtain a non-singular matrix **P** which satisfies the equation $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$ for any two matrices **A** and **B**. However, it is possible to obtain the matrix **P** when **A** or **B** is a diagonal matrix. Thus, our interest is to find a similarity matrix **P** such that for a given matrix **A**, we have

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} \quad \text{or} \quad \mathbf{PDP}^{-1} = \mathbf{A}$$

where **D** is a diagonal matrix. If such a matrix exists, then we say that the matrix **A** is *diagonalizable*.

Diagonalizable matrices

A matrix $\mathbf{A}$ is diagonalizable, if it is similar to a diagonal matrix, that is there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix. Since, similar matrices have the same eigenvalues, the diagonal elements of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$. A necessary and sufficient condition for the existence of $\mathbf{P}$ is given in the following theorem.

Theorem 3.11 A square matrix $\mathbf{A}$ of order n is diagonalizable if and only if it has n linearly independent eigenvectors.

Proof We shall prove the case that if $\mathbf{A}$ has n linearly independent eigenvectors, then $\mathbf{A}$ is diagonalizable. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct) of the matrix $\mathbf{A}$ in the same order, that is the eigenvector $\mathbf{x}_j$ corresponds to the eigenvalue λ_j, $j = 1, 2, \ldots, n$. Let

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] \quad \text{and} \quad \mathbf{D} = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

be the diagonal matrix with eigenvalues of $\mathbf{A}$ as its diagonal elements. The matrix $\mathbf{P}$ is called the *modal matrix* of $\mathbf{A}$ and $\mathbf{D}$ is called the *spectral matrix* of $\mathbf{A}$. We have

$$\mathbf{AP} = \mathbf{A}[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = (\mathbf{Ax}_1, \mathbf{Ax}_2, \ldots, \mathbf{Ax}_n)$$

$$= (\lambda_1\mathbf{x}_1, \lambda_2\mathbf{x}_2, \ldots, \lambda_n\mathbf{x}_n) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)\mathbf{D} = \mathbf{PD}. \tag{3.48}$$

Since the columns of $\mathbf{P}$ are linearly independent, the rank of $\mathbf{P}$ is n and therefore the matrix P is invertible. Premultiplying both sides in Eq. (3.48) by $\mathbf{P}^{-1}$, we obtain

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^{-1}\mathbf{PD} = \mathbf{D} \tag{3.49}$$

which implies that $\mathbf{A}$ is similar to $\mathbf{D}$. Therefore, the matrix of eigenvectors $\mathbf{P}$ reduces a matrix $\mathbf{A}$ to its diagonal form.

Postmultiplying both sides in Eq. (3.48) by $\mathbf{P}^{-1}$, we obtain

$$\mathbf{A} = \mathbf{PDP}^{-1}. \tag{3.50}$$

Remark 20

(a) A square matrix $\mathbf{A}$ of order n has always n linearly indendent eigenvectors when its eigenvalues are distinct. The matrix may also have n linearly independent eigenvectors even when some eigenvalues are repeated (see Example 3.42(iii)). Therefore, there is no restriction imposed on the eigenvalues of the matrix $\mathbf{A}$ in Theorem 3.11.

(b) From Eq. (3.50), we obtain

$$\mathbf{A}^2 = \mathbf{AA} = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}^2\mathbf{P}^{-1}.$$

Repeating the pre-multiplication (post-multiplication) m times, we get

$$\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1} \text{ for any positive integer } m.$$

Therefore, if $\mathbf{A}$ is diagonalizable, so is $\mathbf{A}^m$.

(c) If $\mathbf{D}$ is a diagonal matrix of order n, and

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{bmatrix}, \text{ then } \mathbf{D}^m = \begin{bmatrix} \lambda_1^m & & & \mathbf{0} \\ & \lambda_2^m & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n^m \end{bmatrix}$$

for any positive integer m. If $Q(\mathbf{D})$ is a polynomial in $\mathbf{D}$, then we get

$$Q(\mathbf{D}) = \begin{bmatrix} Q(\lambda_1) & & & \mathbf{0} \\ & Q(\lambda_2) & & \\ & & \ddots & \\ \mathbf{0} & & & Q(\lambda_n) \end{bmatrix}.$$

Now, let a matrix $\mathbf{A}$ be diagonalizable. Then, we have

$$\mathbf{A} = \mathbf{PDP}^{-1} \quad \text{and} \quad \mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$$

for any positive integer m. Hence, we obtain

$$Q(\mathbf{A}) = \mathbf{P}Q(\mathbf{D})\mathbf{P}^{-1}$$

for any matrix polynomial $Q(\mathbf{A})$.

Example 3.44 Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

is diagonalizable. Hence, find $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. Then, obtain the matrix $\mathbf{B} = \mathbf{A}^2 + 5\mathbf{A} + 3\mathbf{I}$.

Solution The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0, \quad \text{or} \quad \lambda = 1, 2, 3.$$

Since the matrix $\mathbf{A}$ has three distinct eigenvalues, it has three linearly independent eigenvectors and hence it is diagonalizable.

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is the solution of the system

$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ The solution is } \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is the solution of the system

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ The solution is } \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is the solution of the system

$$(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ The solution is } \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Hence, the modal matrix is given by

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{P}^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

It can be verified that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \text{diag }(1, 2, 3)$.

We have $\mathbf{D} = \text{diag }(1, 2, 3)$, $\mathbf{D}^2 = \text{diag }(1, 4, 9)$.

Therefore, $$\mathbf{A}^2 + 5\mathbf{A} + 3\mathbf{I} = \mathbf{P}(\mathbf{D}^2 + 5\mathbf{D} + 3\mathbf{I})\mathbf{P}^{-1}$$

Now, $$\mathbf{D}^2 + 5\mathbf{D} + 3\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix}.$$

Hence, we obtain

$$\mathbf{A}^2 + 5\mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 25 & 8 & -8 \\ -18 & 9 & 18 \\ -2 & 8 & 19 \end{bmatrix}.$$

Example 3.45 Examine whether the matrix $\mathbf{A}$, where $\mathbf{A}$ is given by

(i) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$. (ii) $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

is diagonalizable. If so, obtain the matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

Solution

(i) The characteristic equation of the matrix $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(2-\lambda)(2-\lambda) - 2] - [2 - 2(2-\lambda)] = (1-\lambda)(2-\lambda)(2-\lambda) = 0,$$

or $\lambda = 1, 2, 2$. We first find the eigenvectors corresponding to the repeated eigenvalue $\lambda = 2$. We have the system

$$(\mathbf{A}-2\mathbf{I})\mathbf{x}=\begin{bmatrix}-1 & 2 & 2\\ 0 & 0 & 1\\ -1 & 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

Since the rank of the coefficient matrix is 2, it has one linearly independent eigenvector. We obtain another linearly independent eigenvector corresponding to the eigenvalue $\lambda = 1$. Since the matrix **A** has only two linearly independent eigenvectors, the matrix is not diagonalizable.

(ii) The characteristic equation of the matrix **A** is given by

$$|\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & -\lambda\end{vmatrix}=0 \text{ or } \lambda^3+\lambda^2-21\lambda-45=0, \text{ or } \lambda=5,-3,-3.$$

Eigenvector corresponding to the eigenvalue $\lambda = 5$ is the solution of the system

$$(\mathbf{A}-5\mathbf{I})\mathbf{x}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

A solution of this system is $[1, 2, -1]^T$.

Eigenvectors corresponding to $\lambda = -3$ are the solutions of the system

$$(\mathbf{A}+3\mathbf{I})\mathbf{x}=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \quad \text{or} \quad x_1+2x_2-3x_3=0.$$

The rank of the coefficient matrix is 1. Therefore, the system has two linearly independent solutions. We use the equation $x_1 + 2x_2 - 3x_3 = 0$ to find two linearly independent eigenvectors. Taking $x_3 = 0, x_2 = 1$, we obtain the eigenvector $[-2, 1, 0]^T$ and taking $x_2 = 0, x_3 = 1$, we obtain the eigenvector $[3, 0, 1]^T$. The given 3×3 matrix has three linearly independent eigenvectors. Therefore, the matrix **A** is diagonalizable. The modal matrix **P** is given by

$$\mathbf{P}=\begin{bmatrix}1 & -2 & 3\\ 2 & 1 & 0\\ -1 & 0 & 1\end{bmatrix} \text{ and } \mathbf{P}^{-1}=\frac{1}{8}\begin{bmatrix}1 & 2 & -3\\ -2 & 4 & 6\\ 1 & 2 & 5\end{bmatrix}.$$

It can be verified that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \text{diag}\,(5, -3, -3)$.

Example 3.46 The eigenvectors of a 3×3 matrix **A** corresponding to the eigenvalues 1, 1, 3 are $[1, 0, -1]^T$, $[0, 1, -1]^T$ and $[1, 1, 0]^T$ respectively. Find the matrix **A**.

Solution We have

$$\text{modal matrix } \mathbf{P}=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & -1 & 0\end{bmatrix} \text{ and the spectral matrix } \mathbf{D}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}.$$

We find that $\mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

Therefore, $\mathbf{A} = \mathbf{P\,D\,P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.5.3 Special Matrices

In this section we define some special matrices and study the properties of the eigenvalues and eigenvectors of these matrices. These matrices have applications in many areas. We first give some definitions.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ be two vectors of dimension n in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then we define the following:

Inner Product (dot product) of vectors Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^n$. Then

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i \tag{3.51}$$

is called the *inner product* of the vectors $\mathbf{x}$ and $\mathbf{y}$ and is a scalar. The inner product is also denoted by $< \mathbf{x}, \mathbf{y} >$. In this case $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$. Note that $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{C}^n$, then the inner product of these vectors is defined as

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\,\overline{\mathbf{y}} = \sum_{i=1}^{n} x_i \overline{y}_i \quad \text{and} \quad \mathbf{y} \cdot \mathbf{x} = \mathbf{y}^T\,\overline{\mathbf{x}} = \sum_{i=1}^{n} y_i \overline{x}_i$$

where $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ are complex conjugate vectors of $\mathbf{x}$ and $\mathbf{y}$ respectively. Note that $\mathbf{x} \cdot \mathbf{y} = \overline{\mathbf{y} \cdot \mathbf{x}}$. It can be easily verified that

$$(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$$

for any vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and scalars α, β.

Length (norm of a vector) Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then

$$\| \mathbf{x} \| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

is called the *length* or the *norm* of the vector $\mathbf{x}$.

Unit vector The vector $\mathbf{x}$ is called a *unit vector* if $\| \mathbf{x} \| = 1$. If $\mathbf{x} \neq \mathbf{0}$, then the vector $\mathbf{x}/\| \mathbf{x} \|$ is always a unit vector.

Orthogonal vectors The vectors $\mathbf{x}$ and $\mathbf{y}$ for which $\mathbf{x} \cdot \mathbf{y} = 0$ are said to be *orthogonal vectors*.

Orthonormal vectors The vectors of $\mathbf{x}$ and $\mathbf{y}$ for which

$$\mathbf{x} \cdot \mathbf{y} = 0 \quad \text{and} \quad \| \mathbf{x} \| = 1, \| \mathbf{y} \| = 1$$

are called orthonormal vectors. If $\mathbf{x}$, $\mathbf{y}$ are any vectors and $\mathbf{x} \cdot \mathbf{y} = 0$, then $\mathbf{x}/\| \mathbf{x} \|$, $\mathbf{y}/\| \mathbf{y} \|$ a orthonormal.

For example, the set of vectors

(i) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form an orthonormal set in $\mathbb{R}^3$.

(ii) $\begin{pmatrix} 3i \\ -4i \\ 0 \end{pmatrix}, \begin{pmatrix} -4i \\ 3i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1+i \end{pmatrix}$ form an orthogonal set in $\mathbb{C}^3$ and $\begin{pmatrix} 3i/5 \\ -4i/5 \\ 0 \end{pmatrix}, \begin{pmatrix} -4i/5 \\ 3i/5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ (1+i)/\sqrt{2} \end{pmatrix}$

form an orthonormal set in $\mathbb{C}^3$.

Orthonormal and unitary system of vectors Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be m vectors in $\mathbb{R}^n$. Then, th set of vectors forms an *orthonormal system* of vectors, if

$$\mathbf{x}_i \cdot \mathbf{x}_j = \mathbf{x}_i^T \mathbf{x}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be m vectors in $\mathbb{C}^n$. Then, this set of vectors forms an *unitary system* of vector if

$$\mathbf{x}_i \cdot \mathbf{x}_j = \mathbf{x}_i^T \, \overline{\mathbf{x}}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

In section 3.2.2, we have defined symmetric, skew-symmetric, Hermitian and skew-Hermitian matrices We now define a few more special matrices.

Orthogonal matrices A real matrix $\mathbf{A}$ is *orthogonal* if $\mathbf{A}^{-1} = \mathbf{A}^T$. A simple example is

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Unitary matrices A complex matrix $\mathbf{A}$ is *unitary* if $\mathbf{A}^{-1} = (\overline{\mathbf{A}})^T$, or $(\overline{\mathbf{A}})^{-1} = \mathbf{A}^T$. If $\mathbf{A}$ is real, then unitary matrix is same as orthogonal matrix. We note the following.

1. If $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices, then $\alpha\mathbf{A} + \beta\mathbf{B}$ is also Hermitian for any real scalars α, β, since

$$(\overline{\alpha\mathbf{A} + \beta\mathbf{B}})^T = (\alpha\overline{\mathbf{A}} + \beta\overline{\mathbf{B}})^T = \alpha\overline{\mathbf{A}}^T + \beta\overline{\mathbf{B}}^T = \alpha\mathbf{A} + \beta\mathbf{B}.$$

2. Eigenvalues and eigenvectors of $\overline{\mathbf{A}}$ are the conjugates of the eigenvalues and eigenvectors of $\mathbf{A}$, since

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \quad \text{gives} \quad \overline{\mathbf{A}}\,\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}.$$

3. The inverse of a unitary (orthogonal) matrix is unitary (orthogonal). We have $\mathbf{A}^{-1} = \overline{\mathbf{A}}^T$. Let $\mathbf{B} = \mathbf{A}^{-1}$. Then

$$\mathbf{B}^{-1} = \mathbf{A} = (\overline{\mathbf{A}}^T)^{-1} = [(\overline{\mathbf{A}})^{-1}]^T = [\overline{(\mathbf{A}^{-1})}]^T = \overline{\mathbf{B}}^T.$$

We now establish some important results.

Theorem 3.12 An orthogonal set of vectors is linearly independent.

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be an orthogonal set of vectors, that is $\mathbf{x}_i \cdot \mathbf{x}_j = 0$, $i \neq j$. Consider the vector equation

$$\mathbf{x} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \ldots + \alpha_m \mathbf{x}_m = \mathbf{0} \tag{3.52}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are scalars. Taking the inner product of the vector $\mathbf{x}$ in Eq. (3.52) with $\mathbf{x}_1$, we get

$$\mathbf{x} \cdot \mathbf{x}_1 = (\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \ldots + \alpha_m \mathbf{x}_m) \cdot \mathbf{x}_1 = \mathbf{0} \cdot \mathbf{x}_1 = 0$$

or

$$\alpha_1(\mathbf{x}_1 \cdot \mathbf{x}_1) = 0 \quad \text{or} \quad \alpha_1 \| \mathbf{x}_1 \|^2 = 0.$$

Since $\| \mathbf{x}_1 \|^2 \neq 0$, we get $\alpha_1 = 0$. Similarly, taking the inner products of $\mathbf{x}$ with $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_m$ successively, we find that $\alpha_2 = \alpha_3 = \ldots = \alpha_m = 0$. Therefore, the set of orthogonal vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ is linearly independent.

Theorem 3.13 The eigenvalues of

(i) an Hermitian matrix are real.
(ii) a skew-Hermitian matrix are zero or pure imaginary.
(iii) an unitary matrix are of magnitude 1.

Proof Let λ be an eigenvalue and $\mathbf{x}$ be the corresponding eigenvector of the matrix $\mathbf{A}$. We have $\mathbf{Ax} = \lambda \mathbf{x}$. Premultiplying both sides by $\overline{\mathbf{x}}^T$, we get

$$\overline{\mathbf{x}}^T \mathbf{Ax} = \lambda\, \overline{\mathbf{x}}^T \mathbf{x} \quad \text{or} \quad \lambda = \frac{\overline{\mathbf{x}}^T \mathbf{Ax}}{\overline{\mathbf{x}}^T \mathbf{x}}. \tag{3.53}$$

Note that $\overline{\mathbf{x}}^T \mathbf{Ax}$ and $\overline{\mathbf{x}}^T \mathbf{x}$ are scalars. Also, the denominator $\overline{\mathbf{x}}^T \mathbf{x}$ is always real and positive. Therefore, the behaviour of λ is governed by the scalar $\overline{\mathbf{x}}^T \mathbf{Ax}$.

(i) Let $\mathbf{A}$ be an Hermitian matrix, that is $\overline{\mathbf{A}} = \mathbf{A}^T$. Now,

$$(\overline{\overline{\mathbf{x}}^T \mathbf{Ax}}) = \mathbf{x}^T \overline{\mathbf{A}}\, \overline{\mathbf{x}} = \mathbf{x}^T \mathbf{A}^T \overline{\mathbf{x}} = (\mathbf{x}^T \mathbf{A}^T \overline{\mathbf{x}})^T = \overline{\mathbf{x}}^T \mathbf{Ax}$$

since $\overline{\mathbf{x}}^T \mathbf{A}^T \mathbf{x}$ is a scalar. Therefore, $\overline{\mathbf{x}}^T \mathbf{Ax}$ is real. From Eq. (3.53), we conclude that λ is real.

(ii) Let $\mathbf{A}$ be a skew-Hermitian matrix, that is $\mathbf{A}^T = -\overline{\mathbf{A}}$. Now,

$$(\overline{\overline{\mathbf{x}}^T \mathbf{Ax}}) = \mathbf{x}^T \overline{\mathbf{A}}\, \overline{\mathbf{x}} = -\mathbf{x}^T \mathbf{A}^T \overline{\mathbf{x}} = -(\mathbf{x}^T \mathbf{A}^T \overline{\mathbf{x}})^T = -\overline{\mathbf{x}}^T \mathbf{Ax}$$

since $\mathbf{x}^T \mathbf{A}^T \overline{\mathbf{x}}$ is a scalar. Therefore, $\overline{\mathbf{x}}^T \mathbf{Ax}$ is zero or pure imaginary. From Eq. (3.53), we conclude that λ is zero or pure imaginary.

(iii) Let $\mathbf{A}$ be an unitary matrix, that is $\mathbf{A}^{-1} = (\overline{\mathbf{A}})^T$. Now, from

$$\mathbf{Ax} = \lambda \mathbf{x} \quad \text{or} \quad \overline{\mathbf{A}}\, \overline{\mathbf{x}} = \overline{\lambda}\, \overline{\mathbf{x}} \tag{3.54}$$

we get

$$(\overline{\mathbf{A}}\, \overline{\mathbf{x}})^T = (\overline{\lambda}\, \overline{\mathbf{x}})^T \quad \text{or} \quad \overline{\mathbf{x}}^T \overline{\mathbf{A}}^T = \overline{\lambda}\, \overline{\mathbf{x}}^T.$$

or
$$\overline{\mathbf{x}}^T \mathbf{A}^{-1} = \overline{\lambda}\, \overline{\mathbf{x}}^T. \tag{3.55}$$

Using Eqs. (3.54) and (3.55), we can write

$$(\overline{\mathbf{x}}^T \mathbf{A}^{-1})(\mathbf{A}\mathbf{x}) = (\overline{\lambda}\, \overline{\mathbf{x}}^T)(\lambda \mathbf{x}) = |\lambda|^2 \overline{\mathbf{x}}^T \mathbf{x}$$

or
$$\overline{\mathbf{x}}^T \mathbf{x} = |\lambda|^2 \overline{\mathbf{x}}^T \mathbf{x}$$

Since $\mathbf{x} \neq \mathbf{0}$, we have $\overline{\mathbf{x}}^T \mathbf{x} \neq 0$. Therefore, $|\lambda|^2 = 1$, or $\lambda = \pm 1$. Hence, the result.

Remark 21

From Theorem 3.13, we conclude that the eigenvalues of

(i) a symmetric matrix are real.

(ii) a skew-symmetric matrix are zero or pure imaginary.

(iii) an orthogonal matrix are of magnitude 1 and are real or complex conjugate pairs.

Theorem 3.14 The column vectors (and also row vectors) of an unitary matrix form an unitary system of vectors.

Proof Let $\mathbf{A}$ be an unitary matrix of order n, with column vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Then

$$\mathbf{A}^{-1}\mathbf{A} = \overline{\mathbf{A}}^T \mathbf{A} = \begin{bmatrix} \overline{\mathbf{x}}_1^T \\ \overline{\mathbf{x}}_2^T \\ \vdots \\ \overline{\mathbf{x}}_n^T \end{bmatrix} [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = \begin{bmatrix} \overline{\mathbf{x}}_1^T \mathbf{x}_1 & \overline{\mathbf{x}}_1^T \mathbf{x}_2 & \ldots & \overline{\mathbf{x}}_1^T \mathbf{x}_n \\ \overline{\mathbf{x}}_2^T \mathbf{x}_1 & \overline{\mathbf{x}}_2^T \mathbf{x}_2 & \ldots & \overline{\mathbf{x}}_2^T \mathbf{x}_n \\ \vdots & & & \\ \overline{\mathbf{x}}_n^T \mathbf{x}_1 & \overline{\mathbf{x}}_n^T \mathbf{x}_2 & \ldots & \overline{\mathbf{x}}_n^T \mathbf{x}_n \end{bmatrix} = \mathbf{I}$$

Therefore,
$$\overline{\mathbf{x}}_i^T \mathbf{x}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

Hence, the column vectors of $\mathbf{A}$ form an unitary system. Since the inverse of an unitary matrix is also an unitary matrix and the columns of $\mathbf{A}^{-1}$ are the conjugate of the rows of $\mathbf{A}$, we conclude that the row vectors of $\mathbf{A}$ also form an unitary system.

Remark 22

(a) From Theorem 3.14, we conclude that the column vectors (and also the row vectors) of an orthogonal matrix form an orthonormal system of vectors.

(b) A symmetric matrix of order n has n linearly independent eigenvectors and hence is diagonalizable.

Example 3.47 Show that the matrices $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues and for distinct eigenvalues the eigenvectors corresponding to $\mathbf{A}$ and $\mathbf{A}^T$ are mutually orthogonal.

Solution We have

$$|\mathbf{A} - \lambda\mathbf{I}| = |(\mathbf{A}^T)^T - \lambda\mathbf{I}^T| = |[\mathbf{A}^T - \lambda\mathbf{I}]^T| = |\mathbf{A}^T - \lambda\mathbf{I}|.$$

Since $\mathbf{A}$ and $\mathbf{A}^T$ have the same characteristic equation, they have the same eigenvalues.

Let λ and μ be two distinct eigenvalues of $\mathbf{A}$. Let $\mathbf{x}$ be the eigenvector corresponding to the

eigenvalue λ for $\mathbf{A}$ and $\mathbf{y}$ be the eigenvector corresponding to the eigenvalue μ for $\mathbf{A}^T$. We hav $\mathbf{Ax} = \lambda\mathbf{x}$. Premultiplying by $\mathbf{y}^T$, we get

$$\mathbf{y}^T\mathbf{Ax} = \lambda\,\mathbf{y}^T\mathbf{x}. \tag{3.56}$$

We also have $\quad \mathbf{A}^T\mathbf{y} = \mu\mathbf{y}, \text{ or } (\mathbf{A}^T\mathbf{y})^T = (\mu\mathbf{y})^T \text{ or } \mathbf{y}^T\mathbf{A} = \mu\mathbf{y}^T.$

Postmultiplying by $\mathbf{x}$, we get

$$\mathbf{y}^T\mathbf{Ax} = \mu\,\mathbf{y}^T\mathbf{x} \tag{3.57}$$

Subtracting Eqs. (3.56) and (3.57), we obtain

$$(\lambda - \mu)\,\mathbf{y}^T\mathbf{x} = 0.$$

Since $\lambda \neq \mu$, we obtain $\mathbf{y}^T\mathbf{x} = 0$. Therefore, the vectors $\mathbf{x}$ and $\mathbf{y}$ are mutually orthogonal.

3.5.4 Quadratic Forms

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be an arbitrary vector in $\mathbb{R}^n$. A real *quadratic form* is an homogeneou expression of the form

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j \tag{3.58}$$

in which the total power in each term is 2. Expanding, we can write

$$\begin{aligned} Q &= a_{11}x^2 + (a_{12} + a_{21})\,x_1x_2 + \ldots + (a_{1n} + a_{n1})x_1x_n \\ &\quad + a_{22}x_2^2 + (a_{23} + a_{32})x_2x_3 + \ldots + (a_{2n} + a_{n2})x_2x_n \\ &\quad + \ldots + a_{nn}x_n^2 \\ &= \mathbf{x}^{\mathrm{T}}\,\mathbf{A}\,\mathbf{x} \end{aligned} \tag{3.59}$$

using the definition of matrix multiplication. Now, set $b_{ij} = (a_{ij} + a_{ji})/2$. The matrix $\mathbf{B} = (b_{ij})$ is symmetric since $b_{ij} = b_{ji}$. Further, $b_{ij} + b_{ji} = a_{ij} + a_{ji}$. Hence, Eq. (3.59) can be written as

$$Q = \mathbf{x}^T\mathbf{B}\mathbf{x}$$

where $\mathbf{B}$ is a symmetric matrix and $b_{ij} = (a_{ij} + a_{ji})/2$.

For example, for $n = 2$, we have

$$b_{11} = a_{11},\ b_{12} = b_{21} = (a_{12} + a_{21})/2 \quad \text{and} \quad b_{22} = a_{22}.$$

Example 3.48 Obtain the symmetric matrix $\mathbf{B}$ for the quadratic form

(i) $Q = 2x_1^2 + 3x_1x_2 + x_2^2$.

(ii) $Q = x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3 - 5x_2^2 + 4x_3^2$.

Solution

(i) $\quad a_{11} = 2,\ a_{12} + a_{21} = 3$ and $a_{22} = 1$. Therefore,

$$b_{11} = a_{11} = 2,\ b_{12} = b_{21} = \frac{1}{2}(a_{12} + a_{21}) = \frac{3}{2} \text{ and } b_{22} = a_{22} = 1.$$

Therefore, $$\mathbf{B} = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}.$$

(ii) $a_{11} = 1$, $a_{12} + a_{21} = 2$, $a_{13} + a_{31} = -4$, $a_{23} + a_{32} = 6$, $a_{22} = -5$, $a_{33} = 4$. Therefore,

$b_{11} = a_{11} = 1$, $b_{12} = b_{21} = \frac{1}{2}(a_{12} + a_{21}) = 1$, $b_{13} = b_{31} = \frac{1}{2}(a_{13} + a_{31}) = -2$,

$b_{23} = b_{32} = \frac{1}{2}(a_{23} + a_{32}) = 3$, $b_{22} = a_{22} = -5$, $b_{33} = a_{33} = 4$.

Therefore, $$\mathbf{B} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}.$$

If **A** is a complex matrix, then the quadratic form is defined as

$$Q = \sum_{i=1}^{n} \sum_{i=1}^{n} a_{ij}\, \bar{x}_i x_j = \bar{\mathbf{x}}^T \mathbf{A}\mathbf{x} \tag{3.61}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an arbitrary vector in $\mathbb{C}^n$. However, this quadratic form is usually define for an Hermitian matrix **A**. Then, it is called a *Hermitian form* and is always real.

For example, consider the Hermitian matrix $\mathbf{A} = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$. The quadratic form becomes

$$Q = \bar{\mathbf{x}}^T \mathbf{A}\mathbf{x} = [\bar{x}_1, \bar{x}_2] \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= |x_1|^2 + (1+i)\bar{x}_1 x_2 + (1-i)x_1 \bar{x}_2 + 2|x_2|^2$$

$$= |x_1|^2 + (\bar{x}_1 x_2 + x_1 \bar{x}_2) + i(\bar{x}_1 x_2 - x_1 \bar{x}_2) + 2|x_2|^2.$$

Now, $\bar{x}_1 x_2 + x_1 \bar{x}_2$ is real and $\bar{x}_1 x_2 - x_1 \bar{x}_2$ is imaginary. For example if $x_1 = p_1 + iq_1$, $x_2 = p_2 + iq_2$ we obtain

$$\bar{x}_1 x_2 + x_1 \bar{x}_2 = 2(p_1 p_2 + q_1 q_2) \text{ and } \bar{x}_1 x_2 - x_1 \bar{x}_2 = 2i(p_1 q_2 - p_2 q_1).$$

We can also write

$$(\bar{x}_1 x_2 + \bar{x}_2 x_1) + i(\bar{x}_1 x_2 - x_1 \bar{x}_2) = 2[(p_1 p_2 + q_1 q_2) - (p_1 q_2 - p_2 q_1)]$$

$$= 2\,\mathrm{Re}\,[(1+i)\bar{x}_1 x_2]$$

Therefore, $Q = |x_1|^2 + 2\,\mathrm{Re}\,[(1+i)\bar{x}_1 x_2] + |x_2|^2$.

Positive definite matrices

Let $\mathbf{A} = (a_{ij})$ be a square matrix. Then, the matrix **A** is said to be *positive definite* if

$$Q = \bar{\mathbf{x}}^T \mathbf{A}\mathbf{x} > 0 \text{ for any vector } \mathbf{x} \neq \mathbf{0} \text{ and } \bar{\mathbf{x}}^T \mathbf{A}\mathbf{x} = 0, \text{ if and only if } \mathbf{x} = \mathbf{0}.$$

If $\mathbf{A}$ is real, then $\mathbf{x}$ can be taken as real.

Positive definite matrices have the following properties.

1. The eigenvalues of a positive definite matrix are all real and positive.
   This is easily proved when $\mathbf{A}$ is a real matrix. From Eq. (3.53), we have
$$\lambda = (\mathbf{x}^T\mathbf{A}\mathbf{x})/(\mathbf{x}^T\mathbf{x})$$
   Since $\mathbf{x}^T\mathbf{x} > 0$ and $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$, we obtain $\lambda > 0$. If $\mathbf{A}$ is Hermitian, then $\overline{\mathbf{x}}^T\mathbf{A}\mathbf{x}$ is real and λ is real (see Theorem 3.13). Therefore, if the Hermitian form $Q > 0$, then the eigenvalues are real and positive.
2. All the leading minors of $\mathbf{A}$ are positive.

Remark 23

(a) If $\mathbf{A}$ is Hermitian and strictly diagonally dominant ($|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, i = 1, 2, \ldots, n$) with positive real elements on the diagonal, then $\mathbf{A}$ is positive definite.

(b) If $\overline{\mathbf{x}}^T\mathbf{A}\mathbf{x} \geq 0$, then the matrix $\mathbf{A}$ is called *semi-positive definite.*

(c) A matrix $\mathbf{A}$ is called *negative definite* if $(-\mathbf{A})$ is positive definite. All the eigenvalues of a negative definite matrix are real and negative.

Example 3.49 Examine which of the following matrices are positive definite.

(a) $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, (b) $\mathbf{A} = \begin{bmatrix} 3 & -2i \\ 2i & 4 \end{bmatrix}$, (c) $\mathbf{A} = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 3 \end{bmatrix}$.

Solution

(a) (i)
$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = [x_1, x_2]\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + 3x_1x_2 + 4x_2^2$$
$$= 3\left(x_1 + \frac{1}{2}x_2\right)^2 + \frac{13}{4}x_2^2 > 0 \quad \text{for all} \quad \mathbf{x} \neq \mathbf{0}.$$

(ii) eigenvalues of $\mathbf{A}$ are 2 and 5 which are both positive.

(iii) leading minors $|3| = 3$, $\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10$ are both positive.

Hence, the matrix $\mathbf{A}$ is positive definite (it is not necessary to show all the three parts).

(b)
$$Q = \overline{\mathbf{x}}^T\mathbf{A}\mathbf{x} = [\overline{x}_1, \overline{x}_2]\begin{bmatrix} 3 & -2i \\ 2i & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\overline{x}_1, \overline{x}_2]\begin{bmatrix} 3x_1 - 2ix_2 \\ 2ix_1 + 4x_2 \end{bmatrix}$$
$$= 3x_1\overline{x}_1 - 2i\overline{x}_1x_2 + 2ix_1\overline{x}_2 + 4x_2\overline{x}_2.$$

Taking $x_1 = p_1 + iq_1$ and $x_2 = p_2 + iq_2$ and simplifying, we get
$$Q = 3(p_1^2 + q_1^2) + 4(p_2^2 + q_2^2) + 4(p_1q_2 - p_2q_1)$$
$$= p_1^2 + q_1^2 + 2p_2^2 + 2q_2^2 + 2(p_2 - q_1)^2 + 2(p_1 + q_2)^2 > 0\cdot$$

Therefore, the given matrix is positive definite.

Note that **A** is Hermitian, strictly diagonally dominant ($3 > |-2i|, 4 > |2i|$) with positive rea diagonal entries. Therefore, **A** is positive definite (see Remark 23(a).)

(c) $$Q = \bar{\mathbf{x}}^T\mathbf{A}\mathbf{x} = [\bar{x}_1, \bar{x}_2, \bar{x}_3]\begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [\bar{x}_1, \bar{x}_2, \bar{x}_3]\begin{bmatrix} x_1 + ix_3 \\ x_2 \\ -ix_1 + 3x_3 \end{bmatrix}$$

$$= x_1\bar{x}_1 + i\bar{x}_1x_3 + x_2\bar{x}_2 - ix_1\bar{x}_3 + 3x_3\bar{x}_3$$

$$= |x_1|^2 + |x_2|^2 + 3|x_3|^2 + i(\bar{x}_1x_3 - x_1\bar{x}_3)$$

Taking $x_1 = p_1 + iq_1$, $x_2 = p_2 + iq_2$, $x_3 = p_3 + iq_3$ and simplifying, we obtain

$$Q = (p_1^2 + q_1^2) + (p_2^2 + q_2^2) + 3(p_3^2 + q_3^2) - 2(p_1q_3 - p_3q_1)$$
$$= (p_1 - q_3)^2 + (p_3 + q_1)^2 + (p_2^2 + q_2^2) + 2(p_3^2 + q_3^2) > 0.$$

Therefore, the matrix **A** is positive definite. It can be verified that the eigenvalues of are 1, 2, 2 which are all positive.

Example 3.50 Let **A** be a real square matrix. Show that the matrix $\mathbf{A}^T\mathbf{A}$ has real and positiv eigenvalues.

Solution Since $(\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T\mathbf{A}$, the matrix $\mathbf{A}^T\mathbf{A}$ is symmetric. Therefore, the eigenvalues of $\mathbf{A}^T\mathbf{A}$ ar all real. Now,

$$\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \mathbf{y}^T\mathbf{y}, \text{ where } \mathbf{A}\mathbf{x} = \mathbf{y}.$$

Since $\mathbf{y}^T\mathbf{y} > 0$ for any vector $\mathbf{y} \neq \mathbf{0}$, the matrix $\mathbf{A}^T\mathbf{A}$ is positive definite and hence all the eigenvalue of $\mathbf{A}^T\mathbf{A}$ are positive. Therefore, all the eigenvalues of $\mathbf{A}^T\mathbf{A}$ are real and positive.

Exercise 3.4

Verify the Cayley-Hamilton theorem for the matrix **A**. Find $\mathbf{A}^{-1}$, if it exists, where **A** is as given in Problems to **6**.

1. $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$.
2. $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$
3. $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \\ 3 & 1 & -1 \end{bmatrix}$.

4. $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.
5. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix}$.
6. $\begin{bmatrix} 1 & i & i \\ i & 1 & i \\ i & i & 1 \end{bmatrix}$.

Find all the eigenvalues and the corresponding eigenvectors of the matrices given in Problems **7** to **18**. Which of the matrices are diagonalizable?

7. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$.
8. $\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$.
9. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

10. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{bmatrix}$.

11. $\begin{bmatrix} 1 & 1 & i \\ 1 & 0 & i \\ -i & -i & 1 \end{bmatrix}$.

12. $\begin{bmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{bmatrix}$.

13. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

14. $\begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 1 & -2 & 1 \\ -1 & 1 & -2 & 1 \end{bmatrix}$.

15. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

16. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{bmatrix}$.

17. $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

18. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Show that the matrices given in Problems **19** to **24** are diagonalizable. Find the matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

19. $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$.

20. $\begin{bmatrix} -3 & -2 & 1 \\ -2 & 0 & 4 \\ -6 & -3 & 5 \end{bmatrix}$.

21. $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix}$.

22. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

23. $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$.

24. $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

Find the matrix **A** whose eigenvalues and the corresponding eigenvectors are as given in Problems **25** to **30**.

25. Eigenvalues: 2, 2, 4; Eigenvectors: $(-2, 1, 0)^T$, $(-1, 0, 1)^T$, $(1, 0, 1)^T$.
26. Eigenvalues: 1, – 1, 2; Eigenvectors: $(1, 1, 0)^T$, $(1, 0, 1)^T$, $(3, 1, 1)^T$.
27. Eigenvalues: 1, 2, 3; Eigenvectors: $(1, 2, 1)^T$, $(2, 3, 4)^T$, $(1, 4, 9)^T$.
28. Eigenvalues: 1, 1, 1; Eigenvectors: $(-1, 1, 1)^T$, $(1, -1, 1)^T$, $(1, 1, -1)^T$.
29. Eigenvalues: 0, – 1, 1; Eigenvectors: $(-1, 1, 0)^T$, $(1, 0, -1)^T$, $(1, 1, 1)^T$.
30. Eigenvalues: 0, 0, 3; Eigenvectors: $(1, 2, -1)^T$, $(-2, 1, 0)^T$, $(3, 0, 1)^T$.
31. Let a 4×4 matrix **A** have eigenvalues 1, – 1, 2, – 2. Find the value of the determinant of the matrix $\mathbf{B} = 2\mathbf{A} + \mathbf{A}^{-1} - \mathbf{I}$.
32. Let a 3×3 matrix **A** have eigenvalues 1, 2, – 1. Find the trace of the matrix $\mathbf{B} = \mathbf{A} - \mathbf{A}^{-1} + \mathbf{A}^2$.
33. Show that the matrices **A** and $\mathbf{P}^{-1}\mathbf{AP}$ have the same eigenvalues.
34. Let **A** and **B** be square matrices of the same order. Then, show that **AB** and **BA** have the same eigenvalues but different eigenvectors.
35. Show that the matrices $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{BA}^{-1}$ have the same eigenvalues but different eigenvectors.
36. An $n \times n$ matrix **A** is *nilpotent* if for some positive integer k, $\mathbf{A}^k = \mathbf{0}$. Show that all the eigenvalues of a nilpotent matrix are zero.
37. If **A** is an $n \times n$ diagonalizable matrix and $\mathbf{A}^2 = \mathbf{A}$, then show that each eigenvalue of **A** is 0 or 1.

38. Show that the matrix $\mathbf{A} = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$, $a \neq b$, is transformed to a diagonal matrix $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, where $\mathbf{P}$ is of the form $\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\tan 2\theta = \dfrac{2h}{a-b}$.

39. Let $\mathbf{A}$ be similar to $\mathbf{B}$. Then show that (i) $\mathbf{A}^{-1}$ is similar to $\mathbf{B}^{-1}$, (ii) $\mathbf{A}^m$ is similar to $\mathbf{B}^m$ for any positive integer m, (iii) $|\mathbf{A}| = |\mathbf{B}|$.

40. Let $\mathbf{A}$ and $\mathbf{B}$ be symmetric matrices of the same order. Then, show that $\mathbf{AB}$ is symmetric if and only if $\mathbf{AB} = \mathbf{BA}$.

41. For any square matrix $\mathbf{A}$, show that $\mathbf{A}^T\mathbf{A}$ is symmetric.

42. Let $\mathbf{A}$ be a non-singular matrix. Show that $\mathbf{A}^T\mathbf{A}^{-1}$ is symmetric if and only if $\mathbf{A}^2 = (\mathbf{A}^T)^2$.

43. If $\mathbf{A}$ is a symmetric matrix and $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, then show that $\mathbf{P}$ is an orthogonal matrix.

44. Show that the product of two orthogonal matrices of the same order is also an orthogonal matrix.

45. Find the conditions that a matrix $\mathbf{A} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is orthogonal.

46. If $\mathbf{A}$ is an orthogonal matrix, show that $|\mathbf{A}| = \pm 1$.

47. Prove that the eigenvectors of a symmetric matrix corresponding to distinct eigenvalues are orthogonal.

48. A matrix $\mathbf{A}$ is called a *normal matrix* if $\mathbf{A}\overline{\mathbf{A}}^T = \overline{\mathbf{A}}^T\mathbf{A}$. Show that the Hermitian, skew-Hermitian and unitary matrices are normal.

49. If a matrix $\mathbf{A}$ can be diagonalized using an orthogonal matrix then show that $\mathbf{A}$ is symmetric.

50. Suppose that a matrix $\mathbf{A}$ is both unitary and Hermitian. Then, show that $\mathbf{A} = \mathbf{A}^{-1}$.

51. If $\mathbf{A}$ is a symmetric matrix and $\mathbf{x}^T\mathbf{Ax} > 0$ for every real vector $\mathbf{x} \neq \mathbf{0}$, then show that $\overline{\mathbf{z}}^T\mathbf{Az}$ is real and positive for any complex vector $\mathbf{z} \neq \mathbf{0}$.

52. Show that an unitary transformation $\mathbf{y} = \mathbf{Ax}$, where $\mathbf{A}$ is an unitary matrix preserves the value of the inner product.

53. Prove that a real 2×2 symmetric matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite if and only if $a > 0$ (1×1 leading minor) and $ac - b^2 > 0$ (2×2 leading minor).

54. Show that the matrix $\begin{bmatrix} 2 & 1 & 3 \\ -3 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}$ is positive definite.

55. Show that the matrix $\begin{bmatrix} -3 & -2 & 1 \\ -2 & 0 & 4 \\ -6 & -3 & 5 \end{bmatrix}$ is not positive definite.

Find the symmetric or the Hermitian matrix $\mathbf{A}$ for the quadratic forms given in Problems **56** to **60**.

56. $x_1^2 - 2x_1x_2 + 4x_2x_3 - x_2^2 + x_3^2$.

57. $3x_1^2 + 2x_1x_2 - 4x_1x_3 + 8x_2x_3 + x_2^2$.

58. $x_1^2 + 2ix_1x_2 - 8x_1x_3 + 4ix_2x_3 + 4x_3^2$.

59. $x_1^2-(2+4i)x_1x_2-(4-6i)x_2x_3+x_2^2$.

60. $2x_1^2-3x_2^2+(6+8i)x_1x_2+(4-2i)x_2x_3$.

3.6 Answers and Hints

Exercise 3.1

3. $\mathbf{A}^{-1}=\frac{1}{5}\begin{bmatrix}-3 & 2 & 2\\ 2 & -3 & 2\\ 2 & 2 & -3\end{bmatrix}$.

4. $\mathbf{A}^{-1}=\frac{1}{11}\begin{bmatrix}-3 & 4 & 5\\ 9 & -1 & -4\\ 5 & -3 & -1\end{bmatrix}$.

8. (i) $|\mathbf{A}\ adj\ (\mathbf{A})|=\text{diag}\ (|\mathbf{A}|,|\mathbf{A}|,\ldots,|\mathbf{A}|)=|\mathbf{A}|^n$ (use property 10 of determinants). Therefore, $|Adj\ (\mathbf{A})|=|\mathbf{A}|^{n-1}$.

(ii) Let $\mathbf{B}=adj\ (\mathbf{A})$. Since $\mathbf{B}^{-1}=adj\ (\mathbf{B})/|\mathbf{B}|$, we have $\mathbf{B}\ adj\ (\mathbf{B})=|\mathbf{B}|\ \mathbf{I}$. Therefore,

$$adj\ (\mathbf{A})\ adj\ (adj\ (\mathbf{A}))=|\ adj\ (\mathbf{A})\ |\mathbf{I}=|\mathbf{A}|^{n-1}\mathbf{I}.$$

Premultiplying by $\mathbf{A}$ and using $adj\ (\mathbf{A})=\mathbf{A}^{-1}|\mathbf{A}|$, we get

$$\mathbf{A}[\mathbf{A}^{-1}|\mathbf{A}|]\ adj\ (adj\ (\mathbf{A}))=|\mathbf{A}|^{n-1}\mathbf{AI}\quad\text{or}\quad adj\ (adj\ (\mathbf{A}))=|\mathbf{A}|^{n-2}\mathbf{A}.$$

9. $|\mathbf{AA}^{-1}|=|\mathbf{A}||\mathbf{A}^{-1}|=|\mathbf{I}|$ or $|\mathbf{A}^{-1}|=1/|\mathbf{A}|$.

10. $(\mathbf{BAB}^T)^T=\mathbf{BA}^T\mathbf{B}^T=\mathbf{BAB}^T$.

13. $\mathbf{AB}=\mathbf{BA}\Rightarrow\mathbf{B}^{-1}\mathbf{AB}=\mathbf{A}\Rightarrow\mathbf{B}^{-1}\mathbf{A}=\mathbf{AB}^{-1}$. Similarly, $\mathbf{A}^{-1}\mathbf{B}=\mathbf{BA}^{-1}$.

(i) $(\mathbf{AB}^{-1})^T=(\mathbf{B}^{-1})^T\mathbf{A}^T=(\mathbf{B}^T)^{-1}\mathbf{A}^T=\mathbf{B}^{-1}\mathbf{A}=\mathbf{AB}^{-1}$.

(ii) $(\mathbf{A}^{-1}\mathbf{B})^T=\mathbf{B}^T(\mathbf{A}^{-1})^T=\mathbf{B}^T(\mathbf{A}^T)^{-1}=\mathbf{BA}^{-1}=\mathbf{A}^{-1}\mathbf{B}$.

(iii) $(\mathbf{A}^{-1}\mathbf{B}^{-1})^T=[(\mathbf{BA})^{-1}]^T=[(\mathbf{AB})^{-1}]^T=(\mathbf{A}^T)^{-1}(\mathbf{B}^T)^{-1}=\mathbf{A}^{-1}\mathbf{B}^{-1}$.

14. Premultiply both sides by (i) $\mathbf{I}-\mathbf{A}$, (ii) $\mathbf{I}+\mathbf{A}$.

15. $(\mathbf{PAQ})^{-1}=\mathbf{Q}^{-1}\mathbf{A}^{-1}\mathbf{P}^{-1}=\mathbf{I}\Rightarrow\mathbf{A}^{-1}\mathbf{P}^{-1}=\mathbf{Q}\Rightarrow\mathbf{A}^{-1}=\mathbf{QP}$.

16. Use $(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\ldots)=\mathbf{I}$.

17. $(\mathbf{ABC})(\mathbf{ABC})^{-1}=\mathbf{I}$. Premultiply successively by $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$ and $\mathbf{C}^{-1}$.

21. Multiply C_1 by a, C_2 by b, C_3 by c and take out a from R_1, b from R_2, c from R_3.

27. $\begin{vmatrix}\sin\alpha & \cos\alpha & 0\\ \sin\beta & \cos\beta & 0\\ \sin\gamma & \cos\gamma & 0\end{vmatrix}\begin{vmatrix}\cos a & \cos b & \sin c\\ \sin a & \sin b & \sin c\\ 0 & 0 & 0\end{vmatrix}=0$

28. 1, 2, 3. **29.** 1, 1, 1. **30.** 1, 1, 1.

31. 1, 2, 1. **32.** (i) $k\neq 2$ and $k\neq -3$, (ii) $k=2$, or $k=-3$.

33. $\theta=\pi/6$, or $\theta=\sin^{-1}[(9-\sqrt{161})/4]$.

35. (i) $\lambda\neq 3$, μ arbitrary, (ii) $\lambda=3$, $\mu=10$, (iii) $\lambda=3$, $\mu\neq 10$.

36. 2. **37.** 1. **38.** 2.

39. $|\mathbf{A}|=(p-q)(q-r)(r-p)(p+q+r)$; rank $(\mathbf{A})$ is

(i) 3, if $p\neq q\neq r$ and $p+q+r\neq 0$;

(ii) 2, if $p \neq q \neq r$ and $p + q + r = 0$,

(iii) 2, if exactly two of p, q and r are identical;

(iv) 1, if $p = q = r$.

40. (a) 2; (b) $|\mathbf{A}| = (a_1a_2 + b_1b_2 + c_1c_2)^2$, rank $(\mathbf{A})$ is

(i) 4, if $a_1a_2 + b_1b_2 + c_1c_2 \neq 0$;

(ii) 2, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$, since all determinants of third order have the value zero.

42. Conisder $(\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A} + \mathbf{A}^2 - \ldots + (-1)^{n-1}\mathbf{A}^{n-1}) = \mathbf{I} + (-1)^{n-1}\mathbf{A}^n$. In the limit $n \to \infty$, $\mathbf{A}^n \to 0$. Therefore,

$$(\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A} + \mathbf{A}^2 - \ldots) = \mathbf{I}.$$

43. (i) $Trace\ (\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha \sum_{i=1}^{n} a_{ii} + \beta \sum_{i=1}^{n} b_{ii} = \alpha\ Trace\ (\mathbf{A}) + \beta\ Trace(\mathbf{B})$,

(ii) $Trace\ (\mathbf{AB}) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{i=1}^{n} \sum_{m=1}^{n} b_{im}a_{mi} = Trace\ (\mathbf{BA})$,

(iii) If the result is true, then $Trace\ (\mathbf{AB} - \mathbf{BA}) = Trace\ (\mathbf{I})$ which gives $0 = n$ which is not possible.

44. Result is true for $p = 0$ and 1. Let it be true for $p = k$ and show that it is true for $p = k + 1$. Note that when $\mathbf{BC} = \mathbf{CB}$ and $\mathbf{C}^2 = \mathbf{0}$, we have $\mathbf{CB}^{k+1} = \mathbf{B}^{k+1}\mathbf{C}$ and $\mathbf{CB}^k\mathbf{C} = \mathbf{0}$.

45. Apply the operation $C_1 \leftarrow C_1 + C_2 + \ldots + C_n$ and then the operation $R_i \leftarrow R_i - R_1$, $i = 2, 3, \ldots, n$.

46. None.
47. Symmetric.
48. Skew-symmetric.
49. Hermitian.
50. None
51. Skew-Hermitian.
52. None.
53. Skew-Hermitian.
54. Hermitian.
55. None.

Exercise 3.2

1. Yes.
2. No, 1, 4, 5, 6.
3. No, 1, 4, 5, 6.
4. No, when the scalar α is irrational, property 6 is not satisfied. If the field of scalars is taken only as rationals, then it defines a vector space.
5. Yes, since $\mathbf{1} + \mathbf{x} = 1x = x = \mathbf{x}$ and $\mathbf{x} + \mathbf{1} = x1 = x = \mathbf{x}$, the zero vector $\mathbf{0}$ is $\mathbf{1} = 1$. Define $-\mathbf{x} = 1/x$. Then, $\mathbf{x} + (-\mathbf{x}) = x(1/x) = 1 = \mathbf{1} = \mathbf{0}$. Therefore, negative vector is its reciprocal.
6. No, 8, 10.
7. No, 2, 3, 8, 10.
8. Yes (same arguments as in Problem 5.). $(\alpha + \beta)\ x = x^{\alpha+\beta} = x^{\alpha}x^{\beta} = x^{\alpha} + x^{\beta} = \alpha x + \beta x$.
9. (i) Yes, (ii) No, 1, 6.
10. (i) Yes, (ii) No, 1, 4, 6.
11. (i) Yes, (ii) No, when $\mathbf{x}, \mathbf{y} \in W$, $\mathbf{x} + \mathbf{y} \notin W$, (iii) No, when $\mathbf{x}, \mathbf{y} \in W$, $\mathbf{x} + \mathbf{y} \notin W$, (iv) Yes.
12. (i) No, when $\mathbf{A} \in W$, $\alpha\mathbf{A} \notin W$ for α negative,
    (ii) No, sum of two non-singular matrices need not be non-singular,
    (iii) Yes,
    (iv) No, $\alpha\mathbf{A}$ and $\mathbf{A} + \mathbf{B}$ need not belong to W, $(\mathbf{A} = \mathbf{I}, \mathbf{A}^2 = \mathbf{I} = \mathbf{A}$ but $2\mathbf{A} \neq (2\mathbf{A})^2)$.
13. (i) Yes, (ii) No; let $\alpha = i$. Then $\alpha A = iA \notin W$.
14. (i) No; for $P, Q \in W$, $P + Q \notin W$, (ii) Yes.

(iii) No; for $P, Q \in W$, $\alpha P \notin W$ and also $P + Q \notin W$,

(iv) No, for $P, Q \notin W$ having real roots, $P + Q$ need not have real roots. For example, take $P = 2t^2 - 1$, $Q = -t^2 + 3$.

15. (i) Yes,

(ii) No, $\mathbf{x}, \mathbf{y} \in W$, $\mathbf{x} + \mathbf{y} \notin W$. For example, if $\mathbf{x} = (x_1, x_1, x_1 - 1)$, $\mathbf{y} = (y_1, y_1, y_1 - 1)$; $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_1 + y_1, x_1 + y_1 - 2) \notin W$,

(iii) No, $\mathbf{x} \in W$, $\alpha \mathbf{x} \notin W$, for α negative,

(iv) No, $\mathbf{x} \in W$, $\alpha \mathbf{x} \notin W$, (v) No, $\mathbf{x} \in W$, $\alpha \mathbf{x} \notin W$, (for α a rational number).

16. (i) $\mathbf{u} + 2\mathbf{v} - \mathbf{w}$, (ii) $2\mathbf{u} + \mathbf{v} - \mathbf{w}$,

(iii) $(-33\,\mathbf{u} - 11\,\mathbf{v} + 23\,\mathbf{w})/16$.

17. (i) $\mathbf{u} - 2\mathbf{v} + 2\mathbf{w}$, (ii) $3\mathbf{u} + \mathbf{v} - \mathbf{w}$, (iii) not possible.

18. (i) $3P_1(t) - 2P_2(t) - P_3(t)$, (ii) $4P_1(t) - P_2(t) + 3P_3(t)$.

19. Let $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. Then, $\mathbf{x} = (a, b, c)^T = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$, where $\alpha = (a + b)/2$, $\beta = (a - b)/2$ and $\gamma = c$.

20. Let $S = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$. Then, $\mathbf{E} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C} + \delta\mathbf{D}$, where $\alpha = (-a - b + 2c - 2d)/3$, $\beta = (5a + 2b - 4c + 4d)/3$, $\gamma = (-4a - b + 5c - 2d)/3$ and $\delta = (-2a + b + c - d)/3$.

21. (i) independent, (ii) dependent, (iii) dependent,

(iv) independent, (v) dependent.

22. (i) independent, (ii) dependent, (iii) dependent,

(iv) independent, (v) independent.

24. $(-4, 7, 9) = (1, 2, 3) + 2(-1, 3, 4) - (3, 1, 2)$. The vectors in S are linearly dependent.

25. $t^2 + t + 1 = [-t + (t^2 - 1) + 2(t^2 + 2t + 2)]/3$. The elements in S are linearly independent.

26. (i) dimension: 2, a basis : $\{(1, 0, 0, -1), (0, 1, -1, 0)\}$,

(ii) dimension: 3, a basis: $\{(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1)\}$,

(iii) dimension: 3, a basis: $\{(1, 0, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$,

(iv) dimension: 1, a basis: $\{(1, 1, 1, 1)\}$.

27. The given vectors must be linearly independent.

(i) $k \neq 0, 1, -4/3$, (ii) $k \neq 0$, (iii) $k \neq 0$, (iv) $k \neq -8$.

28. (i) dimension: 4, basis: $\{\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{21}, \mathbf{E}_{22}\}$ where $\mathbf{E}_{rs}$ is the standard basis of order 2,

(ii) dimension: 3, basis: $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$,

(iii) dimension: 1, basis: $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$,

(iv) a 2×2 skew-Hermitian matrix (diagonal elements are 0 or pure imaginary) is given by

$$\mathbf{A} = \begin{pmatrix} ia_1 & b_1 + ib_2 \\ -b_1 + ib_2 & ia_2 \end{pmatrix} = \begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix} + i\begin{pmatrix} a_1 & b_2 \\ b_2 & a_2 \end{pmatrix} = \mathbf{B} + i\,\mathbf{C}$$

where $\mathbf{B}$ is a skew-symmetric and $\mathbf{C}$ is a symmetric matrix,

dimension: 4, basis: $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$,

(v) dimension: 3, basis: $\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$.

(vi) dimension: 3, basis: $\left\{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$.

29. (i) dimension: 3, basis: $\{\mathbf{E}_{11}, \mathbf{E}_{22}, \mathbf{E}_{33}\}$,

(ii) dimension: 6, basis: $\{\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{13}, \mathbf{E}_{22}, \mathbf{E}_{23}, \mathbf{E}_{33}\}$,

(iii) dimension: 6, basis: $\{\mathbf{E}_{11}, \mathbf{E}_{21}, \mathbf{E}_{22}, \mathbf{E}_{31}, \mathbf{E}_{32}, \mathbf{E}_{33}\}$.

where $\mathbf{E}_{rs}$ is the standard basis of order 3.

30. (i) n^2, (ii) n, (iii) $n(n+1)/2$, (iv) $n(n-1)/2$.

31. Not linear, $T(\mathbf{x}) + T(\mathbf{y}) \neq T(\mathbf{x} + \mathbf{y})$.

32. Linear.

33. Not linear, $T(\mathbf{x}) + T(\mathbf{y}) \neq T(\mathbf{x} + \mathbf{y})$.

34. Not linear, $T(1, 0) = 3$, $T(0, 1) = 2$, $T(1,1) = 0 \neq T(1, 0) + T(0, 1)$.

35. Not linear, $T(\mathbf{x}) + T(\mathbf{y}) \neq T(\mathbf{x} + \mathbf{y})$.

36. $ker(T) = (0, 0, 0)^T$, $ran(T) = x(1, 0, 1)^T + y(1, 0, -1)^T + z(0, 1, 0)^T$.

$dim(ker(T)) = 0$, $dim(ran(T)) = 3$.

37. $ker(T) = (0, 0)^T$, $ran(T) = x(2, -1, 3)^T + y(1, 1, 4)^T$. $dim(ker(T)) = 0$, $dim(ran(T)) = 2$.

38. $ker(T) = w(1, -2, 0, 1)^T$,

$ran(T) = x(1, 0, 0)^T + y(1, 0, 1)^T + z(0, 1, 0)^T + w(1, 0, 2)^T$

$= r(1, 0, 0)^T + s(1, 0, 1)^T + z(0, 1, 0)$,

where $r = x - w$, $s = y + 2w$. $dim(ker(T)) = 1$, $dim(ran(T)) = 3$.

39. $ker(T) = x(-3, 1)^T$, $ran(T)$ = real number. $dim(ker(T)) = 1$, $dim(ran(T)) = 1$.

40. $ker(T) = x(1, -3, 0)^T + z(0, 0, 1)^T$, $ran(T)$ = real number. $dim(ker(T)) = 2$, $dim(ran(T)) = 1$.

41. $ker(\mathrm{T}) = x(1, 1)^T$, $ran(T) = x(1, 1)^T - y(1, 1)^T = r(1, 1)^T$, where $r = x - y$.

$dim(ker(T)) = 1$, $dim(ran(T)) = 1$.

42. $ker(T) = x(1, 2, -3)^T$, $ran(T) = x(2, 3)^T + y(-1, 0)^T + z(0, 1)^T$ or $ran(T) = r(-1, 0)^T + s(0, 1)^T$, where $r = y + 2x$, $s = z + 3x$. $dim(ker(T)) = 1$, $dim(ran(T)) = 2$.

43. (i) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, (ii) $\mathbf{A} = \begin{bmatrix} -5 & -8 & -7 \\ 3 & 5 & 4 \end{bmatrix}$.

44. (i) $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$, (ii) $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & -1/2 \\ 1 & 1 & 1/2 \end{bmatrix}$.

45. $\mathbf{A} = \begin{bmatrix} -1/2 & -1/2 & -3/2 \\ -1/2 & -3/2 & -1/2 \\ 0 & -1 & -1 \end{bmatrix}$.

46. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

47. We have $T[\mathbf{v}_1, \mathbf{v}_2] = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]\,\mathbf{A} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}$.

Now, any vector $\mathbf{x} = (x_1, x_2)^T$ in $\mathbb{R}^2$ with respect to the given basis can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

We obtain $\alpha = (-4x_1 + 3x_2)/2$, $\beta = (2x_1 - x_2)/2$. Hence, we have

$$T\mathbf{x} = \alpha T\mathbf{v}_1 + \beta T\mathbf{v}_2 = \alpha \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4\alpha + 5\beta \\ 2\alpha + 3\beta \\ \beta \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -6x_1 + 7x_2 \\ -2x_1 + 3x_2 \\ 2x_1 - x_2 \end{bmatrix}.$$

48. $$T\mathbf{x} = \begin{bmatrix} -x_1 + 2x_2 + 8x_3 \\ -2x_1 + 3x_2 + 12x_3 \end{bmatrix}$$

49. $$T\,P_1(t) = (4x_2 - 5x_1) + 7\,(x_2 - x_1)\,t + (2x_1 - x_2)\,t^2.$$

50. (i) Two degrees of freedom, dimension is 2, a basis is {[3, 1, 0], [– 2, 0, 1]}.
(ii) One degree of freedom, dimension is 1, a basis is {(– 5, 4, 23)}.

Exercise 3.3

1. 3. **2.** 2. **3.** 3. **4.** 2.

5. 2. **6.** 2. **7.** 2. **8.** 3.

9. 4. **10.** 2. **11.** 2. **12.** 3.

13. 2. **14.** 2. **15.** 2.

16. Independent, 3. **17.** Independent, 3. **18.** Dependent, 3.

19. Independent, 3. **20.** Dependent, 2. **21.** Dependent, 3.

22. Dependent, 2. **23.** Dependent, 2. **24.** Independent, 4.

25. Dependent, 2. **26.** [1, 2, 2].

27. $[1+\alpha, -2\alpha, \alpha]$, α arbitrary. **28.** Inconsistent.

29. $[1, 1, 1]$. **30.** $[1, 3, 3]$. **31.** $[3/2, 3/2, 1]$. **32.** $[-1, -1/2, 3/4]$.

33. $[(5+\alpha-4\beta)/3, (1+2\alpha+\beta)/3, \alpha, \beta]$, α, β arbitrary.

34. $[2-\alpha, 1, \alpha, 1]$, α arbitrary. **35.** $[-1/4, 1/4, 1/4, 1/4]$.

36. $[-\alpha, \alpha, \alpha]$, α arbitrary. **37.** $[-15\alpha, 13\alpha, \alpha]$, α arbitrary.

38. $[0, 0, 0]$. **39.** $[-2\alpha/3, 7\alpha/3, -8\alpha/3, \alpha]$, α arbitrary.

40. $[2(\beta-\alpha)/3, -(5\beta+\alpha)/3, \beta, \alpha]$, α, β arbitrary.

41. $[0, 0, 0, 0]$. **42.** $[(2\beta-5\alpha)/4, -(10\beta+\alpha)/4, \beta, \alpha]$, α, β arbitrary.

43. $[(\alpha+5\beta)/3, (4\beta-7\alpha)/3, \beta, \alpha]$, α, β arbitrary.

44. $[(3\beta-5\alpha)/3, (3\beta-4\alpha)/3, \beta, \alpha]$, α, β arbitrary.

45. $[\alpha-3\beta, 5\beta, \beta, \alpha]$, α, β arbitrary.

46. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$. **47.** $\begin{bmatrix} 3 & -5 & 6 \\ -2 & 4 & -5 \\ 1 & -2 & 3 \end{bmatrix}$. **48.** $\frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

49. $\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. **50.** $\begin{bmatrix} -1 & -1/3 & 1/3 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & 1/3 & -1/3 & 0 \end{bmatrix}$.

Exercise 3.4

1. $P(\lambda) = \lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0$; $\mathbf{A}^{-1} = \frac{1}{81}\begin{bmatrix} 1 & 16 & -20 \\ 16 & 13 & 4 \\ -20 & 4 & -5 \end{bmatrix}$.

2. $P(\lambda) = \lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$; $\mathbf{A}^{-1} = \frac{1}{16}\begin{bmatrix} 6 & -4 & -2 \\ 0 & 8 & 0 \\ -2 & -4 & 6 \end{bmatrix}$.

3. $P(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda = 0$; Inverse does not exist.

4. $P(\lambda) = \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$; $\mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 4 & -4 \\ -2 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix}$.

5. $P(\lambda) = \lambda^3 - 5\lambda^2 + 9\lambda - 13 = 0$; $\mathbf{A}^{-1} = \frac{1}{13}\begin{bmatrix} 2 & -3 & -7 \\ 1 & 5 & 3 \\ 5 & -1 & 2 \end{bmatrix}$.

6. $P(\lambda) = \lambda^3 - 3\lambda^2 + 6\lambda - 4 + 2i = 0$; $\mathbf{A}^{-1} = -\frac{1+3i}{10}\begin{bmatrix} i-1 & 1 & 1 \\ 1 & i-1 & 1 \\ 1 & 1 & i-1 \end{bmatrix}$.

7. $\lambda = 1$: $(1, 1, -1)^T$; $\lambda = 2, 2$: $(2, 1, 0)^T$; not diagonalizable.
8. $\lambda = -1$: $(0, -1, 1)^T$; $\lambda = i$: $(1 + i, 1, 1)^T$; $\lambda = -i$: $[1 - i, 1, 1]^T$, diagonalizable.
9. $\lambda = 1, 1, 1$: $[0, 3, -2]^T$; not diagonalizable.
10. $\lambda = 1, 1$: $[0, 1, -1]^T$; $\lambda = 7$: $(6, 7, 5)^T$; not diagonalizable.
11. $\lambda = 0$: $[i, 0, -1]^T$; $\lambda = 1 + \sqrt{3}$: $[1, \sqrt{3} - 1, -i]^T$;
 $\lambda = 1 - \sqrt{3}$: $[1, -(\sqrt{3} + 1), -i]^T$; diagonalizable.
12. $\lambda = -i, -i$: $[1, 0, -1]^T$, $[1, -1, 0]^T$; $\lambda = 2i$: $[1, 1, 1]^T$; diagonalizable.
13. $\lambda = 0, 0, 0, 0$: $[1, 0, 0, 0]^T$; not diagonalizable.
14. $\lambda = 0, 0$: $[1, 0, 0, 1]^T$, $[1, -1, -1, 0]^T$; $\lambda = 2$: $[1, 1, 0, 0]^T$;
 $\lambda = -2$: $[1, 0, 1, 1]^T$; diagonalizable.
15. $\lambda = -1, -1, -1$: $[1, -1, 0, 0]^T$, $[1, 0, -1, 0]^T$, $[1, 0, 0, -1]^T$;
 $\lambda = 3$: $[1, 1, 1, 1]^T$; diagonalizable.
16. $\lambda = -4$: $[1, 1, -1, -1]^T$; $\lambda = 10$: $[1, 1, 1, 1]^T$; $\lambda = \sqrt{2}$: $[\sqrt{2} - 1, 1 - \sqrt{2}, -1, 1]^T$,
 $\lambda = -\sqrt{2}$: $[-(1 + \sqrt{2}), 1 + \sqrt{2}, -1, 1]^T$; diagonalizable.
17. $\lambda = -1, -1$: $[1, 0, 0, 0, -1]^T$, $[0, 1, 0, -1, 0]^T$; $\lambda = 1, 1, 1$: $[1, 0, 0, 0, 1]^T$, $[0, 1, 0, 1, 0]^T$, $[0, 0, 1, 0, 0]^T$; diagonalizable.
18. $\lambda = 1, w, w^2, w^3, w^4$, w is fifth root of unity. Let $\xi_j = w^j$, $j = 0, 1, 2, 3, 4$. $\lambda = \xi_j$: $[1, \xi_j, \xi_j^2, \xi_j^3, \xi_j^4]^T$, $j = 0, 1, 2, 3, 4$; diagonalizable.
19. $\lambda = 2, 2$: $[1, 0, -1]^T$, $[-2, 1, 0]^T$; $\lambda = 4$: $[1, 0, 1]^T$.

$$\mathbf{P} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

20. $\lambda = 1$: $[1, -2, 0]^T$; $\lambda = -1$: $[3, -2, 2]^T$; $\lambda = 2$: $[-1, 3, 1]^T$.

$$\mathbf{P} = \begin{bmatrix} 1 & 3 & -1 \\ -2 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} -8 & -5 & 7 \\ 2 & 1 & -1 \\ -4 & -2 & 4 \end{bmatrix}.$$

21. $\lambda = 0$: $[3, 1, -2]^T$; $\lambda = 2i$: $[3 + i, 1 + 3i, -4]^T$; $\lambda = -2i$: $[3 - i, 1 - 3i, -4]^T$.

$$\mathbf{P} = \begin{bmatrix} 3 & 3+i & 3-i \\ 1 & 1+3i & 1-3i \\ -2 & -4 & -4 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{32}\begin{bmatrix} 24 & -8 & 16 \\ 2i-6 & 2-6i & -8 \\ -2i-6 & 2+6i & -8 \end{bmatrix}.$$

22. $\lambda = 0$: $[1, 0, -1]^T$; $\lambda = 1$: $[-1, -1, 1]^T$, $\lambda = 2$: $[1, 1, 0]^T$.

$$\mathbf{P} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}; \mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

23. $\lambda = 1$: $[3, -1, 3]^T$; $\lambda = 2, 2$: $[2, 0, 1]^T$, $[2, 1, 0]^T$.

$$\mathbf{P} = \begin{bmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}; \mathbf{P}^{-1} = \begin{bmatrix} -1 & 2 & 2 \\ 3 & -6 & -5 \\ -1 & 3 & 2 \end{bmatrix}.$$

24. $\lambda = 1: [1, -1, -1]^T; \lambda = 2: [0, 1, 1]^T, \lambda = -2: [8 - 5, 7]^T$.

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 8 \\ -1 & 1 & -5 \\ -1 & 1 & 7 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{12}\begin{bmatrix} 12 & 8 & -8 \\ 12 & 15 & -3 \\ 0 & -1 & 1 \end{bmatrix}.$$

25. $$\mathbf{P} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 1 & 2 & 1 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

26. $$\mathbf{P} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} 6 & -5 & -7 \\ 1 & 0 & -1 \\ 3 & -3 & -4 \end{bmatrix}.$$

27. $$\mathbf{P} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 9 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{12}\begin{bmatrix} -11 & 14 & -5 \\ 14 & -8 & 2 \\ -5 & 2 & 1 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \frac{1}{12}\begin{bmatrix} 30 & -12 & 6 \\ 2 & 4 & 14 \\ -34 & 4 & 38 \end{bmatrix}.$$

28. $$\mathbf{P} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

29. $$\mathbf{P} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \frac{1}{3}\begin{bmatrix} 0 & 0 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}.$$

30. $$\mathbf{P} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 2 & -3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \frac{1}{8}\begin{bmatrix} 9 & 18 & 45 \\ 0 & 0 & 0 \\ 3 & 6 & 15 \end{bmatrix}.$$

31. Eigenvalues of $\mathbf{B}$ are $2\lambda_j + (1/\lambda_j) - 1$, $j = 1, 2, 3, 4$ or $2, -4, 7/2, -11/2$. $|\mathbf{B}|$ = product of eigenvalues of $\mathbf{B} = 154$.

32. Eigenvalues of $\mathbf{B}$ are $\lambda_j + \lambda_j^2 - (1/\lambda_j)$, $j = 1, 2, 3$ or $1, 11/2, 1$. Trace of $\mathbf{B}$ = sum of eigenvalues of $\mathbf{B} = 15/2$.

33. Premultiply $\mathbf{Ax} = \lambda\mathbf{x}$ by $\mathbf{P}^{-1}$ and substitute $\mathbf{x} = \mathbf{Py}$.

34. Let λ be an eigenvalue and $\mathbf{x}$ be the corresponding eigenvector of $\mathbf{AB}$, that is $\mathbf{ABx} = \lambda\mathbf{x}$. Premultiply by $\mathbf{A}^{-1}$ and substitute $\mathbf{x} = \mathbf{Ay}$. We get $\mathbf{BAy} = \lambda\mathbf{y}$. Therefore, λ is also an eigenvalue of $\mathbf{BA}$ and eigenvectors are related by $\mathbf{x} = \mathbf{Ay}$.

35. Let λ be an eigenvalue and $\mathbf{x}$ be the corresponding eigenvector of $\mathbf{A}^{-1}\mathbf{B}$, that is $\mathbf{A}^{-1}\mathbf{Bx} = \lambda\mathbf{x}$. Premultiply by $\mathbf{A}$ and set $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$. We obtain $\mathbf{BA}^{-1}\mathbf{y} = \lambda\mathbf{y}$. Therefore, λ is also an eigenvalue of $\mathbf{BA}^{-1}$ with the corresponding eigenvector $\mathbf{y} = \mathbf{Ax}$.

36. From $\mathbf{Ax} = \lambda\mathbf{x}$, we obtain $\mathbf{A}^k\mathbf{x} = \lambda^k\mathbf{x} = \mathbf{0}$. Therefore, $\lambda^k = 0$ or $\lambda = 0$, since $\mathbf{x} \neq \mathbf{0}$.

37. Since $\mathbf{A}$ is a diagonalizable matrix, there exists a non-singular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ and the eigenvalues of $\mathbf{A}$ and $\mathbf{D}$ are same. We have $\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} = \mathbf{D}^2$. Since $\mathbf{A}^2 = \mathbf{A}$, we get $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}^2$. Therefore, we obtain $\mathbf{D}^2 - \mathbf{D} = \mathbf{0}$. Thus $\mathbf{D} = \mathbf{0}$ or $\mathbf{D} = \mathbf{I}$. Hence, the eigenvalues of $\mathbf{A}$ are 0 or 1.

38. Simplify the right hand side and set the off-diagonal element to zero.

39. Since $\mathbf{A}$ and $\mathbf{B}$ are similar, we have $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$. From this equation, show that $\mathbf{A}^{-1} = \mathbf{P}^{-1}\mathbf{B}^{-1}\mathbf{P}$ and $\mathbf{A}^m = \mathbf{P}^{-1}\mathbf{B}^m\mathbf{P}$. Also $|\mathbf{A}| = |\mathbf{P}^{-1}||\mathbf{B}||\mathbf{P}| = |\mathbf{B}|$.

40. We have $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{B} = \mathbf{B}^T$. Therefore, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T = \mathbf{BA}$.

41. $(\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T\mathbf{A}$.

42. Let $\mathbf{A}^T\mathbf{A}^{-1}$ be a symmetric matrix. We have $(\mathbf{A}^T\mathbf{A}^{-1})^T = (\mathbf{A}^{-1})^T\mathbf{A} = \mathbf{A}^T\mathbf{A}^{-1}$, or $(\mathbf{A}^{-1})^T\mathbf{A}^2 = \mathbf{A}^T$ or $\mathbf{A}^2 = (\mathbf{A}^T)^2$. Now, let $\mathbf{A}^2 = (\mathbf{A}^T)^2$. We have $\mathbf{AA} = \mathbf{A}^T\mathbf{A}^T \Rightarrow \mathbf{A} = \mathbf{A}^{-1}\mathbf{A}^T\mathbf{A}^T \Rightarrow \mathbf{A}(\mathbf{A}^T)^{-1} = \mathbf{A}^{-1}\mathbf{A}^T$, or $\mathbf{A}(\mathbf{A}^{-1})^T = (\mathbf{A}^{-1}\mathbf{A}^T)^T = \mathbf{A}^{-1}\mathbf{A}^T$. Therefore, $\mathbf{A}^T\mathbf{A}^{-1}$ is symmetric.

43. Since $\mathbf{A}$ is symmetric, we have
$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{-1}\mathbf{A}^T = (\mathbf{PDP}^{-1})^{-1}(\mathbf{PDP}^{-1})^T = (\mathbf{PD}^{-1}\mathbf{P}^{-1})\,[(\mathbf{P}^{-1})^T\mathbf{DP}^T]$, since $\mathbf{D}^T = \mathbf{D}$. This result is true only when $\mathbf{P}^{-1}(\mathbf{P}^{-1})^T = \mathbf{I}$, or $\mathbf{P}^{-1} = \mathbf{P}^T$.

44. Let $\mathbf{A}$ and $\mathbf{B}$ be the orthogonal matrices, that is $\mathbf{A}^{-1} = \mathbf{A}^T$ and $\mathbf{B}^{-1} = \mathbf{B}^T$. Then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T = \mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$.

45. $\mathbf{A}^{-1} = \mathbf{A}^T$ gives $\mathbf{AA}^T = \mathbf{I}$. We obtain the conditions as $l_i^2 + m_i^2 + n_i^2 = 1$, $i = 1, 2, 3$ and $l_1l_2 + m_1m_2 + n_1n_2 = 0$, $l_1l_3 + m_1m_3 + n_1n_3 = 0$, $l_2l_3 + m_2m_3 + n_2n_3 = 0$.

46. Since $\mathbf{A}$ is an orthogonal matrix, we have $\mathbf{A}^{-1} = \mathbf{A}^T$. Hence, $|\mathbf{A}^{-1}| = |\mathbf{A}^T| = |\mathbf{A}|$ or $1/|\mathbf{A}| = |\mathbf{A}| \Rightarrow |\mathbf{A}|^2 = 1$ or $|\mathbf{A}| = \pm 1$.

47. Let λ and μ be two distinct eigenvalues and $\mathbf{x}$, $\mathbf{y}$ be the corresponding eigenvectors. We have $\mathbf{Ax} = \lambda\mathbf{x}$ and $\mathbf{Ay} = \mu\mathbf{y}$. From the first equation, we get $\mathbf{x}^T\mathbf{A}^T = \lambda\mathbf{x}^T$ or $\mathbf{x}^T\mathbf{A} = \lambda\mathbf{x}^T$. Postmultiplying by $\mathbf{y}$, we obtain $\mathbf{x}^T\mathbf{Ay} = \lambda\mathbf{x}^T\mathbf{y}$. From the second equation, we get $\mathbf{x}^T\mathbf{Ay} = \mu\mathbf{x}^T\mathbf{y}$. Subtracting the two results, we obtain $(\lambda - \mu)\mathbf{x}^T\mathbf{y} = 0$, which gives $\mathbf{x}^T\mathbf{y} = 0$ since $\lambda \neq \mu$.

49. There exists an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. Now, $\mathbf{A} = \mathbf{PDP}^{-1} = \mathbf{PDP}^T$, since $\mathbf{P}$ is orthogonal. We have $\mathbf{A}^{\mathrm{T}} = (\mathbf{PDP}^T)^T = \mathbf{PD}^T\mathbf{P}^T = \mathbf{PDP}^T = \mathbf{A}$, since a diagonal matrix is always symmetric.

51. Let $\mathbf{z} = \mathbf{U} + i\mathbf{V}$, where $\mathbf{U} \neq \mathbf{0}$, $\mathbf{V} \neq \mathbf{0}$ be real vectors. Then

$$\overline{\mathbf{z}}^T\mathbf{Az} = (\mathbf{U}^T\mathbf{AU} + \mathbf{V}^T\mathbf{AV}) + i(\mathbf{U}^T\mathbf{AV} - \mathbf{V}^T\mathbf{AU}) = \mathbf{U}^T\mathbf{AU} + \mathbf{V}^T\mathbf{AV} > 0$$

since $\mathbf{U}^T\mathbf{AV} = (\mathbf{U}^T\mathbf{AV})^T = \mathbf{V}^T\mathbf{A}^T\mathbf{U} = \mathbf{V}^T\mathbf{AU}$.

52. Let the vectors $\mathbf{a}$, $\mathbf{b}$ be transformed to vectors $\mathbf{u}$, $\mathbf{v}$ respectively. Then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{u}}^T \cdot \mathbf{v} = (\overline{\mathbf{A}}\,\overline{\mathbf{a}})^T(\mathbf{Ab}) = \overline{\mathbf{a}}^T\overline{\mathbf{A}}^T\mathbf{Ab} = \overline{\mathbf{a}}^T\mathbf{b} = \mathbf{a} \cdot \mathbf{b}.$$

53. $$\mathbf{x}^T\mathbf{Ax} = [x_1, x_2]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2$$

$$= a[(x_1 + bx_2/a)^2 + x_2^2(ac - b^2)/a^2] > 0 \text{ for all } x_1, x_2.$$

Therefore, $a > 0$, $ac - b^2 > 0$.

54. $$\mathbf{x}^T\mathbf{Ax} = [x_1, x_2, x_3]\begin{bmatrix} 2 & 1 & 3 \\ -3 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= 2x_1^2 - 2x_1x_2 + 2x_1x_3 + 4x_2^2 + 2x_3^2 = (x_1 - x_2)^2 + (x_1 + x_3)^2 + 3x_2^2 + x_3^2 > 0.$$

55. All the leading minors are not positive. If can also be verified that all the eigenvalues are not positive.

56. $\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

57. $\begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 4 \\ -2 & 4 & 0 \end{bmatrix}$.

58. $\begin{bmatrix} 1 & i & -4 \\ -i & 0 & 2i \\ -4 & -2i & 4 \end{bmatrix}$.

59. $\begin{bmatrix} 1 & -1-2i & 0 \\ -1+2i & 1 & -2+3i \\ 0 & -2-3i & 0 \end{bmatrix}$.

60. $\begin{bmatrix} 2 & 3+4i & 0 \\ 3-4i & -3 & 2-i \\ 0 & 2+i & 0 \end{bmatrix}$.

Chapter 3

Matrices and Eigenvalue Problems

3.1 Introduction

In modern mathematics, matrix theory occupies an important place and has applications in almost all branches of engineering and physical sciences. Matrices of order $m \times n$ form a vector space and they define linear transformations which map vector spaces consisting of vectors in IR^n or $\mathbb{C}^n$ into another vector space consisting of vector in IR^m or $\mathbb{C}^m$ under a given set of rules of vector addition and scalar multiplication. A matrix does not denote a number and no value can be assigned to it. The usual rules of arithmetic operations do not hold for matrices. The rules defining the operations on matrices are usually called its algebra. In this chapter we shall discuss the matrix algebra and its use in solving linear system of algebraic equation $\mathbf{Ax} = \mathbf{b}$ and solving the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$.

3.2 Matrices

An $m \times n$ matrix is an arrangement of mn objects (not necessarily distinct) in m rows and n columns in the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \tag{3.1}$$

We say that the matrix is of order $m \times n$ (m by n). The objects $a_{11}, a_{12}, \ldots, a_{mn}$ are called the elements of the matrix. Each element of the matrix can be a real or complex number or a function of one more variables or any other object. The element a_{ij} which is common to the ith row and the jth column is called its *general element*. The matrices are usually denoted by boldface uppercase letters **A**, **B**, **C**,... etc. When the order of the matrix is understood, we can simply write $\mathbf{A} = (a_{ij})$. If all the element of a matrix are real, it is called a *real matrix*, whereas if one or more elements of a matrix are complex it is called a *complex matrix*. We define the following type of matrices.

Row Vector A matrix of order $1 \times n$, that is, it has one row and n columns is called a *row vector* or a row matrix of order n and is written as

$$[a_{11}\ a_{12}\ \dots\ a_{1n}], \text{ or } [a_1\ a_2\ \dots\ a_n]$$

in which a_{1j} (or a_j) is the jth element.

Column vector A matrix of order $m \times 1$, that is, it has m rows and one column is called a *column vector* or a *column matrix* of order m and is written as

$$\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}, \text{ or } \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in which b_{j1} (or b_j) is the jth element.

The number of elements in a row/column vector is called its *order*. The vectors are usually denoted by boldface lower case letters **a**, **b**, **c**, ... etc. If a vector has n elements and all its elements are real numbers, then it is called an *ordered n-tuple* in $\mathbb{R}^n$, whereas if one or more elements are complex numbers, then it is called an ordered *n-tuple* in $\mathbb{C}^n$.

Rectangular matrix A matrix **A** of order $m \times n$, $m \neq n$ is called a *rectangular matrix*.

Square matrices A matrix **A** of order $m \times n$ in which $m = n$, that is number of rows is equal to the number of columns is called a square matrix of order n. The elements a_{ii}, that is the elements $a_{11}, a_{22}, \dots, a_{nn}$ are called the *diagonal elements* and the line on which these elements lie is called the *principal diagonal* or the *main diagonal* of the matrix. The elements a_{ij}, when $i \neq j$ are called the *off-diagonal elements*. The sum of the diagonal elements of a square matrix is called the *trace* of the matrix.

Null matrix A matrix **A** of order $m \times n$ in which all the elements are zero is called a *null matrix* or a *zero matrix* and is denoted by **0**.

Diagonal matrix A square matrix **A** in which all the off-diagonal elements a_{ij}, $i \neq j$ are zero is called a diagonal matrix. For example

$$\mathbf{A} = \begin{bmatrix} a_{11} & & & \mathbf{0} \\ & a_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & a_{nn} \end{bmatrix} \text{ is a } \textit{diagonal matrix} \text{ of order } n.$$

A diagonal matrix is denoted by **D**. It is also written as diag $[a_{11}\ a_{22}\ \dots\ a_{nn}]$.

If all the elements of a diagonal matrix of order n are equal, that is $a_{ii} = \alpha$ for all i, then the matrix is called a *scalar matrix* of order n.

If all the elements of a diagonal matrix of order n are 1, then the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$ is called an *unit matrix* or an *identity matrix* of order n.

An identity matrix is denoted by $\mathbf{I}$.

Equal matrices Two matrices $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{p\times q}$ are said to be equal, when

(i) they are of the same order, that is $m = p$, $n = q$ and

(ii) their corresponding elements are equal, that is $a_{ij} = b_{ij}$ for all i, j.

Submatrix A matrix obtained by omitting some rows and or columns from a given matrix $\mathbf{A}$ is called a *submartix* of $\mathbf{A}$. As a convention, the given matrix $\mathbf{A}$ is also taken as the submatrix of $\mathbf{A}$.

3.2.1 Matrix Algebra

The basic operations allowed on matrices are

(i) multiplication of a matrix by a scalar,

(ii) addition/subtraction of two matrices,

(iii) multiplication of two matrices.

Note that there is no concept of dividing a matrix by a matrix. Therefore, the operation **A/B** where **A** and **B** are matrices is not defined.

Multiplication of a matrix by a scalar

Let α be a scalar (real or complex) and $\mathbf{A} = (a_{ij})$ be a given matrix of order $m \times n$. Then

$$\mathbf{B} = \alpha\mathbf{A} = \alpha(a_{ij}) = (\alpha a_{ij}) \quad \text{for all } i \text{ and } j. \tag{3.2}$$

The order of the new matrix $\mathbf{B}$ is same as that of the matrix $\mathbf{A}$.

Addition/subtraction of two matrices

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be two matrices of the same order. Then

$$\mathbf{C} = (c_{ij}) = \mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}), \quad \text{for all } i \text{ and } j \tag{3.3a}$$

and
$$\mathbf{D} = (d_{ij}) = \mathbf{A} - \mathbf{B} = (a_{ij}) - (b_{ij}) = (a_{ij} - b_{ij}), \quad \text{for all } i \text{ and } j. \tag{3.3b}$$

The order of the new matrix $\mathbf{C}$ or $\mathbf{D}$ is the same as that of the matrices $\mathbf{A}$ and $\mathbf{B}$. Matrices of the same order are said to be *conformable* for addition/subtraction.

If $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_p$ are p matrices which are conformable for addition and $\alpha_1, \alpha_2, \ldots, \alpha_p$ are any scalars, then

$$\mathbf{C} = \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \ldots + \alpha_p\mathbf{A}_p \tag{3.4}$$

is called a linear combination of the matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_p$. The order of the matrix $\mathbf{C}$ is same as that of $\mathbf{A}_i$, $i = 1, 2, \ldots, p$.

Properties of the matrix addition and scalar multiplication

Let **A, B, C** be the matrices which are conformable for addition and α, β be scalars. Then

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. (commutative law)
2. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (associative law).
3. $\mathbf{A} + \mathbf{0} = \mathbf{A}$ (**0** is the null matrix of the same order as **A**).
4. $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.
5. $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$.
6. $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$.
7. $\alpha(\beta\mathbf{A}) = \alpha\beta\mathbf{A}$.
8. $1 \times \mathbf{A} = \mathbf{A}$ and $0 \times \mathbf{A} = \mathbf{0}$.

Multiplication of two matrices

The product **AB** of two matrices **A** and **B** is defined only when the number of columns in **A** is equal to the number of rows in **B**. Such matrices are said to be *conformable* for multiplication. Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix and $\mathbf{B} = (b_{ij})$ be an $n \times p$ matrix. Then the product matrix

$$\mathbf{C} = (c_{ij}) = \mathbf{AB} = \underset{m \times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ \boxed{a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}} & & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{n \times p}{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & & & & \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}}$$

is a matrix of order $m \times p$. The general element of the product matrix **C** is given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}. \tag{3.5}$$

In the product **AB**, **B** is said to be pre-multiplied by **A** or **A** is said to be post-multiplied by **B**. If **A** is a row matrix of order $1 \times n$ and **B** is a column matrix of order $n \times 1$, then **AB** is a matrix of order 1×1, that is a single element, and **BA** is a matrix of order $n \times n$.

Remark 1

(a) It is possible that for two given matrices **A** and **B**, the product matrix **AB** is defined but the product matrix **BA** may not be defined. For example, if **A** is a 2×3 matrix and **B** is a 3×4 matrix, then the product matrix **AB** is defined and is a matrix of order 2×4, whereas the product matrix **BA** is not defined.

(b) If both the product matrices **AB** and **BA** are defined, then both the matrices **AB** and **BA** are square matrices. In general $\mathbf{AB} \neq \mathbf{BA}$. Thus, the matrix product is not commutative. If $\mathbf{AB} = \mathbf{BA}$, then the matrices **A** and **B** are said to *commute* with each other.

(c) If $\mathbf{AB} = \mathbf{0}$, then it does not always imply that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. For example, let

A^k = nilpotent
$A^2 = I$ Involutory
$AA^T = I$ Orthogonal
$A^2 = A$ Idempotent

$$\mathbf{A} = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$$

then $$\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 0 & 0 \\ ax+by & 0 \end{bmatrix} \neq \mathbf{AB}.$$

(d) If $\mathbf{AB} = \mathbf{AC}$, it does not always imply that $\mathbf{B} = \mathbf{C}$.

(e) Define $\mathbf{A}^k = \mathbf{A} \times \mathbf{A} \dots \times \mathbf{A}$ (k times). Then, a matrix $\mathbf{A}$ such that $\mathbf{A}^k = \mathbf{0}$ for some positive integer k is said to be *nilpotent*. The smallest value of k for which $\mathbf{A}^k = \mathbf{0}$ is called the *index of nilpotency* of the matrix $\mathbf{A}$.

(f) If $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A}$ is called an *idempotent matrix*.

Properties of matrix multiplication

1. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices of order $m \times n$, $n \times p$ and $p \times q$ respectively, then

   $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (associative law)

   is a matrix of order $m \times q$.

2. If $\mathbf{A}$ is a matrix of order $m \times n$ and $\mathbf{B}, \mathbf{C}$ are matrices of order $n \times p$, then

   $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ (left distributive law).

3. If $\mathbf{A}, \mathbf{B}$ are matrices of order $m \times n$ and $\mathbf{C}$ is a matrix of order $n \times p$, then

   $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$ (right distributive law).

4. If $\mathbf{A}$ is a matrix of order $m \times n$ and $\mathbf{B}$ is a matrix of order $n \times p$, then

   $$\alpha(\mathbf{AB}) = \mathbf{A}(\alpha\mathbf{B}) = (\alpha\mathbf{A})\mathbf{B}$$

   for any scalar α.

3.2.2 Some Special Matrices

We now define some special matrices.

Transpose of a matrix The matrix obtained by interchanging the corresponding rows and columns of a given matrix $\mathbf{A}$ is called the *transpose matrix* of $\mathbf{A}$ and is denoted by $\mathbf{A}^T$ or $\mathbf{A}'$, that is, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}.$$

If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{A}^T$ is an $n \times m$ matrix. Also, both the product matrices $\mathbf{A}^T\mathbf{A}$ and $\mathbf{AA}^T$ are defined, and

$$\mathbf{A}^T\mathbf{A} = (n \times m)(m \times n) \text{ is an } n \times n \text{ square matrix}$$

and $$\mathbf{AA}^T = (m \times n)(n \times m) \text{ is an } m \times m \text{ square matrix.}$$

A column vector **b** can also be written as $[b_1\ b_2\ \dots\ b_n]^T$.

The following results can be easily verified

1. The transpose of a row matrix is a column matrix and the transpose of a column matrix is a row matrix.
2. $(\mathbf{A}^T)^T = \mathbf{A}$.
3. $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$, when the matrices **A** and **B** are conformable for addition.
4. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, when the matrices **A** and **B** are conformable for multiplication.
   If the product $\mathbf{A}_1\ \mathbf{A}_2\ \dots\ \mathbf{A}_p$ is defined, then
   $$[\mathbf{A}_1\ \mathbf{A}_2\ \dots\ \mathbf{A}_p]^T = \mathbf{A}_p^T\ \mathbf{A}_{p-1}^T\ \dots\ \mathbf{A}_1^T.$$

Remark 2

The product of a row vector $\mathbf{a}_i = (a_{i1}\ a_{i2}\ \dots\ a_{in})$ of order $1 \times n$ and a column vector $\mathbf{b}_j = (b_{1j}\ b_{2j}\ \dots\ b_{nj})^T$ of order $n \times 1$ is called the *dot product* or the *inner product* of the vectors $\mathbf{a}_i$ and $\mathbf{b}_j$, that is

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

which is a scalar. In terms of the inner products, the product matrix **C** in Eq. (3.5) can be written as

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_p \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_2 \cdot \mathbf{b}_p \\ \dots & \dots & \dots & \\ \mathbf{a}_m \cdot \mathbf{b}_1 & \mathbf{a}_m \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_m \cdot \mathbf{b}_p \end{bmatrix}. \tag{3.6}$$

Symmetric and skew-symmetric matrices A real square matrix $\mathbf{A} = (a_{ij})$ is said to be

symmetric, if $a_{ij} = a_{ji}$ for all i and j, that is $\mathbf{A} = \mathbf{A}^T$

skew-symmetric, if $a_{ij} = -a_{ji}$ for all i and j, that is $\mathbf{A} = -\mathbf{A}^T$.

Remark 3

(a) In a skew-symmetric matrix $\mathbf{A} = (a_{ij})$, all its diagonal elements are zero.

(b) The matrix which is both symmetric and skew-symmetric must be a null matrix.

(c) For any real square matrix **A**, the matrix $\mathbf{A} + \mathbf{A}^T$ is always symmetric and the matrix $\mathbf{A} - \mathbf{A}^T$ is always skew-symmetric. Therefore, a real square matrix **A** can be written as the sum of a symmetric matrix and a skew-symmetric matrix. That is

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T).$$

Triangular matrices A square matrix $\mathbf{A} = (a_{ij})$ is called a *lower triangular matrix* if $a_{ij} = 0$, whenever $i < j$, that is all elements above the principal diagonal are zero and an *upper triangular matrix* if $a_{ij} = 0$, whenever $i > j$, that is all the elements below the principal diagonal are zero.

Conjugate matrix Let $\mathbf{A} = (a_{ij})$ be a complex matrix. Let $\bar{a}_{ij}$ denote the complex conjugate of a_{ij}. Then, the matrix $\bar{\mathbf{A}} = (\bar{a}_{ij})$ is called the *conjugate matrix* of A.

Hermitian and skew-Hermitian matrices A complex matrix $\mathbf{A}$ is called an *Hermitian matrix* if $\bar{\mathbf{A}} = \mathbf{A}^T$ or $\mathbf{A} = (\bar{\mathbf{A}})^T$ and a *skew-Hermitian* matrix if $\bar{\mathbf{A}} = -\mathbf{A}^T$ or $\mathbf{A} = -(\bar{\mathbf{A}})^T$. Sometimes, a Hermitian matrix is denoted by $\mathbf{A}^H$ or $\mathbf{A}^*$.

Remark 4

(a) If $\mathbf{A}$ is a real matrix, then an Hermitian matrix is same as a symmetric matrix and a skew-Hermitian matrix is same as a skew-symmetric matrix.

(b) In an Hermitian matrix, all the diagonal elements are real (let $a_{jj} = x_j + iy_j$; then $a_{jj} = \bar{a}_{jj}$ gives $x_j + iy_j = x_j - iy_j$ or $y_j = 0$ for all j).

(c) In a skew-Hermitian matrix, all the diagonal elements are either 0 or pure imaginary (let $a_{jj} = x_j + iy_j$; then $a_{jj} = -\bar{a}_{jj}$ gives $x_j + iy_j = -(x_j - iy_j)$ or $x_j = 0$ for all j).

(d) For any complex square matrix $\mathbf{A}$, the matrix $\mathbf{A} + \bar{\mathbf{A}}^T$ is always an Hermitian matrix and the matrix $\mathbf{A} - \bar{\mathbf{A}}^T$ is always a skew-Hermitian matrix. Therefore, a complex square matrix $\mathbf{A}$ can be written as the sum of an Hermitian matrix and a skew-Hermitian matrix, that is

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \bar{\mathbf{A}}^T) + \frac{1}{2}(\mathbf{A} - \bar{\mathbf{A}}^T).$$

Example 3.1 Let $\mathbf{A}$ and $\mathbf{B}$ be two symmetric matrices of the same order. Show that the matrix $\mathbf{AB}$ is symmetric if and only if $\mathbf{AB} = \mathbf{BA}$, that is the matrices $\mathbf{A}$ and $\mathbf{B}$ commute.

Solution Since the matrices $\mathbf{A}$ and $\mathbf{B}$ are symmetric, we have

$$\mathbf{A}^T = \mathbf{A} \quad \text{and} \quad \mathbf{B}^T = \mathbf{B}.$$

Let $\mathbf{AB}$ be symmetric. Then

$$(\mathbf{AB})^T = \mathbf{AB}, \quad \text{or} \quad \mathbf{B}^T\mathbf{A}^T = \mathbf{AB}, \quad \text{or} \quad \mathbf{BA} = \mathbf{AB}.$$

Now, let $\mathbf{AB} = \mathbf{BA}$. Taking transpose on both sides, we get

$$(\mathbf{AB})^T = (\mathbf{BA})^T = \mathbf{A}^T\mathbf{B}^T = \mathbf{AB}.$$

Hence, the result.

3.2.3 Determinants

With every square matrix $\mathbf{A}$ of order n, we associate a determinant of order n which is denoted by $det(\mathbf{A})$ or $|\mathbf{A}|$. The determinant has a value and this value is real if the matrix $\mathbf{A}$ is real and may be real or complex, if the matrix is complex. A determinant of order n is defined as

$$det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} a_{ij} A_{ij}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} A_{ij} \tag{3.7}$$

where M_{ij} and A_{ij} are the *minors* and *cofactors* of a_{ij} respectively.

We give now some important properties of determinants.

1. If all the elements of a row (or column) are zero then the value of the determinant is zero.
2. $|\mathbf{A}| = |\mathbf{A}^T|$.
3. If any two rows (or columns) are interchanged, then the value of the determinant is multiplied by (-1).
4. If the corresponding elements of two rows (or columns) are proportional to each other, then the value of the determinant is zero.
5. If each element of a row (or column) is multiplied by a scalar α then the value of the determinant is multiplied by the scalar α. Therefore, if β is a factor of each element of a row (or column), then this factor β can be taken out of the determinant.

   Note that when we multiply a matrix by a scalar α, then every element of the matrix is multiplied by α. Therefore, $|\alpha\mathbf{A}| = \alpha^n |\mathbf{A}|$ where $\mathbf{A}$ is a matrix of order n.
6. If a non-zero constant multiple of the elements of some row (or column) is added to the corresponding elements of some other row (or column), then the value of the determinant remains unchanged.
7. $|\mathbf{A} + \mathbf{B}| \neq |\mathbf{A}| + |\mathbf{B}|$, in general.

Remark 5

When the elements of the jth row are multiplied by a non-zero constant k and added to the corresponding elements of the ith row, we denote this operation as $R_i \leftarrow R_i + kR_j$, where R_i is the ith row of $|\mathbf{A}|$. The elements of the jth row remain unchanged whereas the elements of the ith row get changed. This operation is called an *elementary row operation*. Similarly, the operation $C_i \leftarrow C_i + kC_j$, where C_i is the ith column of $|\mathbf{A}|$, is called the *elementary column operation*. Therefore, under elementary row (or column) operations, the value of a determinant is unchanged.

Product of two determinants

If $\mathbf{A}$ and $\mathbf{B}$ are two square matrices of the same order, then

$$|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|.$$

Since $|\mathbf{A}| = |\mathbf{A}^T|$, we can multiply two determinants in any one of the following ways

(i) row by row, (ii) column by column,
(iii) row by column, (iv) column by row.

The value of the determinant is same in each case.

Rank of a matrix

The *rank* of a matrix **A**, denoted by r or $r(\mathbf{A})$ is the order of the largest non-zero minor of $|\mathbf{A}|$. Therefore, the rank of a matrix is the largest value of r, for which there exists at least one $r \times r$ submartix of **A** whose determinant is not zero. Thus, for an $m \times n$ matrix $r \le \min(m, n)$. For a square matrix **A** of order n, the rank $r = n$ if $|\mathbf{A}| \ne 0$, otherwise $r < n$. The rank of a null matrix is zero and if the rank of matrix is 0, then it must be a null matrix.

Example 3.2 Find all values of μ for which rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$$

is equal to 3.

Solution Since the matrix **A** is of order 4, $r(\mathbf{A}) \le 4$. Now, $r(\mathbf{A}) = 3$, if $|\mathbf{A}| = 0$ and there is at least one submatrix of order 3 whose determinant is not zero. Expanding the determinant through the elements of first row, we get

$$|\mathbf{A}| = \mu \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 11 & -6 & 1 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 \\ 0 & \mu & -1 \\ -6 & -6 & 1 \end{vmatrix} = \mu\Big[\mu(\mu - 6) + 11\Big] - 6$$

$$= \mu^3 - 6\mu^2 + 11\mu - 6 = (\mu - 1)(\mu - 2)(\mu - 3).$$

Setting $|\mathbf{A}| = 0$, we obtain $\mu = 1, 2, 3$. For $\mu = 1, 2, 3$, the determinant of the leading third order submatrix

$$|\mathbf{A}_1| = \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 0 & 0 & \mu \end{vmatrix} = \mu^3 \ne 0.$$

Hence, $r(\mathbf{A}) = 3$, when $\mu = 1$ or 2 or 3. For other values of μ, $r(\mathbf{A}) = 4$.

3.2.4 Inverse of a Square Matrix

Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n. Then, **A** is called a

(i) *singular matrix* if $|\mathbf{A}| = 0$,

(ii) *non-singular matrix* if $|\mathbf{A}| \ne 0$.

In other words, a square matrix of order n is singular if its rank $r(\mathbf{A}) < n$ and non-singular if its rank $r(\mathbf{A}) = n$. A square non-singular matrix **A** of order n is said to be *invertible*, if there exists a non-singular square matrix **B** of order n such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \tag{3.8}$$

where $\mathbf{I}$ is an identity matrix of order n. The matrix $\mathbf{B}$ is called the *inverse matrix* of $\mathbf{A}$ and we write $\mathbf{B} = \mathbf{A}^{-1}$ or $\mathbf{A} = \mathbf{B}^{-1}$. Hence, we say that $\mathbf{A}^{-1}$ is the inverse of the matrix $\mathbf{A}$, if

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}. \tag{3.9}$$

The inverse, $\mathbf{A}^{-1}$ of the matrix $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\, adj(\mathbf{A}) \tag{3.10}$$

where $adj(\mathbf{A})$ = adjoint matrix of $\mathbf{A}$
= transpose of the matrix of cofactors of $\mathbf{A}$.

Remark 6

(a) $$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

We have

$$(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{I}.$$

Pre-multiplying both sides first by $\mathbf{A}^{-1}$ and then by $\mathbf{B}^{-1}$ we obtain

$$\mathbf{B}^{-1}\mathbf{A}^{-1}(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\ \mathbf{B}(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \text{ or } (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

In general, we have $(\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_p)^{-1} = \mathbf{A}_p^{-1}\,\mathbf{A}_{p-1}^{-1} \ldots \mathbf{A}_1^{-1}$.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are non-singular matrices, then $\mathbf{AB}$ is also a non-singular matrix.

(c) If $\mathbf{AB} = \mathbf{0}$ and $\mathbf{A}$ is a non-singular matrix, then $\mathbf{B}$ must be null matrix, since $\mathbf{AB} = \mathbf{0}$ can be pre-multiplied by $\mathbf{A}^{-1}$. If $\mathbf{B}$ is non-singular matrix, then $\mathbf{A}$ must be a null matrix, since $\mathbf{AB} = \mathbf{0}$ can be post-multiplied by $\mathbf{B}^{-1}$.

(d) If $\mathbf{AB} = \mathbf{AC}$ and $\mathbf{A}$ is a non-singular matrix, then $\mathbf{B} = \mathbf{C}$ (see Remark 1(d)).

(e) $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$, in general.

Properties of inverse martices

1. If $\mathbf{A}^{-1}$ exists, then it is unique.
2. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
3. $(\mathbf{A}^{T})^{-1} = (\mathbf{A}^{-1})^{T}$. (From $(\mathbf{AA}^{-1})^{T} = \mathbf{I}^{T} = \mathbf{I}$, we get $(\mathbf{A}^{-1})^{T}\mathbf{A}^{T} = \mathbf{I}$. Hence, the result).
4. Let $\mathbf{D}$ = diag $(d_{11}, d_{22}, \ldots, d_{nn})$, $d_{ii} \neq 0$. Then, $\boldsymbol{D}^{-1}$ = diag $(1/d_{11}, 1/d_{22}, \ldots 1/d_{nn})$.
5. The inverse of a non-singular upper or lower triangular matrix is respectively an upper or a lower triangular matrix.
6. The inverse of a non-singular symmetric matrix is a symmetric matrix.
7. $(\mathbf{A}^{-1})^{n} = \mathbf{A}^{-n}$ for any positive integer n.

Example 3.3 Show that the matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I} = \mathbf{0}$ where $\mathbf{I}$ is an identity matrix of order 3. Hence, find the matrix (i) $\mathbf{A}^{-1}$ and (ii) $\mathbf{A}^{-2}$.

Solution We have

$$\mathbf{A}^2 = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}.$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix}.$$

Substituting in $\mathbf{B} = \mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I}$, we get

$$\mathbf{B} = \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix} - \begin{bmatrix} 24 & -6 & -30 \\ 90 & 6 & -30 \\ 30 & 24 & 54 \end{bmatrix} + \begin{bmatrix} 22 & 0 & -11 \\ 55 & 11 & 0 \\ 0 & 11 & 33 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$$

(i) Premultiplying $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I} = \mathbf{0}$ by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{A}^3 - 6\mathbf{A}^{-1}\mathbf{A}^2 + 11\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1} = \mathbf{0}$$

or $$\mathbf{A}^{-1} = \mathbf{A}^2 - 6\mathbf{A} + 11\,\mathbf{I}$$

$$= \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix} - \begin{bmatrix} 12 & 0 & -6 \\ 30 & 6 & 0 \\ 0 & 6 & 18 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}.$$

(ii) $$\mathbf{A}^{-2} = (\mathbf{A}^{-1})^2 = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -11 & 10 \\ -160 & 61 & -55 \\ 55 & -21 & 19 \end{bmatrix}.$$

We can also write

$$\mathbf{A}^{-2} = (\mathbf{A}^{-1})\,(\mathbf{A}^{-1}) = \mathbf{A} - 6\,\mathbf{I} + 11(\mathbf{A}^{-1}).$$

3.2.5 Solution of $n \times n$ Linear System of Equations

Consider the system of n equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n. \end{aligned} \tag{3.11}$$

In matrix form, we can write the system of equations (3.11) as

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.12}$$

where $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and **A, b, x** are respectively called the *coefficient matrix*, the right hand side column vector and the solution vector. If $\mathbf{b} \neq \mathbf{0}$, that is, at least one of the elements $b_1, b_2, \ldots, b_n$ is not zero, then the system of equations is called *non-homogeneous*. If $\mathbf{b} = \mathbf{0}$, then the system of equations is called *homogeneous*. The system of equations is called *consistent* if it has at least one solution and *inconsistent* if it has no solution.

Non-homogeneous system of equations

The non-homogeneous system of equations $\mathbf{Ax} = \mathbf{b}$ can be solved by the following methods.

Matrix method

Let **A** be non-singular. Pre-multiplying $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^{-1}$, we obtain

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{3.13}$$

The system of equations is consistent and has a unique solution. If $\mathbf{b} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$ (trivial solution) is the only solution.

Cramer's rule

Let **A** be non-singular. The Cramer's rule for the solution of $\mathbf{Ax} = \mathbf{b}$ is given by

$$x_i = \frac{|\mathbf{A}_i|}{|\mathbf{A}|}, \quad i = 1, 2, \ldots, n \tag{3.14}$$

where $|\mathbf{A}_i|$ is the determinant of the matrix $\mathbf{A}_i$ obtained by replacing the ith column of **A** by the right hand side column vector **b**.

We discuss the following cases.

Case 1 When $|\mathbf{A}| \neq 0$, the system of equations is consistent and the unique solution is obtained by using Eq. (3.14).

Case 2 When $|\mathbf{A}| = 0$ and one or more of $|\mathbf{A}_i|$, $i = 1, 2, \ldots, n$, are not zero, then the system of equations has no solution, that is the system is inconsistent.

Case 3 When $|\mathbf{A}| = 0$ and all $|\mathbf{A}_i| = 0$, $i = 1, 2, \ldots, n$, then the system of equations is consistent and has infinite number of solutions. The system of equations has at least a one-parameter family of solutions.

Homogeneous system of equations

Consider the homogeneous system of equations

$$\mathbf{Ax} = \mathbf{0}. \tag{3.15}$$

Trivial solution $\mathbf{x} = \mathbf{0}$ is always a solution of this system.

If $\mathbf{A}$ is non-singular, then again $\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{0} = \mathbf{0}$ is the solution.

Therefore, a homogeneous system of equations is always consistent. We conclude that non-trivial solutions for $\mathbf{Ax} = \mathbf{0}$ exist if and only if $\mathbf{A}$ is singular. In this case, the homogeneous system of equations has infinite number of solutions.

Example 3.4 Show that the system of equations

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

has a unique solution. Solve this system using (i) matrix method, (ii) Cramer's rule.

Solution We find that

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1 + 3) - 2(-1 - 1) + 1\,(3 - 1) = 10 \neq 0.$$

Therefore, the coefficient matrix $\mathbf{A}$ is non-singular and the given system of equations has a unique solution. Let $\mathbf{x} = [x, y, z]^T$.

(i) We obtain
$$\mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

Therefore,
$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Hence, $x = 2$, $y = -1$ and $z = 1$.

(ii) We have
$$|\mathbf{A}_1| = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 4(1 + 3) - 0 + 2(3 - 1) = 20.$$

$$|\mathbf{A}_2| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 1(0 + 6) - 2(4 - 2) + 1(-12 - 0) = -10.$$

$$|\mathbf{A}_3| = \begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1(2 - 0) - 2(-2 - 4) + 1(0 - 4) = 10.$$

Therefore,
$$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = 2,\ y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = -1,\ z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = 1.$$

Example 3.5 Show that the system of equations

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

has infinite number of solutions. Hence, find the solutions.

Solutions We find that

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix} = 0, \quad |\mathbf{A}_1| = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{vmatrix} = 0,$$

$$|\mathbf{A}_2| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 0, \quad |\mathbf{A}_3| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix} = 0.$$

Therefore, the system of equations has infinite number of solutions. Using the first two equations

$$x_1 - x_2 = 3 - 3x_3$$
$$2x_1 + 3x_2 = 2 - x_3$$

and solving, we obtain $x_1 = (11 - 10\,x_3)/5$ and $x_2 = (5x_3 - 4)/5$ where x_3 is arbitrary. This solution satisfies the third equation.

Example 3.6 Show that the system of equations

$$\begin{bmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix}$$

is inconsistent.

Solution We find that

$$|\mathbf{A}| = \begin{vmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{vmatrix} = 0, \quad |\mathbf{A}_1| = \begin{vmatrix} 6 & 9 & 3 \\ 2 & 3 & 1 \\ 7 & 6 & 2 \end{vmatrix} = 0, \quad |\mathbf{A}_2| = \begin{vmatrix} 4 & 6 & 3 \\ 2 & 7 & 1 \\ 2 & 7 & 2 \end{vmatrix} = 6.$$

Since $|\mathbf{A}| = 0$ and $|\mathbf{A}_2| \neq 0$, the system of equations is inconsistent.

Example 3.7 Solve the homogeneous system of equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -2 \\ 4 & 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution We find that $|\mathbf{A}| = 0$. Hence, the given system has infinite number of solutions. Solving the first two equations

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3z \\ 2z \end{bmatrix}$$

we obtain $x = 13z$, $y = -8z$ where z is arbitrary. This solution satisfies the third equation.

Exercise 3.1

1. Given the matrices $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, verify that

   (i) $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$, (ii) $|\mathbf{A} + \mathbf{B}| \neq |\mathbf{A}| + |\mathbf{B}|$.

2. If $\mathbf{A}^T = [1, -5, 7]$, $\mathbf{B} = [3, 1, 2]$, verify that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

3. Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I} = \mathbf{0}$. Hence, find $\mathbf{A}^{-1}$.

4. Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^3 - 6\mathbf{A}^2 + 5\mathbf{A} + 11\,\mathbf{I} = \mathbf{0}$. Hence, find $\mathbf{A}^{-1}$.

A (adj A) = |A|

5. For the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$, verify that

   (i) $\boxed{[adj\,(\mathbf{A})]^T = adj\,(\mathbf{A}^T),}$ (ii) $\boxed{[adj\,(\mathbf{A})]^{-1} = adj\,(\mathbf{A}^{-1}).}$

6. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that

   (i) $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$, (ii) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

7. For the matrices $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 0 & 9 \end{bmatrix}$, verify that

   (i) $adj\,(\mathbf{AB}) = adj\,(\mathbf{A})\,adj\,(\mathbf{B})$, (ii) $(\mathbf{A}+\mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$.

8. For any non-singular matrix $\mathbf{A} = (a_{ij})$ of order n, show that

   (i) $\boxed{|adj\,(\mathbf{A})| = |\mathbf{A}|^{n-1},}$ (ii) $\boxed{adj\,(adj(\mathbf{A})) = |\mathbf{A}|^{n-2}\,\mathbf{A}.}$

9. For any non-singular matrix $\mathbf{A}$, show that $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$.

10. For any symmetric matrix $\mathbf{A}$, show that $\mathbf{BAB}^T$ is symmetric, where $\mathbf{B}$ is any matrix for which the product matrix $\mathbf{BAB}^T$ is defined.

11. If $\mathbf{A}$ is a symmetric matrix, prove that $(\mathbf{BA}^{-1})^T(\mathbf{A}^{-1}\mathbf{B}^T)^{-1} = \mathbf{I}$ where $\mathbf{B}$ is any matrix for which the product matrices are defined.

12. If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices, then prove that

    (i) $\mathbf{A} + \mathbf{B}$ is symmetric, (ii) $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ are both symmetric,

    (iii) $\mathbf{AB} - \mathbf{BA}$ is skew-symmetric. → AB = BA

13. If $\mathbf{A}$ and $\mathbf{B}$ are non-singular commutative and symmetric matrices, then prove that

    (i) $\mathbf{AB}^{-1}$, (ii) $\mathbf{A}^{-1}\mathbf{B}$, (iii) $\mathbf{A}^{-1}\mathbf{B}^{-1}$

    are symmetric.

14. Let $\mathbf{A}$ be a non-singular matrix. Show that

    (i) if $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots + \mathbf{A}^n = \mathbf{0}$, then $\mathbf{A}^{-1} = \mathbf{A}^n$,

    (ii) if $\mathbf{I} - \mathbf{A} + \mathbf{A}^2 - \ldots + (-1)^n\mathbf{A}^n = \mathbf{0}$, then $\mathbf{A}^{-1} = (-1)^{n-1}\mathbf{A}^n$.

→ 15. Let $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{A}$ be non-singular square matrices of order n and $\mathbf{PAQ} = \mathbf{I}$, then show that $A^{-1} = QP$.

16. If $\mathbf{I} - \mathbf{A}$ is a non-singular matrix, then show that

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots$$

    assuming that the series on the right hand side converges.

17. For any three non-singular matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, each of order n, show that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

Solve the following system of equation:

18. $x - y + z = 2, \quad 2x + 3y - z = 5, \quad x + y - z = 0.$

19. $x + 2y + 3z = 6, \quad 2x + 4y + z = 7, \quad 3x + 2y + 9z = 14.$

20. $-x + y + 2z = 2, \quad 3x - y + z = 3, \quad -x + 3y + 4z = 6.$

21. $2x - z = 1, \quad 5x + y = 7, \quad y + 3z = 5.$

22. Determine the values of k for which the system of equations

$$x - ky + z = 0, \quad kx + 3y - kz = 0, \quad 3x + y - z = 0$$

has (i) only trivial solution, (ii) non-trivial solution.

23. Find the value of θ for which the system of equations

$2(\sin\theta)\, x + y - 2z = 0, \quad 3x + 2(\cos 2\theta)\, y + 3z = 0, \quad 5x + 3y - z = 0$ has a non-trivial solution.

24. If the system of equations $x + ay + az = 0$, $bx + y + bz = 0$, $cx + cy + z = 0$, where a, b, c, are non-zero and non-unity, has a non-trivial solution, then show that

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} = -1.$$

25. Find the values of λ and μ for which the system of equations

$$x + 2y + z = 6, \quad x + 4y + 3z = 10, \quad x + 4y + \lambda z = \mu$$

has (i) a unique solution, (ii) infinite number of solution, (iii) no solution.

Find the rank of the matrix **A**, where **A** is given by

26. $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ **27.** $\begin{bmatrix} 1 & 3 & -4 \\ -1 & -3 & 4 \\ 2 & 6 & -8 \end{bmatrix}$ **28.** $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ p^3 & q^3 & r^3 \end{bmatrix}$. **30.** (a) $\begin{bmatrix} 2 & 1 & 5 & -1 \\ -1 & 2 & 5 & 3 \\ 3 & 2 & 9 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & c_1 & -b_1 & a_2 \\ -c_1 & 0 & a_1 & b_2 \\ b_1 & -a_1 & 0 & c_2 \\ -a_2 & -b_2 & -c_2 & 0 \end{bmatrix}$

31. Prove that if **A** is an Hermitian matrix then i**A** is a Skew-Hermitian matrix and if **A** is a Skew-Hermitian matrix, then i**A** is a Hermitian martix.

32. Prove that if **A** is a real matrix and $\mathbf{A}^n \to \mathbf{0}$ as $n \to \infty$, then $\mathbf{I} + \mathbf{A}$ is invertible.

33. Let **A**, **B** be $n \times n$ real matrices. Then, show that

(i) Trace $(\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha$ Trace $(\mathbf{A}) + \beta$ Trace $(\mathbf{B})$ for any scalars α and β.

(ii) Trace $(\mathbf{AB})$ = Trace $(\mathbf{BA})$, (iii) $\mathbf{AB} - \mathbf{BA} = \mathbf{I}$ is never true.

34. If **B**, **C** are $n \times n$ matrices, $\mathbf{A} = \mathbf{B} + \mathbf{C}$, $\mathbf{BC} = \mathbf{CB}$ and $\mathbf{C}^2 = \mathbf{0}$, then show that $\mathbf{A}^{p+1} = \mathbf{B}^p\,[\mathbf{B} + (p + 1)\,\mathbf{C}]$ for any positive integer p.

35. Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n, such that $a_{ij} = d$, $i \neq j$ and $a_{ij} = c$, $i = j$. Then, show that $|\mathbf{A}| = (c - d)^{n-1}\,[c + (n - 1)\,d]$.

Identity the following matrices as symmetric, skew-symmetric, Hermitian, skew-Hermitian or none of these:

36. $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & 4 \\ -3 & -4 & 6 \end{bmatrix}$.

37. $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.

38. $\begin{bmatrix} 0 & b & c \\ -b & 0 & e \\ -c & -e & 0 \end{bmatrix}$.

39. $\begin{bmatrix} 1 & 2+4i & 1-i \\ 2-4i & -5 & 3-5i \\ 1+i & 3+5i & 6 \end{bmatrix}$.

40. $\begin{bmatrix} 1 & 2+4i & 1-i \\ -2+4i & -5 & 3-5i \\ -1-i & 3-5i & 6 \end{bmatrix}$.

41. $\begin{bmatrix} 0 & 2+4i & 1-i \\ -2+4i & 0 & 3-5i \\ -1-i & -3-5i & 0 \end{bmatrix}$.

42. $\begin{bmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{bmatrix}$

43. $\begin{bmatrix} 0 & -i & 1+i \\ -i & -2i & 0 \\ -1+i & 0 & i \end{bmatrix}$.

44. $\begin{bmatrix} 1 & -1 & i \\ -1 & 0 & 1-i \\ -i & 1+i & 2 \end{bmatrix}$.

45. $\begin{bmatrix} 1 & 2i & -i \\ -2i & i & 1 \\ i & 1 & 2 \end{bmatrix}$.

3.3 Vector Spaces

Let V be a non-empty set of certain objects, which may be vectors, matrices, functions or some other objects. Each object is an element of V and is called a vector. The elements of V are denoted by **a, b, c, u, v,** etc. Assume that the two algebraic operations

(i) vector addition and (ii) scalar multiplication

are defined on elements of V.

If the vector addition is defined as the usual addition of vectors, then

$$\mathbf{a} + \mathbf{b} = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

If the scalar multiplication is defined as the usual scalar multiplication of a vector by the scaler α, then

$$\alpha\mathbf{a} = \alpha(a_1, a_2, \ldots, a_n) = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n).$$

The set V defines a vector space if for any elements **a, b, c** in V and any scalars α, β the following properties (axioms) are satisfied.

Properties (axioms) with respect to vector addition

1. $\mathbf{a} + \mathbf{b}$ is in V.
2. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. (commutative law)
3. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$. (associative law)
4. $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$. (existence of a unique zero element in V)
5. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$. (existence of additive inverse or negative vector in V)

Properties (axioms) with respect to scalar multiplication

6. $\alpha\mathbf{a}$ is in V.

7. $(\alpha + \beta)\,\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$. (left distributive law)

8. $(\alpha\beta)\mathbf{a} = \alpha(\beta\mathbf{a})$.

9. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$. (right distributive law)

10. $1\mathbf{a} = \mathbf{a}$. (existence of multiplicative identity)

The properties defined in **1** and **6** are called the *closure* properties. When these two properties are satisfied, we say that the vector space is closed under the vector addition and scalar multiplication. The vector addition and scalar multiplication defined above need not always be the usual addition and multiplication operators. Thus, *the vector space depends not only on the set V of vectors, but also on the definition of vector addition and scalar multiplication on V.*

If the elements of V are real, then it is called a *real vector space* when the scalars α, β are real numbers, whereas V is called a *complex vector space,* if the elements of V are complex and the scalars α, β may be real or complex numbers or if the elements of V are real and the scalars α, β are complex numbers.

Remark 7

(a) If even one of the above properties is not satisfied, then V is not a vector space. We usually check the closure properties first before checking the other properties.

(b) The concepts of length, dot product, vector product etc. are not part of the properties to be satisfied.

(c) The set of real numbers and complex numbers are called *fields* of scalars. We shall consider vector space only on the fields of scalars. In an advanced course on linear algebra, vector spaces over arbitrary fields are considered.

(d) The vector space $V = \{\mathbf{0}\}$ is called a trivial vector space.

The following are some examples of vector spaces under the usual operations of vector addition and scalar multiplication.

1. The set V of real or complex numbers.
2. The set of real valued continuous functions f on any closed interval $[a, b]$. The $\mathbf{0}$ vector defined in property **4** is the zero function.
3. The set of polynomials P_n of degree less than or equal to n.
4. The set V of n-tuples in $\mathbb{R}^n$ or $\mathbb{C}^n$.
5. The set V of all $m \times n$ matrices. The element $\mathbf{0}$ defined in property **4** is the null matrix of order $m \times n$.

The following are some examples which are not vector spaces. Assume that usual operations of vector addition and scalar multiplication are being used.

1. The set V of all polynomials of degree n. Let P_n and Q_n be two polynomials of degree n in V. Then, $\alpha P_n + \beta Q_n$ need not be a polynomial of degree n and thus may not be in V. For example, if $P_n = x^n + a_1x^{n-1} + \dots + a_n$ and $Q_n = -x^n + b_1x^{n-1} + \dots + b_n$, then $P_n + Q_n$ is a polynomial of degree $(n - 1)$.

2. The set V of all real-valued functions of one variable x, defined and continuous on the closed interval $[a, b]$ such that the value of the function at c, $a \le c \le b$, is some non-zero constant p. For example, let $f(x)$ and $g(x)$ be two elements in V. Now, $f(c) = g(c) = p$. Since $f(c) + g(c) = 2p$, $f(x) + g(x)$ is not in V. Note that if $p = 0$, then V forms a vector space.

Example 3.8 Let V be the set of all polynomials, with real coefficients, of degree n, where addition is defined by $\mathbf{a} + \mathbf{b} = \mathbf{ab}$ and under usual scalar multiplication. Show that V is not a vector space.

Solution let P_n and Q_n be two elements in V. Now, $P_n + Q_n = (P_n)(Q_n)$ is a polynomial of degree $2n$, which is not in V. Therefore, V does not define a vector space.

Example 3.9 Let V be the set of all ordered pairs (x, y), where x, y are real numbers. Let $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ be two elements in V. Define the addition as

$$\mathbf{a} + \mathbf{b} = (x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$$

and the scalar multiplication as

$$\alpha(x_1, y_1) = (\alpha x_1/3, \alpha y_1/3).$$

Show that V is not a vector space.

Solution We illustrate the properties that are not satisfied.

(i) $(x_2, y_2) + (x_1, y_1) = (2x_2 - 3x_1, y_2 - y_1) \neq (x_1, y_1) + (x_2, y_2)$.

Therefore, property **2** (commutative law) does not hold.

(ii)
$$\begin{aligned}((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (2x_1 - 3x_2, y_1 - y_2) + (x_3, y_3)\\ &= (4x_1 - 6x_2 - 3x_3, y_1 - y_2 - y_3)\\ (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (2x_2, -3x_3, y_2 - y_3)\\ &= (2x_1, -6x_2 + 9x_3, y_1 - y_2 + y_3).\end{aligned}$$

Therefore, property **3** (associative law) does not satisfied.

Hence, V is not a vector space.

Example 3.10 Let V be the set of all ordered pairs (x, y), where x, y are real number. Let $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ be two elements in V. Define the addition as

$$\mathbf{a} + \mathbf{b} = (x_1, y_1) + (x_2, y_2) + = (x_1 x_2, y_1 y_2)$$

and the scalar multiplication as

$$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1).$$

Show that V is not a vector space.

Solution Note that $(1, 1)$ is an element of V. From the given definition of vector addition, we find that

$$(x_1, y_1) + (1, 1) = (x_1, y_1).$$

This is true only for the element $(1, 1)$. Therefore, the element $(1, 1)$ plays the role of $\mathbf{0}$ element as defined in property **4**.

Now, there exists the element $(1/x_1, 1/y_1)$ such that $(x_1, y_1) + (1/x_1 + 1/y_1) = (1, 1)$. The element $(1/x_1, 1/y_1)$ plays the role of additive inverse.

Therefore, property **5** is satisfied.

Now, let $\alpha = 1$, $\beta = 2$ be any two scalars. We have

$$(\alpha + \beta)(x_1, y_1) = 3(x_1, y_1) = (3x_1, 3y_1)$$

and $\quad \alpha(x_1, y_1) + \beta(x_1, y_1) = 1(x_1, y_1) + 2(x_1, y_1) = (x_1, y_1) + (2x_1, 2y_1) = (2x_1^2, 2y_1^2).$

Therefore, $(\alpha + \beta)(x_1, y_1) \neq \alpha(x_1, y_1) + \beta(x_1, y_1)$ and property **7** is not satisfied.

Similarly, it can be shown that property **9** is not satisfied. Hence, V is not a vector space.

3.3.1 Subspaces

Let V be an arbitrary vector space defined under a given vector addition and scalar multiplication. A non-empty subset W of V, such that W is also a vector space under the same two operations of vector addition and scalar multiplication, is called a *subspace* of V. Thus, W is also closed under the two given algebraic operations on V. As a convention, the vector space V is also taken as a subspace of V.

Remark 8

To show that W is a subspace of a vector space V, it is not necessary to verify all the 10 properties as given in section 3.3. If it is shown that W is closed under the given definition of vector addition and scalar multiplication, then the properties **2, 3, 7, 8, 9** and **10** are automatically satisfied because these properties are valid for all elements in V and hence are also valid for all elements in W. Thus, we need to verify the remaining properties, that is, the existence of the zero element and the additive inverse in W.

Consider the following examples:

1. Let V be the set of n-tuples $(x_1, x_2, \ldots x_n)$ in $\mathbb{R}^n$ with usual addition and scalar multiplication. Then
   - **(i)** W consisting of n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_1 = 0$ is a subspace of V.
   - **(ii)** W consisting of n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_1 \geq 0$ is not a subspace of V, since W is not closed under scalar multiplication ($\alpha\mathbf{x}$, when α is a negative real number, is not in W).
   - **(iii)** W consisting of n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_2 = x_1 + 1$ is not a subspace of V, since W is not closed under addition.

     (Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with $x_2 = x_1 + 1$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ with $y_2 = y_1 + 1$ be two elements in W. Then

     $$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

     is not in W as $x_2 + y_2 = x_1 + y_1 + 2 \neq x_1 + y_1 + 1$).
2. Let V be the set of all real polynomials P of degree $\leq m$ with usual addition and scalar multiplication. Then
   - **(i)** W consisting of all real polynomials of degree $\leq m$ with $P(0) = 0$ is a subspace of V.

(ii) W consisting of all real polynomials of degree $\leq m$ with $P(0) = 1$ is not a subspace of V, since W is not closed under addition (if P and $Q \in W$, then $P + Q \notin W$).

(iii) W consisting of all polynomials of degree $\leq m$ with real positive coefficients is not a subspace of V since W is not closed under scalar multiplication (if P is an element of W, then $-P \notin W$).

3. Let V be the set of all $n \times n$ real square matrices with usual matrix addition and scalar multiplication. Then

(i) W consisting of all symmetric/skew-symmetric matrices of order n is a subspace of V.

(ii) W consisting of all upper/lower triangular matrices of order n is a subspace of V.

(iii) W consisting of all $n \times n$ matrices having real positive elements is not a subspace of V since W is not closed under scalar multiplication (if $\mathbf{A}$ is an element of W, then $-\mathbf{A} \notin W$).

4. Let V be the set of all $n \times n$ complex matrices with usual matrix addition and scalar multiplication. Then

(i) W consisting of all Hermitian matrices of order n forms a vector space when scalars are real numbers and does not form a vector space when scalars are complex numbers (W is not closed under scalar multiplication).

Let $$\mathbf{A} = \begin{pmatrix} a & x+iy \\ x-iy & b \end{pmatrix} \in W.$$

Let $\alpha = i$. We get $$\alpha\mathbf{A} = i\mathbf{A} = \begin{pmatrix} ai & xi-y \\ xi+y & bi \end{pmatrix} \notin W.$$

(ii) W consisting of all skew-Hermitian matrices of order n forms a vector space when scalars are real numbers and does not form a vector space when scalars are complex numbers.

Let $$\mathbf{A} = \begin{pmatrix} i & x+iy \\ -x+iy & 2i \end{pmatrix} \in W.$$

Let $\alpha = i$. We get $$i\mathbf{A} = \begin{pmatrix} -1 & ix-y \\ -ix-y & -2 \end{pmatrix} \notin W.$$

Example 3.11 Let F and G be subspaces of a vector space V such that $F \cap G = \{\mathbf{0}\}$. The sum of F and G is written as $F + G$ and is defined by

$$F + G = \{\mathbf{f} + \mathbf{g} : \mathbf{f} \in F, \mathbf{g} \in G\}.$$

Show that $F + G$ is a subspace of V assuming the usual definition of vector addition and scalar multiplication.

Solution Let $W = F + G$ and $\mathbf{f} \in F$, $\mathbf{g} \in G$. Since $\mathbf{0} \in F$, and $\mathbf{0} \in G$ we have $\mathbf{0} + \mathbf{0} = \mathbf{0} \in W$. Let $\mathbf{f}_1 + \mathbf{g}_1$ and $\mathbf{f}_2 + \mathbf{g}_2$ belong to W where $\mathbf{f}_1, \mathbf{f}_2 \in F$ and $\mathbf{g}_1, \mathbf{g}_2 \in G$. Then

$$(\mathbf{f}_1 + \mathbf{g}_1) + (\mathbf{f}_2 + \mathbf{g}_2) = (\mathbf{f}_1 + \mathbf{f}_2) + (\mathbf{g}_1 + \mathbf{g}_2) \in F + G = W.$$

Also, for any scalar α, $\alpha(\mathbf{f} + \mathbf{g}) = \alpha\mathbf{f} + \alpha\mathbf{g} \in F + G = W$.
Therefore, $W = F + G$ is a subspace of V.

We now state an important result on subspaces.

Theorem 3.1 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be any r elements of a vector space V under usual vector addition and scalar multiplication. Then, the set of all linear combinations of these elements, that is the set of all elements of the form

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_r\mathbf{v}_r \tag{3.16}$$

is a subspace of V, where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are scalars.

Spanning set Let S be a subset of a vector space V and suppose that every element in V can be obtained as a linear combination of the elements taken from S. Then S is said to be the *spanning set* for V. We also say that S spans V.

Example 3.12 Let V be the vector space of all 2×2 real matrices. Show that the sets

(i) $$S = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

(ii) $$S = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}$$

span V.

Solution Let $\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary element of V.

(i) We write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since every element of V can be written as a linear combination of the elements of S, the set S spans the vector space V.

(ii) We need to determine the scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Equating the corresponding elements, we obtain the system of equations

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = a, \quad \alpha_2 + \alpha_3 + \alpha_4 = b,$$

$$\alpha_3 + \alpha_4 = c, \quad \alpha_4 = d.$$

The solution of this system of equations is

$$\alpha_4 = d, \quad \alpha_3 = c - d, \quad \alpha_2 = b - c, \quad \alpha_1 = a - b.$$

Therefore, we can write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a - b)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (b - c)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (c - d)\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since every element of V can be written as a linear combination of the elements of S, the set S spans the vector space V.

Example 3.13 Let V be the vector space of all polynomials of degree ≤ 3. Determine whether or not the set

$$S = \{t^3, t^2 + t, t^3 + t + 1\}$$

spans V ?

Solution Let $P(t) = \alpha t^3 + \beta t^2 + \gamma t + \delta$ be an arbitrary element in V. We need to find whether or not there exist scalars a_1, a_2, a_3 such that

$$\alpha t^3 + \beta t^2 + \gamma t + \delta = a_1 t^3 + a_2(t^2 + t) + a_3(t^3 + t + 1)$$

$$\alpha t^3 + \beta t^2 + \gamma t + \delta = (a_1 + a_3)t^3 + a_2 t^2 + (a_2 + a_3)\ t + a_3.$$

Comparing the coefficients of various powers of t, we get

$$a_1 + a_3 = \alpha, \quad a_2 = \beta, \quad a_2 + a_3 = \gamma, \quad a_3 = \delta.$$

The solution of the first three equations is given by

$$a_1 = \alpha + \beta - \gamma, \quad a_2 = \beta, \quad a_3 = \gamma - \beta.$$

Substituting in the last equation, we obtain $\gamma - \beta = \delta$, which may not be true for all elements in V. For example, the polynomial $t^3 + 2t^2 + t + 3$ does not satisfy this condition and therefore, it cannot be written as a linear combination of the elements of S. Therefore, S does not span the vector space V.

3.3.2 Linear Independence of Vectors

Let V be a vector space. A finite set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of the elements of V is said to be *linearly dependent* if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{0}. \tag{3.17}$$

If Eq. (3.17) is satisfied only for $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, then the set of vectors is said to be *linearly independent.*

The above definition of linear dependence of $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ can be written alternately as follows.

Theorem 3.2 The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent if and only if at least one element of the set is a linear combination of the remaining elements.

Remark 9
Eq. (3.17) gives a homogeneous system of algebraic equations. Non-trivial solutions exist if *det*(coefficient matrix) = 0, that is the vectors are linearly dependent in this case. If the *det*(coefficient matrix) $\neq 0$, then by Cramer's rule, $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ and the vectors are linearly independent.

Example 3.14 Let $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (0, 1, -1)$ and $\mathbf{v}_3 = (0, 0, 1)$ be elements of IR^3. Show that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Solution We consider the vector equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}.$$

Substituting for $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, we obtain

$$\alpha_1(1, -1, 0) + \alpha_2(0, 1, -1) + \alpha_3(0, 0, 1) = \mathbf{0}$$

or

$$(\alpha_1, -\alpha_1 + \alpha_2, -\alpha_2 + \alpha_3) = \mathbf{0}.$$

Comparing, we obtain $\alpha_1 = 0$, $-\alpha_1 + \alpha_2 = 0$ and $-\alpha_2 + \alpha_3 = \mathbf{0}$. The solution of these equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, the given set of vectors is linearly independent.

Alternative

$$det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1 \neq 0.$$

Therefore, the given vectors are linearly independent.

Example 3.15 Let $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (0, 1, -1)$, $\mathbf{v}_3 = (0, 2, 1)$ and $\mathbf{v}_4 = (1, 0, 3)$ be elements of IR^3. Show that the set of vectors $\{\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\}$ is linearly dependent.

Solution The given set of elements will be linearly dependent if there exist scalars α_1, α_2, α_3, α_4, not all zero, such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 = \mathbf{0}. \tag{3.18}$$

Substituting for $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ and comparing, we obtain

$$\alpha_1 + \alpha_4 = 0, \quad -\alpha_1 + \alpha_2 + 2\alpha_3 = 0, \quad -\alpha_2 + \alpha_3 + 3\alpha_4 = 0.$$

The solution of this system of equations is

$$\alpha_1 = -\alpha_4, \quad \alpha_2 = 5\alpha_4/3, \quad \alpha_3 = -4\alpha_4/3, \quad \alpha_4 \text{ arbitrary}.$$

Substituting in Eq. (3.18) and cancelling α_4, we obtain

$$-\mathbf{v}_1 + \frac{5}{3}\mathbf{v}_2 - \frac{4}{3}\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}.$$

Hence, there exist scalars not all zero, such that Eq. (3.18) is satisfied. Therefore, the set of vectors is linearly dependent.

3.3.3 Dimension and Basis

Let V be a vector space. If for some positive integer n, there exists a set S of n linearly independent elements of V and if every set of $n + 1$ or more elements in V is linearly dependent, then V is said to have *dimension n*. Then, we write $dim\ (V) = n$. Thus, the maximum number of linearly independent elements of V is the dimension of V. The set S of n linearly independent vectors is called the *basis* of V. Note that a vector space whose only element is zero has dimension zero.

Theorem 3.3 Let V be a vector space of dimension n. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the linearly independent elements of V. Then, every other element of V can be written as a linear combination of these elements. Further, this representation is unique.

Proof Let $\mathbf{v}$ be an element of V. Then, the set $\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent as it has $n + 1$ elements. Therefore, there exist scalars $\alpha_0, \alpha_1, \ldots, \alpha_n$, not all zero, such that

$$\alpha_0\mathbf{v} + \alpha_1\mathbf{v}_1 + \ldots + \alpha_n\mathbf{v}_n = \mathbf{0}. \tag{3.19}$$

Now, $\alpha_0 \neq 0$. Because, if $\alpha_0 = 0$, we get $\alpha_1\mathbf{v}_1 + \ldots + \alpha_n\mathbf{v}_n = \mathbf{0}$ and since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, we get $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$. This implies that the set of $n + 1$ elements $\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly independent, which is not possible as the dimension of V is n.
Therefore, we obtain from Eq. (3.19)

$$\mathbf{v} = \sum_{i=1}^{n} (-\alpha_i/\alpha_0)\mathbf{v}_i. \tag{3.20}$$

Hence, $\mathbf{v}$ is a linear combination of n linearly independent vectors of V.
Now, let there be two representations of $\mathbf{v}$ given by

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n \quad \text{and} \quad \mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_n\mathbf{v}_n$$

where $b_i \neq a_i$ for at least one i. Subtracting these two equations, we get

$$\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \ldots + (a_n - b_n)\mathbf{v}_n.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ are linearly independent, we get

$$a_i - b_i = 0 \quad \text{or} \quad a_i = b_i,\ i = 1, 2, \ldots, n.$$

Therefore, both the representations of $\mathbf{v}$ are same and the representation of $\mathbf{v}$ given by Eq. (3.20) is unique.

Remark 10

- **(a)** A set of $(n + 1)$ vectors in IR^n is linearly dependent.
- **(b)** A set of vectors containing $\mathbf{0}$ as one of its elements is linearly dependent as $\mathbf{0}$ is the linear combination of any set of vectors.

Theorem 3.4 Let V be an n-dimensional vector space. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, $k < n$ are linearly independent elements of V, then there exist elements $\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \ldots, \mathbf{v}_n$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V.

Proof There exists an element $\mathbf{v}_{k+1}$ such that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly independent. Otherwise, every element of V can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and therefore V has dimension $k < n$. This argument can be continued. If $n > k + 1$, we keep adding elements $\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \ldots, \mathbf{v}_n$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V.

Since all the elements of a vector space V of dimension n can be represented as linear combinations of the n elements in the basis of V, the basis of V spans V. However, there can be many basis for the same vector space. For example, consider the vector space IR^3. Each of the following set of vectors

(i) $[1, -1, 0], [0, 1, -1], [0, 0, 1]$

(ii) $[1, -1, 0], [0, 0, 1], [1, 2, 3]$

(iii) $[1, 0, 0], [0, 1, 0], [0, 0, 1]$

are linearly independent and therefore forms a basis in IR^3. Some of the standard basis are the following.

1. If V consists of n-tuples in IR^n, then

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \mathbf{e}_n = (0, 0, \ldots, 0, 1)$$

is called a standard basis in IR^n.

2. If V consists of all $m \times n$ matrices, then

$$\mathbf{E}_{rs} = \begin{bmatrix} 0 & \ldots & 0 & \ldots & 0 \\ \vdots & & & & \\ 0 & \ldots & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & \ldots & 0 & \ldots & 0 \end{bmatrix}, \ r = 1, 2, \ldots, m \text{ and } s = 1, 2, \ldots, n$$

where 1 is located in the (r, s) location, that is the rth row and the sth column, is called its standard basis.

For example, if V consists of all 2×3 matrices, then any matrix $\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$ in V can be written as

$$\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = a\mathbf{E}_{11} + b\mathbf{E}_{12} + c\mathbf{E}_{13} + x\mathbf{E}_{21} + y\mathbf{E}_{22} + z\mathbf{E}_{23}$$

where $$\mathbf{E}_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{E}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ etc.}$$

3. If V consists of all polynomials $P(t)$ of degree $\leq n$, then $\{1, t, t^2, \ldots, t^n\}$ is taken as its standard basis.

Example 3.16 Determine whether the following set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis in IR^3, where

(i) $\mathbf{u} = (2, 2, 0)$, $\mathbf{v} = (3, 0, 2)$, $\mathbf{w} = (2, -2, 2)$

(ii) $\mathbf{u} = (0, 1, -1)$, $\mathbf{v} = (-1, 0, -1)$, $\mathbf{w} = (3, 1, 3)$.

Solution If the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis in $\mathbb{R}^3$, then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ must be linearly independent. Let $\alpha_1, \alpha_2, \alpha_3$ be scalars. Then, the only solution of the equation

$$\alpha_1\mathbf{u} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w} = \mathbf{0} \tag{3.21}$$

must be $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

(i) Using Eq. (3.21), we obtain the system of equations

$$2\alpha_1 + 3\alpha_2 + 2\alpha_3 = 0, \quad 2\alpha_1 - 2\alpha_3 = 0 \quad \text{and} \quad 2\alpha_2 + 2\alpha_3 = 0.$$

The solution of this system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent and they form a basis in $\mathbb{R}^3$.

(ii) Using Eq. (3.21), we obtain the system of equations

$$-\alpha_2 + 3\alpha_3 = 0, \quad \alpha_1 + \alpha_3 = 0, \quad \text{and} \quad -\alpha_1 - \alpha_2 + 3\alpha_3 = 0.$$

The solution of this system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent and they form a basis in $\mathbb{R}^3$.

Example 3.17 Find the dimension of the subspace of $\mathbb{R}^4$ spanned by the set $\{(1\ 0\ 0\ 0), (0\ 1\ 0\ 0), (1\ 2\ 0\ 1), (0\ 0\ 0\ 1)\}$. Hence find its basis.

Solution The dimension of the subspace is ≤ 4. If it is 4, then the only solution of the vector equation

$$\alpha_1(1\ 0\ 0\ 0) + \alpha_2(0\ 1\ 0\ 0) + \alpha_3(1\ 2\ 0\ 1) + \alpha_4(0\ 0\ 0\ 1) = \mathbf{0} \tag{3.22}$$

should be $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Comparing, we obtain the system of equations

$$\alpha_1 + \alpha_3 = 0, \quad \alpha_2 + 2\alpha_3 = 0, \quad \alpha_3 + \alpha_4 = 0.$$

The solution of this system of equations is given by

$$\alpha_1 = \alpha_4, \quad \alpha_2 = 2\alpha_4, \quad \alpha_3 = -\alpha_4, \quad \text{where} \quad \alpha_4 \text{ is arbitrary.}$$

Hence, the vector equation (3.22) is satisfied for non-zero values of $\alpha_1, \alpha_2, \alpha_3$, and α_4. Therefore, the dimension of the set is less than 4.

Now, consider any three elements of the set, say $(1\ 0\ 0\ 0)$, $(0\ 1\ 0\ 0)$ and $(1\ 2\ 0\ 1)$. Consider the vector equation

$$\alpha_1(1\ 0\ 0\ 0) + \alpha_2(0\ 1\ 0\ 0) + \alpha_3(1\ 2\ 0\ 1) = \mathbf{0}. \tag{3.23}$$

Comparing, we obtain the system of equations

$$\alpha_1 + \alpha_3 = 0, \ \alpha_2 + 2\alpha_3 = 0 \quad \text{and} \quad \alpha_3 = 0.$$

The solution of this system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Hence, these three elements are linearly independent. Therefore, the dimension of the given subspace is 3 and the basis is the set of vectors $\{(1\ 0\ 0\ 0), (0\ 1\ 0\ 0), (1\ 2\ 0\ 1)\}$. We find that the fourth vector can be written as

$$(0\ 0\ 0\ 1) = (1\ 0\ 0\ 0) - 2(0\ 1\ 0\ 0) + 1(1\ 2\ 0\ 1).$$

Example 3.18 Let $\mathbf{u} = \{(a, b, c, d)$, such that $a + c + d = 0, b + d = 0\}$ be a subspace of IR^4. Find the dimension and the basis of the subspace.

Solution $\mathbf{u}$ satisfies the closure properties. From the given equations, we have

$$a + c + d = 0 \text{ and } b + d = 0 \quad \text{or} \quad a = -c - d \text{ and } b = -d.$$

We have two free parameters, say, c and d. Therefore, the dimension of the given subspace is 2. Choosing $c = 0, = 1$ and $c = 1, d = 0$ we may write a basis as $\{(-1\ -1\ 0\ 1), (-1\ 0\ 1\ 0)\}$.

3.34 Linear transformations

Let A and B be two arbitrary sets. A rule that assigns to elements of A exactly one element of B is called a *function* or a *mapping* or a *transformation*. Thus, a transformation maps the elements of A into the elements of B. The set A is called the *domain* of the transformation. We use capital letters T, S etc. to demote a transformation. If T is a transformation from A into B, we write

$$T : A \to B. \tag{3.24}$$

For each element $\mathbf{a} \in A$, we get a unique element $\mathbf{b} \in B$. we write $\mathbf{b} = T(\mathrm{a})$ or $\mathbf{b} = T\mathbf{a}$ and $\mathbf{b}$ is called the image of $\mathbf{a}$ under the mapping T. The collection of all such images in B is called the range or the image set of the transformation T.

In this section, we shall discuss mapping from a vector space into a vector space. Let V and W be two vector spaces, both real or complex, over the same field F of scalars. Let T be a mapping from V into W. The mapping T is said to be a *linear transformation* or a *linear mapping*, if it satisfies the following two properties:

(i) For every scalar α and every element $\mathbf{v}$ in V

$$T(\alpha \mathbf{v}) = \alpha T(\mathbf{v}). \tag{3.25}$$

(ii) For any two elements $\mathbf{v}_1$, $\mathbf{v}_2$ in V

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + (\mathbf{v}_2). \tag{3.26}$$

Since V is a vector space, the product $\alpha\mathbf{v}$ and the sum $\mathbf{v}_1 + \mathbf{v}_2$ are defined and are elements in V. Then, T defines a mapping from V into W. Since $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are in W, the product $\alpha T(\mathbf{v})$ and the sum $T(\mathbf{v}_1) + (\mathbf{v}_2)$ are in W. The conditions given in Eqs. (3.25) and (3.26) are equivalent to

$$T(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = T(\alpha \mathbf{v}_1) + T(\beta \mathbf{v}_2) = \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2)$$

for $\mathbf{v}_1$ and $\mathbf{v}_2$ in V and any scalars α, β.

Let V be a vector space of dimension n and let the set $\{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n\}$ be its basis. Then, any element $\mathbf{v}$ in V can be written as a linear combination of the elements $\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_n$.

Remark 11

A linear transformation is completely determined by its action on basis vectors of a vector space.

Letting $\alpha = 0$ in Eq. (3.25), we find that for every element $\mathbf{v}$ in V

$$T(0\mathbf{v}) = T(\mathbf{0}) = 0T(\mathbf{v}) = \mathbf{0}.$$

Therefore, the zero element in V is mapped into zero element in W by the linear transformation T. The collection of all elements $w = T(\mathbf{v})$ is called the *range* of T and is written as $ran(T)$. The set of all elements of V that are mapped into the zero element by the linear transformation T is called the *kernel* or the *null-space* of T and is denoted by $ker(T)$. Therefore, we have

$$ker(T) = \{\mathbf{v} \mid T(\mathbf{v}) = \mathbf{0}\} \quad \text{and} \quad ran(T) = \{T(\mathbf{v}) \mid \mathbf{v} \in V\}.$$

Thus, the null space of T is a subspace of V and the range of T is a subspace of W.

The dimension of $ran(T)$ is called the rank(T) and the dimension of $ker(T)$ is called the *nullity* of T. We have the following result.

Theorem 3.5 If T has rank r and the dimension of V is n, then the nullity of T is $n - r$, that is

$$\text{rank } (T) + \text{nullity} = n = dim\ (V).$$

We shall discuss the linear transformation only in the context of matrices.

Let $\mathbf{A}$ be an $m \times n$ real (or complex) matrix. Let the rows of $\mathbf{A}$ represent the elements in $\mathbb{R}^n$ (or $\mathbb{C}^n$) and the columns of $\mathbf{A}$ represent the elements in $\mathbb{R}^m$ (or $\mathbb{C}^m$). If $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{Ax}$ is in $\mathbb{R}^m$. Thus, an $m \times n$ matrix maps the elements in $\mathbb{R}^n$ into the elements in $\mathbb{R}^m$. We write

$$T = \mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m, \quad \text{and} \quad T\mathbf{x} = \mathbf{Ax}.$$

The mapping $\mathbf{A}$ is a linear transformation. The range of T is a linear subspace of $\mathbb{R}^m$ and the kernel of T is a linear subspace of $\mathbb{R}^n$.

Remark 12

Let T_1 and T_2 be linear transformations from V into W. We define the sum $T_1 + T_2$ to be the transformation S such that

$$S\mathbf{v} = T_1\mathbf{v} + T_2\mathbf{v}, \quad \mathbf{v} \in V.$$

Then, $T_1 + T_2$ is a linear transformation and $T_1 + T_2 = T_2 + T_1$.

Example 3.19 Let T be a a linear transformation defined by

$$T\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \quad T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

Find $T\left[\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix}\right]$.

Solution The matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are linearly independent and hence form a basis in the space of 2×2 matrices. We write for any scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, not all zero

$$\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + a_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \end{bmatrix}.$$

Comparing the elements and solving the resulting system of equations, we get $\alpha_1 = 4$, $\alpha_2 = 1$, $\alpha_3 = -2$, $\alpha_4 = 5$. Since T is a linear transformation, we get

$$T\left[\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix}\right] = \alpha_1 T\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] + \alpha_2 T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] + \alpha_3 T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] + \alpha_4 T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= 4\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 20 \\ 36 \end{pmatrix}.$$

Example 3.20 For the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = (1, 3)^T$, $\mathbf{x}_2 = (4, 6)^T$, are in IR^2, find the matrix of linear transformation $T : \mathrm{IR}^2 \to \mathrm{IR}^3$, such that

$$T\mathbf{x}_1 = (-2 \quad 2 \quad -7)^T \quad \text{and} \quad T\mathbf{x}_2 = (-2 \quad -4 \quad -10)^T.$$

Solution The transformation T maps column vector in IR^2 into column vectors in IR^3. Therefore, T must be a matrix $\mathbf{A}$ of order 3×2. Let

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

Therefore, we have

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -10 \end{bmatrix}.$$

Multiplying and comparing the corresponding elements, we get

$$\begin{aligned} a_1 + 3b_1 &= -2, & 4a_1 + 6b_1 &= -2, \\ a_2 + 3b_2 &= 2, & 4a_2 + 6b_2 &= -4, \\ a_3 + 3b_3 &= -7, & 4a_3 + 6b_3 &= -10 \end{aligned}$$

Solving these equations, we obtain

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -4 & 2 \\ 2 & -3 \end{bmatrix}.$$

Example 3.21 Let T be a linear transformation from IR^3 into IR^2, where

$T\mathbf{x} = \mathbf{A}\mathbf{x}$, $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = (x\ y\ z)^T$. Find $ker(T)$, $ran(T)$ and their dimensions.

Solution To find $ker(T)$, we need to determine all $\mathbf{v} = (v_1\ v_2\ v_3)^T$ such that $T\mathbf{v} = \mathbf{0}$. Now, $T\mathbf{v} = \mathbf{A}\mathbf{v} = \mathbf{0}$ gives the equations

$$v_1 + v_2 = 0, \quad -v_1 + v_3 = 0$$

whose solution is $v_1 = -v_2 = v_3$. Therefore $\mathbf{v} = v_1[1\ -1\ 1]^T$.
Hence, dimension of $ker(T)$ is 1.
Now, $ran(T)$ is defined as $\{T(\mathbf{v}) \mid \mathbf{v} \in V\}$. We have

$$T(\mathbf{v}) = \mathbf{A}\mathbf{v} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\begin{bmatrix} v_1 + v_2 \\ -v_1 + v_3 \end{bmatrix}$$

$$= v_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + v_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the dimension of $ran(T)$ is 2.

Example 3.22 Find the matrix of a linear transformation T from IR^3 into IR^3 such that

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix},\quad T\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix},\quad T\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 5 \end{pmatrix},$$

Solution The transformation T maps elements in IR^3 into IR^3. Therefore, the transformation is a matrix of order 3×3. Let this matrix be written as

$$T = \mathbf{A} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

We determine the elements of the matrix $\mathbf{A}$ such that

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix},\ \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix},\ \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}.$$

Equating the elements and solving the resulting equations, we obtain

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ -15/2 & 3 & 13/2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Example 3.23 Let T be a transformation from IR^3 into IR^1 defined by

$$T(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2.$$

Show that T is not a linear transformation.

Solution Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be any two elements in IR^3. Then

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3).$$

We have

$$T(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2, \quad T(\mathbf{y}) = y_1^2 + y_2^2 + y_3^2$$

$$T(\mathbf{x} + \mathbf{y}) = (x_1 + y_1)^2 + (x_2 + y_2)^2 + (x_3 + y_3)^2 \neq T(\mathbf{x}) + T(\mathbf{y}).$$

Therefore, T is not a linear transformation.

Matrix representation of a linear transformation

We observe from the earlier discussion that a matrix $\mathbf{A}$ of order $m \times n$ is a linear transformation which maps the elements in IR^n into the elements in IR^m. Now, let T be a linear transformation from a finite dimensional vector space into another finite dimensional vector space over the same field F. We shall now show that with this linear transformation, we may associate a matrix $\mathbf{A}$.

Let V and W be respectively, n-dimensional and m-dimensional vector spaces over the same field F. Let T be a linear transformation such that $T : V \rightarrow W$. Let

$$\mathbf{x} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}, \; \mathbf{y} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$$

be the ordered basis of V and W respectively. Let $\mathbf{v}$ be an arbitrary element in V and $\mathbf{w}$ be an arbitrary element in W. Then, there exist scalars, $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_m$, not all zero, such that

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n \tag{3.27i}$$

$$\mathbf{w} = \beta_1\mathbf{w}_1 + \beta_2\mathbf{w}_2 + \ldots + \beta_m\mathbf{w}_m \tag{3.27 ii}$$

and

$$\mathbf{w} = T\mathbf{v} = T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n)$$

$$= \alpha_1 T\mathbf{v}_1 + \alpha_2 T\mathbf{v}_2 + \ldots + \alpha_n T\mathbf{v}_n \tag{3.27 iii}$$

Since every element $T\mathbf{v}_i$, $i = 1, 2, \ldots, n$ is in W, it can be written as a linear combination of the basis vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ in W. That is, there exist scalars a_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ not all zero, such that

$$T\mathbf{v}_i = a_{1i}\mathbf{w}_1 + a_{2i}\mathbf{w}_2 + \ldots + a_{mi}\mathbf{w}_m$$

$$= [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m]\,[a_{1i}, a_{2i}, \ldots, a_{mi}]^T, \; i = 1, 2, \ldots, n. \tag{3.27 iv}$$

Hence, we can write

$$T[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n] = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m]\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \quad (3.27\ \text{v})$$

or
$$T\mathbf{x} = \mathbf{y}\mathbf{A}$$
where A is the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}. \quad (3.27\ \text{vi})$$

The $m \times n$ matrix **A** is called the matrix representation of T or the matrix of T with respect to the ordered basis **x** and **y**. It may be observed that **x** is a basis of the vector space V, on which T acts and **y** is the basis of the vector space W that contains the range of T. Therefore, the matrix representation of T depends not only on T but also on the basis **x** and **y**. For a given linear transformation T, the elements a_{ij} of the matrix $\mathbf{A} = (a_{ij})$ are determined from (3.27 v), using the given basis vectors in **x** and **y**. From (3. 27 iii), we have (using 3.27iv)

$$\begin{aligned} \mathbf{w} &= \alpha_1(a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \ldots + a_{m1}\mathbf{w}_m) + \alpha_2(a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \ldots + a_{m2}\mathbf{w}_m) \\ &\quad + \ldots + \alpha_n(a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \ldots + a_{mn}\mathbf{w}_m) \\ &= (\alpha_1 a_{11} + \alpha_2 a_{12} + \ldots + \alpha_n a_{1n})\,\mathbf{w}_1 + (\alpha_1 a_{21} + \alpha_2 a_{22} + \ldots + \alpha_n a_{2n})\mathbf{w}_2 \\ &\quad + \ldots + (\alpha_1 a_{m1} + \alpha_2 a_{m2} + \ldots + \alpha_n a_{mn})\mathbf{w}_m \\ &= \beta_1\mathbf{w}_1 + \beta_2\mathbf{w}_2 + \ldots + \beta_m\mathbf{w}_m \end{aligned}$$

where
$$\beta_i = \alpha_1 a_{i1} + \alpha_2 a_{i2} + \ldots + \alpha_n a_{in}, \quad i = 1, 2, \ldots, m.$$

Hence,
$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

or
$$\boldsymbol{\beta} = \mathbf{A}\boldsymbol{\alpha}$$

where the matrix **A** is as defined in (3.27 vi) and

$$\boldsymbol{\beta} = [\beta_1, \beta_2, \ldots, \beta_m]^{\mathrm{T}}, \ \boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^{\mathrm{T}}.$$

For a given ordered basis vectors **x** and **y** of vector spaces V and W respectively, and a linear transformation $T : V \to W$, the matrix **A** obtained from (3.27 v) is unique. We prove this result as follows:

Let $\mathbf{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be two matrices each of order $m \times n$ such that

$$T\mathbf{x} = \mathbf{y}\mathbf{A} \quad \text{and} \quad T\mathbf{x} = \mathbf{y}\boldsymbol{B}.$$

Therefore, we have

$$\mathbf{y}\mathbf{A} = \mathbf{y}\boldsymbol{B}$$

or
$$\sum_{i=1}^{m} \mathbf{w}_i a_{ij} = \sum_{i=1}^{m} \mathbf{w}_i b_{ij}, \quad j = 1, 2, \ldots, n.$$

Since $\boldsymbol{Y} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is a given basis, we obtain $a_{ij} = b_{ij}$ for all i and j and hence $\mathbf{A} \equiv \boldsymbol{B}$.

Example 3.24 Let $T : \mathrm{IR}^3 \to \mathrm{IR}^2$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ y-z \end{pmatrix}.$$

Determine the matrix of the linear transformation T, with respect to the ordered basis

(i) $\mathbf{x} = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$ in IR^3 and $\mathbf{y} = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ in IR^2

(standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ in IR^3 and $\boldsymbol{e}_1, \boldsymbol{e}_2$ in IR^2).

(ii) $\mathbf{x} = \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$ in IR^3 and $\mathbf{y} = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ in IR^2.

Solution Let $V = \mathrm{IR}^3$, $W = \mathrm{IR}^2$. Let $\mathbf{x} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\mathbf{y} = \{\mathbf{w}_1, \mathbf{w}_2\}$.

(i) We have
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We obtain
$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0), \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(1),$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(-1)$$

Using the notation given in (3.27 v), that is $T\mathbf{x} = \mathbf{y}\boldsymbol{A}$, we write

$$T\,[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = [\mathbf{w}_1, \mathbf{w}_2]\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

or
$$T\left[\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Therefore, the matrix of the linear transformation T with respect to the given basis vectors is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

(ii) We have
$$\mathbf{v}_1 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix}1\\1\end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix}1\\-1\end{pmatrix}.$$

We obtain
$$T\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}(1) + \begin{pmatrix}1\\-1\end{pmatrix}(1), \quad T\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}(0) + \begin{pmatrix}1\\-1\end{pmatrix}(1)$$

$$T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}(1) + \begin{pmatrix}1\\-1\end{pmatrix}(0).$$

Using (3.27 v), that is $T\mathbf{x} = \mathbf{y}\mathbf{A}$, we write

$$T\left[\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right]\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Therefore, the matrix of the linear transformation T with respect to the given basis vectors is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Exercise 3.2

Discuss whether V defined in problems **1** to **10** is a vector space. If V is not a vector space, state which of the properties are not satisfied.

1. Let V be the set of the real polynomials of degree $\leq m$ and having 2 as a root with the usual addition and scalar multiplication.
2. Let V be the set of all real polynomials of degree 4 or 6 with the usual addition and scalar multiplication.
3. Let V be the set of all real polynomials of degree ≥ 4 with the usual addition and scalar multiplication.
4. Let V be the set of all rational numbers with the usual addition and scalar multiplication.
5. Let V be the set of all positive real numbers with addition defined as $x + y = xy$ and usual scalar multiplication.
6. Let V be the set of all ordered pairs (x, y) in $\mathbb{R}^2$ with vector addition defined as $(x, y) + (u, v) = (x + u, y + v)$ and scalar multiplication defined as $\alpha(x, y) = (3\alpha x, y)$.
7. Let V be the set of all ordered triplets (x, y, z), $x, y, z \in \mathbb{R}$, with vector addition defined as
$$(x, y, z) + (u, v, w) = (3x + 4u, y - 2v, z + w)$$
and scalar multiplication defined as
$$\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z/3).$$
8. Let V be the set of all positive real numbers with addition defined as $x + y = xy$ and scalar multiplication defined as $\alpha x = x^{\alpha}$.
9. Let V be the set of all positive real valued continuous functions f on $[a, b]$ such that

   (i) $\int_a^b f(x)\,dx = 0$ and (ii) $\int_a^b f(x)\,dx = 2$ with usual addition and scalar multiplication.
10. Let V be the set of all solutions of the

    (i) homogeneous linear differential equation $y'' - 3y' + 2y = 0$.

    (ii) non-homogeneous linear differential equation $y'' - 3y' + 2y = x$.

    under the usual addition and scalar multiplication.

Is W a subspace of V in problems **11** to **15**? If not, state why?

11. Let V be the set of all 3×1 real matrices with usual matrix addition and scalar multiplication and W consisting of all 3×1 real matrices of the from

    (i) $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$, (ii) $\begin{bmatrix} a \\ a \\ a^2 \end{bmatrix}$, (iii) $\begin{bmatrix} a \\ b \\ 2 \end{bmatrix}$, (iv) $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$.
12. Let V be the set of all 3×3 real matrices with the usual matrix addition and scalar multiplication and W consisting of all 3×3 matrices $\mathbf{A}$ which

    (i) have positive elements, (ii) are non-singular,

    (iii) are symmetric, (iv) $\mathbf{A}^2 = \mathbf{A}$.
13. Let V be the set of all 2×2 complex matrices with the usual matrix addition and scalar multiplication and W consisting of all matrices with the usual addition and scalar multiplication and W consisting of all martices of the form $\begin{bmatrix} z & x+iy \\ x-iy & u \end{bmatrix}$, where x, y, z, u are real numbers and (i) scalars are real numbers, (ii) scalars are complex numbers.

14. Let V consist of all real polynomials of degree ≤ 4 with the usual polynomial addition and scalar multiplication and W consisting of polynomials of degree ≤ 4 having

(i) constant term 1, (ii) coefficient of t^2 as 0,

(iii) coefficient of t^3 as 1, (iv) only real roots.

15. Let V be the vector space of all triplets of the form (x_1, x_2, x_3) in IR^3 with the usual addition and scalar multiplication and W be the set of triplets of the form (x_1, x_2, x_3) such that

(i) $x_1 = 2x_2 = 3x_3$, (ii) $x_1 = x_2 = x_3 + 1$,

(iii) $x_1 \geq 0$, x_2, x_3 arbitrary, (iv) $x_1^2 + x_2^2 + x_3^2 \leq 4$. (v) x_3 is an integer.

16. Let $\mathbf{u} = (1, 2, -1)$, $\mathbf{v} = (2, 3, 4)$ and $\mathbf{w} = (1, 5, -3)$. Determine whether or not $\mathbf{x}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, where $\mathbf{x}$ is given by

(i) (4, 3, 10), (ii) (3, 2, 5) (iii) (–2, 1, –5).

17. Let $\mathbf{u} = (1, -2, 1, 3)$, $\mathbf{v} = (1, 2, -1, 1)$ and $\mathbf{w} = (2, 3, 1, -1)$. Determine whether or not $\mathbf{x}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, where $\mathbf{x}$ is given by

(i) (3, 0, 5, –1), (ii) (2, –7, 1, 11), (iii) (4, 3, 0, 3).

18. Let $P_1(t) = t^2 - 4t - 6$, $P_2(t) = 2t^2 - 7t - 8$, $P_3(t) = 2t - 3$, Write $P(t)$ as a linear combination of $P_1(t)$, $P_2(t)$, $P_3(t)$, when

(i) $P(t) = -t^2 + 1$, (ii) $P(t) = 2t^2 - 3t - 25$.

19. Let V be the set of all 3×1 real matrices. Show that the set

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ spans } V,$$

20. Let V be the set of all 2×2 real matrices. Show that the set

$$S = \left\{ \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} \right\} \text{ spans } V.$$

21. Examine whether the following vectors in $\mathrm{IR}^3/\mathbb{C}^3$ are linearly independent.

(i) (2, 2, 1), (1, –1, 1), (1, 0, 1), (ii) (1, 2, 3), (3, 4, 5), (6, 7, 8),

(iii) (0, 0, 0), (1, 2, 3), (3, 4, 5), (iv) $(2, i, -1)$, $(1, -3, i)$, $(2i, -1, 5)$,

(v) (1, 3, 4), (1, 1, 0), (1, 4, 2), (1, –2, 1).

22. Examine whether the following vectors in IR^4 are linearly independent.

(i) (4, 1, 2, –6), (1, 1, 0, 3), (1, –1, 0, 2), (–2, 1, 0, 3),

(ii) (1, 2, 3,1), (2, 1, –1, 1), (4, 5, 5, 3), (5, 4, 1, 3),

(iii) (1, 2, 3, 4), (2, 0, 1, –2), (3, 2, 4, 2),

(iv) (1, 1, 0, 1), (1, 1, 1, 1), (–1, –1, 1, 1), (1, 0, 0, 1),

(v) (1, 2, 3, –1), (0, 1, –1, 2), (1, 5, 1, 8), (–1, 7, 8, 3).

23. If $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are linearly independent vectors in IR^3, then show that

(i) $\mathbf{x} + \mathbf{y}$, $\mathbf{y} + \mathbf{z}$, $\mathbf{z} + \mathbf{x}$; (ii) $\mathbf{x}$, $\mathbf{x} + \mathbf{y}$, $\mathbf{x} + \mathbf{y} + \mathbf{z}$

are also linearly independent in IR^3.

24. Write (– 4, 7, 9) as a linear combination of the elements of the set S: {(1, 2, 3), (– 1, 3, 4), (3, 1, 2)}. Show that S is not a spanning set in IR^3.

25. Write $t^2 + t + 1$ as a linear combination of the elements of the set S: $\{3t, t^2 - 1, t^2 + 2t + 2\}$. Show that S is the spanning set for all polynomials of degree 2 and can be taken as its basis.

26. Let V be the set of all vectors in $\mathbb{R}^4$ and S be a subset of V consisting of all vectors of the form

(i) $(x, y, -y, -x)$, (ii) (x, y, z, w) such that $x + y + z - w = 0$,

(iii) $(x, 0, z, w)$, (iv) (x, x, x, x).

Find the dimension and the basis of S.

27. For what values of k do the following set of vectors form a basis in $\mathbb{R}^3$?

(i) $\{(k, 1 - k, k), (0, 3k - 1, 2), (-k, 1, 0)\}$,

(ii) $\{(k, 1, 1), (0, 1, 1), (k, 0, k)\}$,

(iii) $\{(k, k, k), (0, k, k), (k, 0, k)\}$,

(iv) $\{(1, k, 5), (1, -3, 2), (2, -1, 1)\}$.

28. Find the dimension and the basis for the vector space V, when V is the set of all 2×2 (i) real matrices (ii) symmetric matrices, (iii) skew-symmetric matrices, (iv) skew-Hermitian matrices, (v) real matrices $\mathbf{A} = (a_{ij})$ with $a_{11} + a_{22} = 0$, (vi) real matrices $\mathbf{A} = (a_{ij})$ with $a_{11} + a_{12} = 0$.

29. Find the dimension and the basis for the vector space V, when V is the set of all 3×3 (i) diagonal matrices (ii) upper triangular matrices, (iii) lower triangular matrices.

30. Find the dimension of the vector space V, when V is the set of all $n \times n$ (i) real matrices, (ii) diagonal matrices, (iii) symmetric matrices (iv) skew-symmetric matrices.

Examine whether the transformation T given in problems **31** to **35** is linear or not. If not linear, state why?

31. $T : \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \end{pmatrix} = x + y + a$, $a \neq 0$, a real constant.

32. $T : \mathbb{R}^3 \to \mathbb{R}^2$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x + z \end{pmatrix}$.

33. $T : \mathbb{R}^1 \to \mathbb{R}^2$; $T(x) = \begin{pmatrix} x^2 \\ 3x \end{pmatrix}$.

34. $T : \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} 0 & x \neq 0, y \neq 0 \\ 2y, & x = 0 \\ 3x, & y = 0. \end{cases}$

35. $T : \mathbb{R}^3 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + x + z$.

Find $ker(T)$ and $ran(T)$ and their dimensions in problems **36** to **42**.

36. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ z \\ x - y \end{pmatrix}$.

37. $T : \mathbb{R}^2 \to \mathbb{R}^3$; $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ y - x \\ 3x + 4y \end{pmatrix}$.

38. $T : \mathbb{R}^4 \to \mathbb{R}^3$; $T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x + y + w \\ z \\ y + 2w \end{pmatrix}$.

39. $T : \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \end{pmatrix} = x + 3y$.

40. $T : \mathbb{R}^3 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + 3y$.

41. $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x - y \end{pmatrix}$.

42. $T : \mathrm{IR}^3 \to \mathrm{IR}^2$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y \\ 3x + z \end{pmatrix}$.

43. Let $T : \mathrm{IR}^3 \to \mathrm{IR}^2$ be a linear transformation defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ x - z \end{pmatrix}$.

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } \mathrm{IR}^3 \text{ and } \mathbf{y} = \left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right\} \text{ in } \mathrm{IR}^2.$$

44. Let V and W be two vector spaces in IR^3. Let $T : V \to W$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ x + y \\ x + y + z \end{pmatrix}.$$

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} \text{ in } V \quad \text{and} \quad \mathbf{y} = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } W.$$

45. Let V and W be two vector spaces in IR^3. Let $T : V \to \mathrm{W}$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} x + z \\ x + y \\ x + y + z \end{pmatrix}.$$

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\} \text{ in } V \quad \text{and} \quad \mathbf{y} = \left\{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\} \text{ in } W$$

46. Let $T : \mathrm{IR}^3 \to \mathrm{IR}^4$ be a linear transformation defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ y + z \\ x + z \\ x + y + z \end{pmatrix}$.

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } \mathrm{IR}^3 \quad \text{and} \quad \mathbf{y} = \left(\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right) \text{ in } \mathrm{IR}^4$$

47. Let $T: \mathrm{IR}^2 \to \mathrm{IR}^3$ be a linear transformation. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$ be the matrix representation of the linear transformation T with respect to the ordered basis vectors $\mathbf{v}_1 = [1, 2]^T$, $\mathbf{v}_2 = [3, 4]^T$ in IR^2 and $\mathbf{w}_1 = [-1, 1, 1]^T$, $\mathbf{w}_2 = [1, -1, 1]^T$, $\mathbf{w}_3 = [1, 1, -1]^T$ in IR^3. Then, determine the linear transformation T.

48. Let $T: \mathrm{IR}^3 \to \mathrm{IR}^2$ be a linear transformation. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & -4 \end{bmatrix}$ be the matrix representation of the linear transformation with respect to the ordered basis vectors $\mathbf{v}_1 = [1, -1, 1]^T$, $\mathbf{v}_2 = [2, 3, -1]^T$, $\mathbf{v}_3 = [1, 1, -1]^T$ in IR^3 and $\mathbf{w}_1 = [1, 1]^T$, $\mathbf{w}_2 = [2, 3]^T$ in IR^2. Then, determine the linear transformation T.

49. Let $T: P_1(t) \to P_2(t)$ be a linear transformation. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 1 \end{bmatrix}$ be the matrix representation of the linear transformation with respect to the ordered basis $[1 + t, t]$ in $P_1(t)$ and $[1 - t, 2t, 2 + 3t - t^2]$ in $P_2(t)$. Then, determine the linear transformation T.

50. Let V be the set of all vectors of the form (x_1, x_2, x_3) in IR^3 satisfying (i) $x_1 - 3x_2 + 2x_3 = 0$; (ii) $3x_1 - 2x_2 + x_3 = 0$ and $4x_1 + 5x_2 = 0$. Find the dimension and basis for V.

3.4 Solution of General linear System of Equations

In section 3.2.5, we have discussed the matrix method and the Cramer's rule for solving a system of n equations in n unknowns, $\mathbf{A}\mathbf{x} = \mathbf{b}$. We assumed that the coefficient matrix $\mathbf{A}$ is non-singular, that is $|\mathbf{A}| \neq 0$, or the rank of the matrix $\mathbf{A}$ is n. The matrix method requires evaluation of n^2 determinants each of order $(n - 1)$, to generate the cofactor matrix, and one determinant of order n, whereas the Cramer's rule requires evaluation of $(n + 1)$ determinants each of order n. Since the evaluation of high order determinants is very time consuming, these methods are not used for large values of n, say $n > 4$. In this section, we discuss a method for solving a general system of m equations in n unknowns, given by

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.28}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

are respectively called the *coefficient matrix*, *right hand side column vector* and the *solution vector*. The order of the matrices $\mathbf{A}$, $\mathbf{b}$, $\mathbf{x}$ are respectively $m \times n$, $m \times 1$ and $n \times 1$.

The matrix

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \tag{3.29}$$

is called the *augmented matrix* and has m rows and $(n + 1)$ columns. The augmented matrix describes completely the system of equations. The solution vector of the system of equations (3.28) is an n-tuple $(x_1, x_2, \ldots, x_n)$ that satisfies all the equations. There are three possibilities:

(i) the system has a unique solution,

(ii) the system has no solution,

(iii) the system has infinite number of solutions.

The system of equations is said to be *consistent*, if it has atleast one solution and *inconsistent*, if it has no solution. Using the concepts of ranks and vector spaces, we now obtain the necessary and sufficient conditions for the existence and uniqueness of the solution of the linear system of equations.

This was taught in class

3.4.1 Existence and Uniqueness of the Solution

Let V_n be a vector space consisting of n-tuples in $\mathbb{R}^n$(or $\mathbb{C}^n$). The row vectors $R_1, R_2, \ldots, R_m$ of the $m \times n$ matrix $\mathbf{A}$ are n-tuples which belong to V_n. Let S be the subspace of V_n generated by the rows of $\mathbf{A}$. Then, S is called the *row-space* of the matrix $\mathbf{A}$ and its dimension is called the *row-rank* of $\mathbf{A}$ and is denoted by $rr(\mathbf{A})$. Therefore,

$$\textit{row-rank} \text{ of } \mathbf{A} = rr(\mathbf{A}) = dim\ (S). \tag{3.30}$$

Similarly, we define the *column-space* of $\mathbf{A}$ and the *column-rank* of $\mathbf{A}$ denoted by $cr(\mathbf{A})$.

Since the row-space of $m \times n$ matrix $\mathbf{A}$ is generated by m row vectors of $\mathbf{A}$, we have dim $(S) \leq m$ and since S is a subspace of V_n, we have $dim\ (S) \leq n$. Therefore, we have

$$rr\,(\mathbf{A}) \leq \min\,(m, n) \quad \text{and} \quad \text{similarly} \quad cr(\mathbf{A}) \leq \min\,(m, n). \tag{3.31}$$

Theorem 3.6 Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix. Then the row-rank and column-rank of $\mathbf{A}$ are same.

Now, we state an important result which is known as the *fundamental theorem of linear algebra*.

Theorem 3.7 The non-homogeneous system of equations $\mathbf{A\,x} = \mathbf{b}$, where $\mathbf{A}$ is an $m \times n$ matrix, has a solution if and only if the matrix $\mathbf{A}$ and the augmented matrix $(\mathbf{A} \mid \mathbf{b})$ have the same rank.

In section 3.2.3, we defined the rank of $m \times n$ matrix $\mathbf{A}$ in terms of the determinants of the submatrices of $\mathbf{A}$. An $m \times n$ matrix has rank r if it has at least one square submatrix of order r which is non-singular and all square submatrices of order greater than r are singular. This approach is very time consuming when n is large. Now, we discuss an alternative procedure to obtain the rank of a matrix.

3.4.2 Elementary Row and Column Operation

The following three operations on a matrix $\mathbf{A}$ are called the *elementary row operations*:

(i) Interchange of any two rows (written as $R_i \sim R_j$).

(ii) Multiplication/division of any row by a non-zero scalar (written as αR_i)

(iii) Adding/subtracting a scalar multiple of any row to another row (written as $R_i \leftarrow R_i + \alpha R_j$, that is α multiples of the elements of the jth row are added to the corresponding elements of the ith row. The elements of the jth row remain unchanged, whereas, the elements of the ith row get changed).

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These operations change the form of **A** but do not change the row-rank of **A** as they do not change the row-space of **A**. A matrix **B** is said to be *row equivalent* to a matrix **A**, if the matrix **B** can be obtained from the matrix **A** by a finite sequence of elementary row operations. Then, we usually write $\mathbf{B} \approx \mathbf{A}$. We observe that

(i) every matrix is row equivalent to itself.

(ii) if **A** is row equivalent to **B**, then **B** is row equivalent to **A**.

(iii) if **A** is row equivalent to **B** and **B** is row equivalent to **C**, then **A** is row equivalent to **C**.

The above operations performed on columns (that is column in place of row) are called *elementary column operations*.

3.4.3 Echelon Form of a Matrix

An $m \times n$ matrix is called a *row echelon* matrix or in *row echelon* form if the number of zeros preceeding the first non-zero entry of a row increases row by row until a row having all zero entries (or no other elimination is possible) is obtained. Therefore, a matrix is in row echelon form if the following are satisfied.

(i) If the ith row contains all zeros, it is true for all subsequent rows.

(ii) If a column contains a non-zero entry of any row, then every subsequent entry in this column is zero, that is, if the ith and $(i + 1)$th rows are both non-zero rows, then the initial non-zero entry of the $(i + 1)$th row appears in a later column than that of the ith row.

(iii) Rows containing all zeros occur only after all non-zero rows.

For example, the following matrices are in row echelon form.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathbf{A} = (a_{ij})$ be a given $m \times n$ matrix. Assume that $a_{11} \neq 0$. If $a_{11} = 0$, we interchange the first row with some other row to make the element in the (1, 1) position as non-zero. Using elementary row operations, we reduce the matrix **A** to its row echelon form (elements of first column below a_{11} are made zero, then elements in the second column below a_{22} are made zero and so on).

Similarly, we define the column echelon form of a matrix.

Rank of A The number of non-zero rows in the row echelon form of a matrix **A** gives the rank of the matrix **A** (that is, the dimension of the row-space of the matrix **A**) and the set of the non-zero rows in the row echelon form gives the basis of the row-space.

Similar results hold for column echelon matrices.

Remark 13

(i) If **A** is a square matrix, then the row-echelon form is an upper triangular matrix and the column echelon form is a lower triangular matrix.

(ii) This approach can be used to examine whether a given set of vectors are linearly independent or not. We form the matrix with each vector as its row (or column) and reduce it to the row (column) echelon form. The given vectors are linearly independent, if the row echelon form has no row with all its elements as zeros. The number of non-zero rows is the dimension of the given set of vectors and the set of vectors consisting of the non-zero rows is the basis.

Example 3.25 Reduce the following matrices to row echelon form and find their ranks.

(i) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$.

Solution Let the given matrix be denoted by **A**. We have

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 + 2R_1 \end{matrix} \approx \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix} R_3 + 2R_2 \approx \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is the row echelon form of **A**. Since the number of non-zero rows in the row echelon form is 2, we get rank (**A**) = 2.

$$\text{(ii) } \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - 8R_1 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \begin{matrix} R_3 + R_2 \\ R_4 - 5R_2 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the number of non-zero rows in the echelon form of **A** is 2, we get rank (**A**) = 2.

Example 3.26 Reduce the following matrices to column echelon form and find their ranks.

(i) $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 4 & -1 & 5 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix}$.

Solution Let the given matrix be denoted by **A**. We have

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix} \begin{matrix} C_2 - C_1/3 \\ C_3 - 7C_1/3 \end{matrix} \approx \begin{bmatrix} 3 & 0 & 0 \\ 1 & 5/3 & 5/3 \\ 4 & -7/3 & -7/3 \\ 2 & 1/3 & 1/3 \end{bmatrix} C_3 - C_2 \approx \begin{bmatrix} 3 & 0 & 0 \\ 1 & 5/3 & 0 \\ 4 & -7/3 & 0 \\ 2 & 1/3 & 0 \end{bmatrix}.$$

Since the column echelon form of **A** has two non-zero columns, rank (**A**) = 2.

$$\text{(ii) } \mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix} \begin{matrix} C_2 - C_1 \\ C_3 + C_1 \\ C_4 - C_1 \end{matrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & -2 \\ 1 & -1 & 2 & 1 \\ 1 & -2 & 4 & 2 \end{bmatrix} \begin{matrix} C_3 + 2C_2 \\ C_4 + C_2 \end{matrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}.$$

Since the column echelon form of **A** has 2 non-zero columns, rank (**A**) = 2.

Example 3.27 Examine whether the following set of vectors is linearly independent. Find the dimension and the basis of the given set of vectors.

(i) (1, 2, 3, 4), (2, 0, 1, – 2), (3, 2, 4, 2),

(ii) (1, 1, 0, 1), (1, 1, 1, 1), (–1, 1, 1, 1), (1, 0, 0, 1),

(iii) (2, 3, 6, –3, 4), (4, 2, 12, –3, 6), (4, 10, 12, –9, 10).

Solution Let each given vector represent a row of a matrix **A**. We reduce **A** to row echelon form. If all the rows of the echelon form have some non-zero elements, then the given set of vectors are linearly independent.

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & 2 & 4 & 2 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & -4 & -5 & -10 \end{bmatrix} R_3 - R_2 \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since all the rows in the row echelon form of **A** are not non-zero, the given set of vectors are linearly dependent. Since the number of non-zero rows is 2, the dimension of the given set of vectors is 2. The basis can be taken as the set of vectors {(1 2 3 4), (0, –4, –5, –10)}.

$$\text{(ii) } \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 + R_1 \\ R_4 - R_1 \end{matrix} \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix} R_2 \sim R_3 \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} R_4 + R_2/2$$

$$\approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{bmatrix} R_4 - R_3/2 \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since all the rows in the row echelon form of **A** are non-zero, the given set of vectors are linearly independent and the dimension of the given set of vectors is 4. The set of vectors {(1, 1, 0, 1), (0, 2, 1, 2), (0, 0, 1, 0), (0, 0, 0, 1)} or the given set itself forms the basis.

(iii) $\mathbf{A} = \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 4 & 2 & 12 & -3 & 6 \\ 4 & 10 & 12 & -9 & 10 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix} \approx \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 4 & 0 & -3 & 2 \end{bmatrix} R_3 + R_4 \approx \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

Since all the rows in the echelon form of **A** are not non-zero, the given set of vectors are linearly dependent. Since the number of non-zero rows is 2, the dimension of the given set of vectors is 2 and its basis can be taken as the set {(2, 3, 6, –3, 4), (0, – 4, 0, 3, –2)}.

3.4.4 Gauss Elimination Method for Non-homogeneous Systems

Consider a non-homogeneous system of m equations in n unknowns

$$\mathbf{Ax} = \mathbf{b} \tag{3.32}$$

where $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$

We assume that at least one element of **b** is not zero. We write the augmented matrix of order $m \times (n + 1)$ as

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

and reduce it to the row echelon form by using elementary row operations. We need a maximum of $(m - 1)$ stages of eliminations to reduce the given augmented matrix to the equivalent row echelon form. This process may terminate at an earlier stage. We then have an equivalent system of the form

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1n} & b_1 \\ 0 & \bar{a}_{22} & \cdots & \bar{a}_{2r} & \cdots & \bar{a}_{2n} & \bar{b}_2 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & a^*_{rr} & \cdots & a^*_{rn} & b^*_r \\ 0 & 0 & \cdots & 0 & \cdots & 0 & b^*_{r+1} \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & b^*_m \end{array}\right]. \tag{3.33}$$

where $r \le m$ and $a_{11} \ne 0$, $\bar{a}_{22} \ne 0, \ldots, a^*_{rr} \ne 0$ are called *pivots*. We have the following cases:

(a) Let $r < m$ and one or more of the elements $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are not zero. Then, rank (**A**) ≠ rank (**A** | **b**) and the system of equations has no solution.

(b) Let $m \geq n$ and $r = n$ (the number of columns in **A**) and $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are all zeros. In this case, rank (**A**) = rank (**A** | **b**) = n and the system of equations has a unique solution. We solve the nth equation for x_n, the $(n-1)$th equation for x_{n-1} and so on. This procedure is called the *back substitution method.*

For example, if we have 10 equations in 5 variables, then the augmented matrix is of order 10 × 6. When rank (**A**) = rank (**A** | **b**) = 5, the system has a unique solution.

(c) Let $r < n$ and $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are all zeros. In this case, r unknowns, $x_1, x_2, \ldots, x_r$ can be determined in terms of the remaining $(n - r)$ unknowns $x_{r+1}, x_{r+2}, \ldots, x_n$ by solving the rth equation for x_r, $(r - 1)$th equation for x_{r-1} and so on. In this case, we obtain an $(n - r)$ parameter family of solutions, that is infinitely many solutions.

Remark 14

(a) We do not, normally use column elementary operations in solving the linear system of equations. When we interchange two columns, the order of the unknowns in the given system of equations is also changed. Keeping track of the order of unknowns is quite difficult.

(b) Gauss elimination method may be written as

$$(\mathbf{A} \mid \mathbf{b}) \xrightarrow[\text{row operations}]{\text{Elementary}} (\mathbf{B} \mid \mathbf{c}).$$

The matrix **B** is the row echelon form of the matrix **A** and **c** is the new right hand side column vector. We obtain the solution vector (if it exists) using the back substitution method.

(c) If **A** is a square matrix of order n, then **B** is an upper triangular matrix of order n.

(d) Gauss elimination method can be used to solve p systems of the form $\mathbf{A}\mathbf{x} = \mathbf{b}_1$, $\mathbf{A}\mathbf{x} = \mathbf{b}_2$, ..., $\mathbf{A}\mathbf{x} = \mathbf{b}_p$ which have the same coefficient matrix but different right hand side column vectors. We form the augmented matrix as $(\mathbf{A} \mid \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p)$, which has m rows and $(n + p)$ columns. Using the elementary row operations, we obtain the row equivalent system $(\mathbf{B} \mid \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p)$, where **B** is the row echelon form of **A**. Now, we solve the systems $\mathbf{B}\mathbf{x} = \mathbf{c}_1$, $\mathbf{B}\mathbf{x} = \mathbf{c}_2$, ..., $\mathbf{B}\mathbf{x} = \mathbf{c}_p$, using the back substitution method.

Remark 15

(a) If at any stage of elimination, the pivot element becomes zero, then we interchange this row with any other row below it such that we obtain a non-zero pivot element. We normally choose the row such that the pivot element becomes largest in magnitude.

(b) For an $n \times n$ system, we require $(n - 1)$ stages of elimination. It is possible to compute the total number of additions, subtractions, multiplications and divisions. This number is called the *operation count* of the method. The operation count of the Gauss elimination method for solving an $n \times n$ system is $n(n^2 + 3n - 1)/3$. For large n, the operation count is approximately $n^3/3$.

Example 3.28 Solve the following systems of equations (if possible) using Gauss elimination method.

(i) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$,

(iii) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

Solution We write the augmented matrix and reduce it to row echelon form by applying elementary row operations.

$$\text{(i)}\ (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{array}\right] \begin{matrix} R_2 - R_1/2 \\ R_3 + R_1/2 \end{matrix} \approx \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 5/2 & -3/2 & 4 \end{array}\right] R_3 + 5R_2/3$$

$$\approx \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 0 & 8/3 & -8/3 \end{array}\right].$$

Using the back substitution method, we obtain the solution as

$$\frac{8}{3}z = -\frac{8}{3}, \quad \text{or} \quad z = -1,$$

$$-\frac{3}{2}y + \frac{5}{2}z = -4, \quad \text{or} \quad y = 1,$$

$$2x + y - z = 4, \quad \text{or} \quad x = 1.$$

Therefore, the system of equations has the unique solution $x = 1$, $y = 1$, $z = -1$.

$$\text{(ii)}\ (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 4 & -2 & 3 & 3 \end{array}\right] \begin{matrix} R_2 - R_1/2 \\ R_3 - 2R_1 \end{matrix} \approx \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & -1 & 1/2 & -1/2 \\ 0 & -2 & 1 & -3 \end{array}\right] R_3 - 2R_2 \approx \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & -1 & 1/2 & -1/2 \\ 0 & 0 & 0 & -2 \end{array}\right].$$

We find that rank (**A**) = 2 and rank (**A** | **b**) = 3. Therefore, the system of equations has no solution.

$$\text{(iii)}\ (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 5 & -2 & 2 & 5 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix} \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] R_3 - R_2 \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system is consistent and has infinite number of solutions. We find that the last equation is satisfied for all values of x, y, z. From the second equation, we get $3y - 3z = 0$, or $y = z$. From the first equation, we get $x - y + z = 1$, or $x = 1$. Therefore, we obtain the solution $x = 1$, $y = z$ and z is arbitrary.

Example 3.29 Solve the following system of equations using Gauss elimination method.

(i) $4x - 3y - 9z + 6w = 0$
$2x + 3y + 3z + 6w = 6$
$4x - 21y - 39z - 6w = -24,$

(ii) $x + 2y - 2z = 1$
$2x - 3y + z = 0$
$5x + y - 5z = 1$
$3x + 14y - 12z = 5.$

Solution We have

$$\text{(i) } (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 2 & 3 & 3 & 6 & 6 \\ 4 & -21 & -39 & -6 & -24 \end{array}\right] \begin{matrix} R_2 - R_1/2 \\ R_3 - R_1 \end{matrix} \approx \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 0 & 9/2 & 15/2 & 3 & 6 \\ 0 & -18 & -30 & -12 & -24 \end{array}\right] R_3 + 4R_2$$

$$= \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 0 & 9/2 & 15/2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system of equations is consistent and has infinite number of solutions. Choose w as arbitrary. From the second equation, we obtain

$$\frac{9}{2}y + \frac{15}{2}z = 6 - 3w, \text{ or } y = \frac{2}{9}\left(6 - 3w - \frac{15}{2}z\right) = \frac{1}{3}(4 - 5z - 2w).$$

From the first equation, we obtain

$$4x = 3y + 9z - 6w = 4 - 5z - 2w + 9z - 6w = 4 + 4z - 8w$$

or

$$x = 1 + z - 2w.$$

Thus, we obtain a two parameter family of solutions

$$x = 1 + z - 2w \quad \text{and} \quad y = (4 - 5z - 2w)/3$$

where z and w are arbitrary.

$$\text{(ii) } (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & -3 & 1 & 0 \\ 5 & 1 & -5 & 1 \\ 3 & 14 & -12 & 5 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \\ R_4 - 3R_1 \end{matrix} \approx \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & -9 & 5 & -4 \\ 0 & 8 & -6 & 2 \end{array}\right] \begin{matrix} R_3 - 9R_2/7 \\ R_4 + 8R_2/7 \end{matrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & 0 & -10/7 & -10/7 \\ 0 & 0 & -2/7 & -2/7 \end{array}\right] R_4 - R_3/5 \approx \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & 0 & -10/7 & -10/7 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The last equation is satisfied for all values of x, y, z. From the third equation, we obtain $z = 1$. Back substitution gives $y = 1$, $x = 1$. Hence, the system of equation has a unique solution $x = 1$, $y = 1$ and $z = 1$. Since $R_4 = (24R_1 - 7R_2 + R_3)/5$, the last equation is redundant.

3.4.5 Gauss-Jordan Method

In this method, we perform elementary row transformations on the augmented matrix $[\mathbf{A} \mid \mathbf{b}]$, where A is a square matrix, and reduce it to the form

$$[\mathbf{A} \mid \mathbf{b}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \mathbf{c}]$$

Where **I** is the identity matrix and **c** is the solution vector. This reduction is equivalent to finding the solution as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. The first step is same as in the Gauss elimination method. From second step onwards, we make elements below and above the pivot as zeros, using elementary row transformations. Finally, we divide each row by its pivot to obtain the form $[\mathbf{I} \mid \mathbf{c}]$. Alternately, at every step, the pivot can be made as 1 before elimination. Then, **c** is the solution vector.

This method is more expensive (larger operation count) than the Gauss elimination. Hence, we do not normally use the Gauss-Jordan method form finding the solution of a system. However, this method is very useful for finding the inverse ($\mathbf{A}^{-1}$) of a matrix **A**. We consider the augmented matrix $[\mathbf{A} \mid \mathbf{I}]$ and reduce it to the form

$$[\mathbf{A} \mid \mathbf{I}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \mathbf{A}^{-1}]$$

using elementary row transformations. If we are solving the system of equations (3.28), then we have $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, and the matrix multiplication in the right hand side gives the solution vector.

Remark 16

If any pivot element at any stage of elimination becomes zero, then we interchange rows as in the Gauss elimination method.

Example 3.30 Using the Gauss-Jordan method, solve the system of equation $\mathbf{A}\,\mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

Solution We perform elementary row transformations on the augmented matrix and reduce it the form $[\mathbf{I} \mid \mathbf{C}]$. We get

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & -3 & 4 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -5 & 4 \\ 0 & 2 & 0 & 1 \end{array}\right] R_2/3$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ R_3 - 2R_2 \end{array} \approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 10/3 & -5/3 \end{array}\right] R_3/(10/3)$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \begin{array}{l} R_1 + 2R_3/3 \\ R_2 + 5R_3/3 \end{array} \approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right].$$

Hence, the solution vector is

$$\mathbf{x} = [1 \quad 1/2 \quad -1/2]^T.$$

Example 3.31 Using Gauss-Jordan method, find the inverse of the matix $\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

Type

Solution We have

$$(\mathbf{A} \mid \mathbf{I}) = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right].$$

The pivot element a_{11} is -1. We make it 1 by multiplying the first row by -1. Therefore,

$$(\mathbf{A} \mid \mathbf{I}) \approx \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 3R_1 \\ R_3 + R_1 \end{array} \approx \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] R_2/2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ R_3 - 2R_2 \end{array} \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] (-R_3)/5$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 3/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right] \begin{array}{l} R_1 - 3R_3/2 \\ R_3 - 7R_3/2 \end{array} \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/10 & 2/10 & 3/10 \\ 0 & 1 & 0 & -13/10 & -2/10 & 7/10 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right].$$

Hence,
$$\mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}.$$

3.4.6 Homogeneous System of Linear Equations

Consider the homogeneous system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{0} \tag{3.34}$$

where $\mathbf{A}$ is an $m \times n$ matrix. The homogenous system is always consistent since $\mathbf{x} = \mathbf{0}$ (trivial solution) is always a solution. In this case, rank $(\mathbf{A})$ = rank $(\mathbf{A} \mid \mathbf{0})$. Therefore, for the homogeneous system to have a non-trivial solution, we require that rank $(\mathbf{A}) < n$. If rank $(\mathbf{A}) = r < n$, we obtain an $(n - r)$ parameter family of solutions which form a vector space of dimension $(n - r)$ as $(n - r)$ parameters can be chosen arbitrarily.

The solution space of the homogeneous system is called the *null space* and its dimension is called the *nullity* of $\mathbf{A}$. Therefore, we obtain the result

$$\text{rank } (\mathbf{A}) + \text{nullity } (\mathbf{A}) = n \text{ (see Theorem 3.5).}$$

Remark 17

(a) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of a linear homogeneous system, then $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ is also a solution of the homogenous system for any scalars α, β. This result does not hold for non-homogenous systems.

(b) A homogeneous system of m equations in n unknowns and $m < n$, always possesses a non-trivial solution.

Theorem 3.8 If a non-homogeneous system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ has solutions, then all these solutions are of the form $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$ where $\mathbf{x}_0$ is any fixed solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_h$ is any solution of the corresponding homogeneous system.

Proof Let $\mathbf{x}$ be any solution and $\mathbf{x}_0$ be any fixed solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$. Therefore, we have

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{A}\mathbf{x}_0 = \mathbf{b}.$$

Subtracting, we get

$$\mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{x}_0 = \mathbf{0}, \quad \text{or} \quad \mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{0}.$$

Thus, the difference $\mathbf{x} - \mathbf{x}_0$ between any solution $\mathbf{x}$ of $\mathbf{A}\mathbf{x} = \mathbf{b}$ and any fixed solution $\mathbf{x}_0$ of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is a solution of the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$, say $\mathbf{x}_h$. Hence, the result.

Remark 18

If the non-homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A}$ is an $m \times n$ matrix $(m \geq n)$ has a unique solution, that is rank $(\mathbf{A}) = n$, then the corresponding homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution, that is $\mathbf{x}_h = \mathbf{0}$.

Example 3.32 Solve the following homogeneous system of equation $\mathbf{A}\mathbf{x} = \mathbf{0}$, where $\mathbf{A}$ is given by

$$\text{(i)} \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & -6 & 1 \end{bmatrix}.$$

Find the rank (A) and nullity (A).

Solution We write the augmented matrix $(\mathbf{A} \mid \mathbf{0})$ and reduce it to row echelon form.

(i) $$(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 0 \end{array}\right] \begin{array}{l} R_2 - R_1/2 \\ R_3 - 3R_1/2 \end{array} \approx \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -3/2 & 0 \\ 0 & 1/2 & 0 \end{array}\right] R_3 + R_2/3 \approx \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Since, rank $(\mathbf{A}) = 2$ = number of unknowns, the system has only a trivial solution. Hence, nullity $(\mathbf{A}) = 0$.

(ii) $$(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array} \approx \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -3 & 4 & 0 \end{array}\right] R_3 - 3R_2 \approx \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right].$$

Since rank $(\mathbf{A}) = 3$ = number of unknowns, the homogeneous system has only a trivial solution. Therefore, nullity $(\mathbf{A}) = 0$.

(iii) $$(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 2 & 3 & 1 & 4 & 0 \\ 3 & 2 & -6 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \approx \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & -1 & -3 & -2 & 0 \end{array}\right] R_3 + R_2 \approx \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, rank $(\mathbf{A}) = 2$ and the number of unknowns is 4. Hence, we obtain a two parameter family of solutions as $x_2 = -3x_3 - 2x_4$, $x_1 = -x_2 + x_3 - x_4 = 4x_3 + x_4$, where x_3 and x_4 are arbitrary. Therefore, nullity $(\mathbf{A}) = 2$.

Exercise 3.3

Using the elementary row operations, determine the ranks of the following matrices.

1. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 2 \\ 5 & -5 & 11 \end{bmatrix}$.

3. $\begin{bmatrix} 2 & 1 & -2 \\ -1 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$.

4. $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & 4 & -13 & -5 \end{bmatrix}$.

5. $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 3 & -1 \\ 8 & 13 & 14 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 7 & 11 & 15 & 19 \\ 9 & 15 & 21 & 27 \end{bmatrix}$.

8. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

9. $\begin{bmatrix} 2 & 0 & -1 & 0 \\ 4 & 1 & 0 & 5 \\ 0 & 1 & 3 & 6 \\ 6 & 1 & -2 & 6 \end{bmatrix}$.

10. $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$.

Using the elementary column operations, determine the rank of the following matrices.

11. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix}$.

12. $\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$.

13. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ -1 & 1 & 3 & -5 \\ 2 & 3 & 4 & 5 \end{bmatrix}$.

14. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$.

15. $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$.

Determine whether the following set of vectors is linearly independent. Find also its dimension.

16. $\{(3, 2, 4), (1, 0, 2), (1, -1, -1)\}$.

17. $\{(2, 2, 1), (1, -1, 1), (1, 0, 1)\}$.

18. $\{(2, 1, 0), (1, -1, 1), (4, 1, 2), (2, -3, 3)\}$.

19. $\{(2, 2, 1), (2, i, -1), (1 + i, -i, 1)\}$.

20. $\{(1, 1, 1), (i, i, i), (1 + i, -1 - i, i)\}$

21. $\{(1, 1, 1, 1), (-1, 1, 1, -1), (1, 0, -1, 1), (1, 1, 0, 1)\}$.

22. $\{(1, 2, 3, 1), (2, 1, -1, 1), (4, 5, 5, 3), (5, 4, 1, 3)\}$.

23. $\{(1, 2, 3, 4), (0, 1, -1, 2), (1, 4, 1, 8), (3, 7, 8, 14)\}$.

24. $\{(1, 1, 0, 1), (1, 1, 1, 1), (4, 4, 1, 1), (1, 0, 0, 1)\}$.

25. $\{(2, 2, 0, 2), (4, 1, 4, 1), (3, 0, 4, 0)\}$.

Determine which of the following systems are consistent and find all the solutions for the consistent systems.

26. $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$.

27. $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

28. $\begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 3 \end{bmatrix}$.

29. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & -9 & 2 \\ 5 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$.

30. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \\ 22 \end{bmatrix}$.

31. $\begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & -3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

32. $\begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$.

33. $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

34. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$.

35. $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Find all the solutions of the following homogeneous systems $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{A}$ is given as the following.

36. $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}$.

37. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & -7 \\ -1 & -2 & 11 \end{bmatrix}$.

38. $\begin{bmatrix} 3 & -11 & 5 \\ 4 & 1 & -10 \\ 4 & 9 & -6 \end{bmatrix}$

39. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 6 & 12 \end{bmatrix}$.

40. $\begin{bmatrix} 2 & -1 & -3 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & -7 & -13 & -1 \\ -1 & 5 & 9 & 1 \end{bmatrix}$.

41. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$.

42. $\begin{bmatrix} 3 & 1 & 1 & 4 \\ 0 & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{bmatrix}$

43. $\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & -2 & -3 \\ 3 & 0 & -5 & -1 \\ 5 & -1 & -7 & -4 \end{bmatrix}$.

44. $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$.

45. $\begin{bmatrix} 1 & 1 & -2 & -1 \\ 2 & 1 & 1 & -2 \\ 3 & 2 & -1 & -3 \\ 4 & 2 & 2 & -4 \end{bmatrix}$.

Using the Gauss-Jordan method, find the inverses of the following matrices.

46. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

47. $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix}$.

48. $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

49. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$.

50. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 4 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

3.5 Eigenvalue Problems

Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n. The matrix $\mathbf{A}$ may be singular or non-singular. Consider the homogeneous system of equations

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \quad \text{or} \quad (\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{3.35}$$

Where λ is a scalar and **I** is an identity matrix of order n. The homogeneous system of equations (3.35) always has a trivial solution. We need to find values of λ for which the homogeneous system (3.35) has non-trivial solutions. The values of λ for which non-trivial solutions of the homogeneous system (3.35) exist, are called the *eigenvalues* or the *characteristic values* of **A** and the corresponding a non-trivial solution vector x are called the *eigenvalues* or the *characteristic values* of **A**. If **x** is a non-trival of the homogeneous system (3.35), then α **x**, where α is any constant is also a solution of the homogeneous system. Hence, an eigenvector is unique only upto a constant multiple. The problem of determining the eigenvalues and the corresponding eigenvector of a square matrix **A** is called and *eigenvalues problem*.

3.5.1 Eigenvalues and Eigenvectors

If the homogeneous system (3.35) has a non-trivial solution, then the rank of the coefficient matrix $(\mathbf{A} - \lambda\mathbf{I})$ is less than n, that is the coefficient matrix must be singular. Therefore,

rank (B) = n
if |B| ≠ 0

$$det\,(\mathrm{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0. \tag{3.36}$$

Expanding the determinant given in Eq. (3.36), we obtain a polynomial of degree n in λ, which is of the form

$$P_n(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| = (-1)^n\,[\lambda^n - c_1\,\alpha^{n-1} + c_2\,\lambda^{n-2} - \ldots + (-1)^n c_n] = 0$$

or

$$\lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} - \ldots + (-1)^n c_n = 0. \tag{3.37}$$

Where $c_1, c_2 \ldots, c_n$ can be expressed in terms of the elements a_{ij} of the matrix **A**. This equation is called the *characteristic equation* of the matrix **A**. The polynomial equation $P_n(\lambda) = 0$ has n roots which can be real or complex, simple or repeated. The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the polynomial equation $P_n(\lambda) = 0$ are called the *eigenvalues*. By using the relation between the roots and the coefficients, We can write

$$\begin{aligned} &\lambda_1 + \lambda_2 + \ldots + \lambda_n = c_1 = a_{11} + a_{22} + \ldots + a_{nn} \\ &\lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_{n-1}\,\lambda_n = c_2 \\ &\vdots \\ &\lambda_1\lambda_2 \ldots \lambda_n = c_n. \end{aligned} \tag{3.38}$$

If we set $\lambda = 0$ in Eq. (3.36), then we get

$$|\mathbf{A}| = (-1)^{2n} c_n = c_n = \lambda_1\lambda_2 \ldots \lambda_n.$$

Therefore, we get

$$\boxed{\text{Sum of eigenvalues = trace (A), and product of eigenvalues = |A|.}}$$

trace(A) = $\lambda_1 + \lambda_2 + \cdots \lambda_n$ $|A| = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n$

The set of the eigenvalues is called the *spectrum* of **A** and the largest eigenvalue in magnitude is called the *spectral radius* of **A** and is denoted by ρ (**A**). If $|\mathbf{A}| = 0$, that is the martix is singular, then from Eq. (3.38), we find that atleast one of the eigenvalues must be zero. Conversely, if one of the eigenvalues is zero, then $|\mathbf{A}| = 0$. Note that if **A** is a diagonal or an upper triangular or a lower triangular matrix, then the diagonal elements of the matrix **A** are the eigenvalues of **A**. (example?)

After determining the eigenvalues λ_i's, we solve the homogeneous system $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x} = \mathbf{0}$ for each λ_i, $i = 1, 2, \ldots, n$ to obtain the corresponding eigenvectors.

Properties of eigenvalues and eigenvectors

Let λ be an eigenvalue of **A** and **x** be its corresponding eigenvector. Then, we have the following results.

1. α **A** has eigenvalue $\alpha\lambda$ and the corresponding eigenvector is **x**.

$$\mathbf{Ax} = \lambda\mathbf{x} \Rightarrow \alpha\,\mathbf{Ax} = (\alpha\lambda)\mathbf{x}.$$

2. $\mathbf{A}^m$ has eigenvalue λ^m and the corresponding eigenvector is **x** for any positive interger m. Pre-multiplying both sides of $\mathbf{Ax} = \lambda\mathbf{x}$ by **A**, we get

$$\mathbf{A}\,\mathbf{Ax} = \mathbf{A}\lambda\mathbf{x} = \lambda\mathbf{Ax} = \lambda(\lambda\mathbf{x}) \text{ or } \mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}.$$

Therefore, $\mathbf{A}^2$ has the eigenvalue λ^2 and the corresponding eigenvector is **x**. Pre-multiplying successively m times, we obtain the result.

3. $\mathbf{A} - k\mathbf{I}$ has the eigenvalue $\lambda - k$, for any scalar k and the corresponding eigenvector is **x**.

$$\mathbf{Ax} = \lambda\mathbf{x} \Rightarrow \mathbf{Ax} - k\,\mathbf{Ix} = \lambda\mathbf{x} - k\mathbf{x}$$

or $$(\mathbf{A} - k\mathbf{I})\,\mathbf{x} = (\lambda - k)\,\mathbf{x}.$$

4. $\mathbf{A}^{-1}$(if it exists) has the eigenvalue $1/\lambda$ and the corresponding eigenvector is **x**. Pre-multiplying both sides of $\mathbf{Ax} = \lambda\mathbf{x}$ by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{Ax} = \lambda\mathbf{A}^{-1}\mathbf{x} \quad \text{or} \quad \mathbf{A}^{-1}\mathbf{x} = (1/\lambda)\,\mathbf{x}.$$

5. $(\mathbf{A} - k\mathbf{I})^{-1}$ has the eigenvalue $1/(\lambda - k)$ and the corresponding eigenvector is **x** for any scalar k.

6. **A** and $\mathbf{A}^T$ have the same eigenvalues (since a determinant can be expanded by rows or by columns) but different eigenvectors, (see Example 3.41).

7. For a real matrix **A**, if $\alpha + i\beta$ is an eigenvalue, then its conjugate $\alpha - i\beta$ is also an eigenvalue (since the characteristic equation has real coefficients). When the matrix **A** is complex, this property does not hold. $\overline{AB} = \overline{A}\,\overline{B}$

We now present an important result which gives the relationship of a matrix **A** and its characteristic equation.

Theorem 3.9 (Cayley-Hamilton theorem) Every square matrix **A** satisfies its own characteristic equation

$$\mathbf{A}^n - c_1\mathbf{A}^{n-1} + \ldots + (-1)^{n-1}c_{n-1}\,\mathbf{A} + (-1)^n c_n\,\mathbf{I} = \mathbf{0}. \tag{3.39}$$

Proof The cofactors of the elements of the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ are polynomials in λ of degree $(n-1)$ or less. Therefore, the elements of the adjoint matrix (transpose of the cofactor matrix) are also polynomials in λ of degree $(n-1)$ or less. Hence, we can express the adjoint matrix as a polynomial in λ whose coefficients $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$ are square matrices of order n having elements as functions of the elements of the matrix $\mathbf{A}$. Thus, we can write

$$adj(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \ldots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n.$$

We also have

$$(\mathbf{A} - \lambda\mathbf{I})\ adj\ (\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}|\ \mathbf{I}.$$

Therefore, we can write for any λ

$$(\mathbf{A} - \lambda\mathbf{I})(\mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \ldots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n)$$
$$= \lambda^n\mathbf{I} - c_1\lambda^{n-1}\mathbf{I} + \ldots + (-1)^{n-1}c_{n-1}\lambda\mathbf{I} + (-1)^n c_n\mathbf{I}$$

Comparing the coefficients of various powers of λ, we obtain

$$-\mathbf{B}_1 = \mathbf{I}$$
$$\mathbf{AB}_1 - \mathbf{B}_2 = c_1\mathbf{I}$$
$$\mathbf{AB}_2 - \mathbf{B}_3 = c_2\mathbf{I}$$
$$\ldots$$
$$\mathbf{AB}_{n-1} - \mathbf{B}_n = (-1)^{n-1}c_{n-1}\mathbf{I}$$
$$\mathbf{AB}_n = (-1)^n c_n\mathbf{I}.$$

Pre-multiplying these equations by $\mathbf{A}^n, \mathbf{A}^{n-1}, \ldots, \mathbf{A}, \mathbf{I}$ respectively and adding, we get

$$\mathbf{A}^n - c_1\mathbf{A}^{n-1} + \ldots + (-1)^{n-1}c_{n-1}\mathbf{A} + (-1)^n c_n\mathbf{I} = \mathbf{0}$$

which proves the theorem.

Remark 19

(a) We can use Eq. (3.39) to find $\mathbf{A}^{-1}$ (if it exists) in terms of the powers of the matrix $\mathbf{A}$.

Pre-multiplying both sides in Eq. (3.39) by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{A}^n - c_1\mathbf{A}^{-1}\mathbf{A}^{n-1} + \ldots + (-1)^{n-1}c_{n-1}\mathbf{A}^{-1}\mathbf{A} + (-1)^n c_n\mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$$

or
$$\mathbf{A}^{-1} = -\frac{(-1)^n}{c_n}[\mathbf{A}^{n-1} - c_1\mathbf{A}^{n-2} + \ldots + (-1)^{n-1}c_{n-1}\mathbf{I}] \tag{3.40}$$

(b) We can use Eq.(3.39) to obtain $\mathbf{A}^n$ in terms of lower powers of $\mathbf{A}$ as

$$\mathbf{A}^n = c_1\mathbf{A}^{n-1} - c_2\mathbf{A}^{n-2} + \ldots + (-1)^{n-1}c_n\mathbf{I}. \tag{3.41}$$

Example 3.33 Verify Cayley-Hamilton theorem for the martrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Also, (i) obtain $\mathbf{A}^{-1}$ and $\mathbf{A}^3$, (ii) find eigenvalues of $\mathbf{A}$, $\mathbf{A}^2$ and verify that eigenvalues of $\mathbf{A}^2$ are squares of those of $\mathbf{A}$, (iii) find the spectral radius of $\mathbf{A}$.

Solution The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\ \{(1-\lambda)^2 - 4\} - 2\ \{-(1-\lambda) - 2\}$$

$$= (1-\lambda)\ (\lambda^2 - 2\lambda - 3) - 2(\lambda - 3) = -\lambda^3 + 3\lambda^2 - \lambda + 3 = 0.$$

Now,

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}.$$

We have

$$-\mathbf{A}^3 + 3\mathbf{A}^2 - \mathbf{A} + 3\mathbf{I} = -\begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix} + 3\begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}. \tag{3.42}$$

Hence, $\mathbf{A}$ satisfies the characteristic equation $-\lambda^3 + 3\lambda^2 - \lambda + 3 = 0$.

(i) From Eq. (3.42), we get

$$\mathbf{A}^{-1} = \frac{1}{3}\ [\mathbf{A}^2 - 3\mathbf{A} + \mathbf{I}] = \frac{1}{3}\left[\begin{pmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 6 & 0 \\ -3 & 3 & 6 \\ 3 & 6 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}.$$

From Eq. (3.42), we get

$$\mathbf{A}^3 = 3\mathbf{A}^2 - \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} -3 & 12 & 12 \\ 0 & 9 & 12 \\ 0 & 18 & 15 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}.$$

(ii) Eigenvalues of **A** are the roots of

$$\lambda^3 - 3\lambda^2 + \lambda - 3 = 0 \text{ or } (\lambda - 3)(\lambda^2 + 1) = 0 \text{ or } \lambda = 3, i, -i.$$

The characteristic equation of $\mathbf{A}^2$ is given by

$$\begin{vmatrix} -1-\lambda & 4 & 4 \\ 0 & 3-\lambda & 4 \\ 0 & 6 & 5-\lambda \end{vmatrix} = (-1 - \lambda)\,[(3 - \lambda)\,(5 - \lambda) - 24] = 0$$

or $\quad (\lambda + 1)\,(\lambda^2 - 8\lambda - 9) = 0$ or $(\lambda + 1)\,(\lambda - 9)\,(\lambda + 1) = 0.$

The eigenvalues of $\mathbf{A}^2$ are $9, -1, -1$ which are the squares of the eigenvalues of **A**.

(iii) The spectral radius of **A** is given by

$$\rho\,(\mathbf{A}) = \text{largest eigenvalue in magnitude} = \max_i |\lambda_i| = 3.$$

Example 3.34 If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then show that $\mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}$ for $n \geq 3$. Hence, find $\mathbf{A}^{50}$.

Solution The characteristic equation of **A** is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1 - \lambda)(\lambda^2 - 1) = 0, \quad \text{or} \quad \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

Using Cayley-Hamilton theorem, we get

$$\mathbf{A}^3 - \mathbf{A}^2 - \mathbf{A} + \mathbf{I} = \mathbf{0}, \quad \text{or} \quad \mathbf{A}^3 - \mathbf{A}^2 = \mathbf{A} - \mathbf{I}.$$

Pre-multiplying both sides successively by **A**, we obtain

$$\mathbf{A}^3 - \mathbf{A}^2 = \mathbf{A} - \mathbf{I}$$
$$\mathbf{A}^4 - \mathbf{A}^3 = \mathbf{A}^2 - \mathbf{A}$$
$$\ldots$$
$$\mathbf{A}^{n-1} - \mathbf{A}^{n-2} = \mathbf{A}^{n-3} - \mathbf{A}^{n-4}$$
$$\mathbf{A}^n - \mathbf{A}^{n-1} = \mathbf{A}^{n-2} - \mathbf{A}^{n-3}.$$

Adding these equations, we get

$$\mathbf{A}^n - \mathbf{A}^2 = \mathbf{A}^{n-2} - \mathbf{I}, \quad \text{or} \quad \mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}, \quad n \geq 3.$$

Using this equation recursively, we get

$$\mathbf{A}^n = (\mathbf{A}^{n-4} + \mathbf{A}^2 - \mathbf{I}) + \mathbf{A}^2 - \mathbf{I} = \mathbf{A}^{n-4} + 2(\mathbf{A}^2 - \mathbf{I})$$
$$= (\mathbf{A}^{n-6} + \mathbf{A}^2 - \mathbf{I}) + 2(\mathbf{A}^2 - \mathbf{I}) = \mathbf{A}^{n-6} + 3(\mathbf{A}^2 - \mathbf{I})$$
$$\ldots$$
$$= \mathbf{A}^{n-(n-2)} + \frac{1}{2}(n-2)(\mathbf{A}^2 - \mathbf{I}) = \frac{n}{2}\mathbf{A}^2 - \frac{1}{2}(n-2)\mathbf{I}.$$

Substituting $n = 50$, we get

$$\mathbf{A}^{50} = 25\mathbf{A}^2 - 24\mathbf{I} = 25\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} - 24\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 25 & 1 & 0\\ 25 & 0 & 1\end{bmatrix}.$$

Example 3. 35 Find the eigenvalues and the corresponding eigenvectors of the following matrices.

(i) $\mathbf{A} = \begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}$, (ii) $\mathbf{A} = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$, (iii) $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$.

Solution

(i) The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix}1-\lambda & 4\\ 3 & 2-\lambda\end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 3\lambda - 10 = 0, \quad \text{or} \quad \lambda = -2, 5.$$

Corresponding to the eigenvalue $\lambda = -2$, we have

$$(\mathbf{A} + 2\mathbf{I})\,\mathbf{x} = \begin{pmatrix}3 & 4\\ 3 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \text{ or } 3x_1 + 4x_2 = 0 \text{ or } x_1 = -\frac{4}{3}x_2.$$

Hence, the eigenvector $\mathbf{x}$ is given by

$$\mathbf{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-4x_2/3\\ x_2\end{bmatrix} = x_2\begin{bmatrix}-4/3\\ 1\end{bmatrix}.$$

Since an eigenvector is unique upto a constant multiple, we can take the eigenvector as $[-4, 3]^T$.

Corresponding to the eigenvalue $\lambda = 5$, we have

$$(\mathbf{A} - 5\mathbf{I})\,\mathbf{x} = \begin{pmatrix}-4 & 4\\ 3 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \text{ or } x_1 - x_2 = 0, \text{ or } x_1 = x_2.$$

Therefore, the eigenvalue is given by $\mathbf{x} = (x_1, x_2)^T = x_1(1, 1)^T$ or $(1, 1)^T$.

(ii) The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix}1-\lambda & 1\\ -1 & 1-\lambda\end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 - 2\lambda + 2 = 0, \quad \text{or} \quad \lambda = 1 \pm i.$$

Corresponding to the eigenvalue $\lambda = 1 + i$, we have

$$[\mathbf{A} - (1 + i)\,\mathbf{I}]\,\mathbf{x} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$-ix_1 + x_2 = 0 \text{ and } -x_1 - ix_2 = 0.$$

Both the equations reduce to $-x_1 - ix_2 = 0$. Choosing $x_2 = 1$, we get $x_1 = -i$. Therefore, the eigenvector is $\mathbf{x} = [-i, 1]^T$.

Corresponding to the eigenvalue $\lambda = 1 - i$, we have

$$[\mathbf{A} - (1 - i)\mathbf{I}]\,\mathbf{x} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$ix_1 + x_2 = 0 \text{ and } -x_1 + ix_2 = 0.$$

Both the equations reduce to $-x_1 + ix_2 = 0$. Choosing $x_2 = 1$, we get $x_1 = i$. Therefore, the eigenvector is $x = [i, 1]^T$.

Remark 20

For a real matrix **A**, the eigenvalues and the corresponding eigenvectors can be complex.

(iii) The characteristic equation of **A** is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad (1 - \lambda)(2 - \lambda)(3 - \lambda) = 0 \quad \text{or} \quad \lambda = 1, 2, 3.$$

Corresponding to the eigenvalue $\lambda = 1$, we have

$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} x_2 + x_3 = 0 \\ x_1 + x_3 = 0 \end{cases}$$

We obtain two equations in three unknowns. One of the variables x_1, x_2, x_3 can be chosen arbitrarily. Taking $x_3 = 1$, we obtain the eigenvector as $[-1, -1, 1]^T$.

Corresponding to the eigenvalue $\lambda = 2$, we have

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $x_1 = 0$, $x_3 = 0$ and x_2 arbitrary. Taking $x_2 = 1$, we obtain the eigenvector as $[0, 1, 0]^T$.

Corresponding to the eigenvalue $\lambda = 3$, we have

$$(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} x_1 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

Choosing $x_3 = 1$, we obtain the eigenvector as $[\,0, -1, 1]^T$.

Example 3.36 Find the eigenvalues and the corresponding eigenvectors of the following matrices.

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{(ii) } \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{(iii) } \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution In each of the above problems, we obtain the characteristic equation as $(1 - \lambda)^3 = 0$. Therefore, the eigenvalues are $\lambda = 1, 1, 1$, a repeated value. Since a 3×3 matrix has 3 eigenvalues, it is important to know, whether the given matrix has 3 linearly independent eigenvectors, or it has lesser number of linearly independent eigenvectors.

Corresponding to the eigenvalue $\lambda = 1$, we obtain the following eigenvectors.

(i) $$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathrm{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_1 \text{ arbitrary.} \end{cases}$$

Choosing $x_1 = 1$, we obtain the solution as $[1, 0, 0]^T$.

Hence, **A** has only one independent eigenvector.

(ii) $$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{cases} x_2 = 0 \\ x_1, x_3 \text{ arbitrary.} \end{cases}$$

Doubt

Taking $x_1 = 0, x_3 = 1$ and $x_1 = 1, x_3 = 0$, we obtain two linearly independent solutions

$$\mathbf{x}_1 = [0, 0, 1]^T, \quad \mathbf{x}_2 = [1, 0, 0]^T.$$

In this case **A** has two linearly independent eigenvectors.

(iii) $$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Doubt

This system is satisfied for arbitrary values of all the three variables. Hence, we obtain three linearly independent eigenvectors, which can be taken as

$$\mathbf{x}_1 = [1, 0, 0]^T, \quad \mathbf{x}_2 = [0, 1, 0]^T, \quad \mathbf{x}_3 = [0, 0, 1]^T.$$

We now state some important results regarding the relationship between the eigenvalues of a matrix and the corresponding linearly independent eigenvectors.

1. Eigenvectors corresponding to distinct eigenvalues are linearly independent.
2. If λ is an eigenvalue of multiplicity m of a square matix $\mathbf{A}$ of order n, then the number of linearly independent eigenvectors associated with λ is given by

$$p = n - r, \quad \text{where} \quad r = \text{rank } (\mathbf{A} - \lambda\mathbf{I}), \ 1 \le p \le m.$$

Remark 21

In Example 3.35, all the eigenvalues are distinct and therefore, the corresponding eigenvectors are linearly indenpendent. In Example 3.36, the eigenvalue $\lambda = 1$ is of multiplicity 3. We find that in

(i) Example 3.36(i), the rank of the matrix $\mathbf{A} - \mathbf{I}$ is 2 and we obtain one linearly independent eigenvector.

(ii) Example 3.36(ii), the rank of the matrix $\mathbf{A} - \mathbf{I}$ is 1 and we obtain two linearly independent eigenvectors.

(iii) Example 3.36(iii), the rank of the matrix $\mathbf{A} - \mathbf{I}$ is 0 and we obtain three linearly independent eigenvectors.

3.5.2 Similar and Diagonalizable Matrices

Similar matrices

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same order. The matrix $\mathbf{A}$ is said to be similar to the matrix $\mathbf{B}$ if there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P} \quad \text{or} \quad \mathbf{PA} = \mathbf{BP}. \tag{3.43}$$

Post-multiplying both sides in Eq. (3.43) by $\mathbf{P}^{-1}$, we get

$$\mathbf{PAP}^{-1} = \mathbf{B}.$$

Therefore, $\mathbf{A}$ is similar to $\mathbf{B}$ if and only if $\mathbf{B}$ is similar to $\mathbf{A}$. The matrix $\mathbf{P}$ is called the *similarity matrix*. The transformation in Eq. (3.43) is called a *similarity transformation*. We now prove a result regarding eigenvalues of similar matrices.

Theorem 3.10 Similar matrices have the same characteristic equation (and hence the same eigenvalues). Further, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then $\mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue λ, where $\mathbf{P}$ is the similarity matrix.

Proof Let λ be an eigenvalue and $\mathbf{x}$ be the corresponding eigenvector of $\mathbf{A}$. That is

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

Pre-multiplying both sides by an invertible matrix $\mathbf{P}^{-1}$, we obtain

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{x} = \lambda\mathbf{P}^{-1}\mathbf{x}.$$

Set $\mathbf{x} = \mathbf{P}\mathbf{y}$. We get

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{y} = \lambda\mathbf{P}^{-1}\mathbf{P}\mathbf{y}, \quad \text{or} \quad (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\mathbf{y} = \lambda\mathbf{y} \quad \text{or} \quad \mathbf{B}\mathbf{y} = \lambda\mathbf{y}.$$

where $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. Therefore, $\mathbf{B}$ has the same eigenvalues as $\mathbf{A}$, that is, the characterstic equation of $\mathbf{B}$ is same as the characteristic equation of $\mathbf{A}$. Now, $\mathbf{A}$ and $\mathbf{B}$ are similar matrices. Therefore, similar matrices have the same characteristic equation (and hence the same eigenvalues). Also, $\mathbf{x} = \mathbf{P}\mathbf{y}$, that is eigenvectors of $\mathbf{A}$ and $\mathbf{B}$ are related by $\mathbf{x} = \mathbf{P}\mathbf{y}$ or $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$.

Remark 22

(a) Theorem 3.10 states that if two matrices are similar, then they have the same characterstic equation and hence the same eigenvalues. However, the converse of this theorem is not true. Two matrices which have the same characteristic equation need not always be similar.

(b) If $\mathbf{A}$ is similar to $\mathbf{B}$ and $\mathbf{B}$ is similar to $\mathbf{C}$, then $\mathbf{A}$ is similar to $\mathbf{C}$.

Let there be two invertible matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P} \quad \text{and} \quad \mathbf{B} = \mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}.$$

Then $$\mathbf{A} = \mathbf{P}^{-1}\mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}\mathbf{P} = \mathbf{R}^{-1}\mathbf{C}\mathbf{R}, \quad \text{where} \quad \mathbf{R} = \mathbf{Q}\mathbf{P}.$$

Example 3.37 Examine whether $\mathbf{A}$ is similar to $\mathbf{B}$, where

Imp

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}, \quad \text{(ii) } \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Solution The given matrices are similar if there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P} \quad \text{or} \quad \mathbf{P}\mathbf{A} = \mathbf{B}\mathbf{P}.$$

Let $\mathbf{P} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We shall determine a, b, c and d such that $\mathbf{PA} = \mathbf{BP}$ and then check whether $\mathbf{P}$ is non-singular.

(i) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 5a-2b & 5a \\ 5c-2d & 5c \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -3a+4c & -3b+4d \end{bmatrix}.$$

Equating the corresponding elements, we obtain the system of equations

$$\begin{aligned} 5a - 2b &= a + 2c, &\quad \text{or} \quad 4a - 2b - 2c &= 0 \\ 5a &= b + 2d, &\quad \text{or} \quad 5a - b - 2d &= 0 \\ 5c - 2d &= -3a + 4c, &\quad \text{or} \quad 3a + c - 2d &= 0 \\ 5c &= -3b + 4d, &\quad \text{or} \quad 3b + 5c - 4d &= 0. \end{aligned}$$

A solution to this system of equations is $a = 1$, $b = 1$, $c = 1$, $d = 2$.

Therefore, we get $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, which is a non-singular matrix. Hence, the matrices $\mathbf{A}$ and $\mathbf{B}$ are similar.

(ii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+d \\ c & d \end{bmatrix}$.

Equating the corresponding elements, we get

$$a = a + c, \quad b = b + d \quad \text{or} \quad c = d = 0.$$

Therefore, we get $\mathbf{P} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, which is a singular matrix.

Since an invertible matrix **P** does not exist, the matrices **A** and **B** are not similar.

It can be verified that the eigenvalues of **A** are 1, 1 whereas the eigenvalues of **B** are 0, 2.

In practice, it is usually difficult to obtain a non-singular matrix **P** which satisfies the equation $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$ for any two matrices **A** and **B**. however, it is possible to obtain the matrix **P** when **A** or **B** is a diagonal matrix. Thus, our interest is to find a similarity matrix **P** such that for a given matrix **A**, we have

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} \quad \text{or} \quad \mathbf{PDP}^{-1} = \mathbf{A}$$

Where **D** is a diagonal matrix. If such a matrix exists, then we say that the matrix **A** is *diagonalizable*.

Diagonalizable matrices

A matrix **A** is diagonalizable, if it is similar to a diagonal matrix, that is there exists an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where **D** is a diagonal matrix. Since, similar matrices have the same eigenvalues, the diagonal elements of **D** are the eigenvalues of **A**. A necessary and sufficient condition for the existence of **P** is given in the following theorem.

Theorem 3. 11 a square matrix **A** of order n is diagonalizable if and only if it has n linearly independent eigenvectors.

Proof We shall prove only the if part, that is the case that if **A** has n linearly independent eigenvectors, then **A** is diagonalizable. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct) of the matrix **A** in the same order, that is the eigenvector $\mathbf{x}_j$ corresponds to the eigenvalues λ_j, $j = 1, 2, \ldots, n$. let

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] \quad \text{and} \quad \mathbf{D} = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

be the diagonal matrix with eigenvalues of **A** as its diagonal elements. The matrix **P** is called the *modal matrix* of **A** and **D** is called the *spectral matrix* of **A**. We have

$$\mathbf{AP} = \mathbf{A}\,[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = (\mathbf{Ax}_1, \mathbf{Ax}_2, \ldots, \mathbf{Ax}_n) \tag{3.44}$$
$$= (\lambda_1\mathbf{x}_1, \lambda_2\mathbf{x}_2, \ldots, \lambda_n\mathbf{x}_n) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)\,\mathbf{D} = \mathbf{PD}.$$

Since the columns of **P** are linearly independent, the rank of **P** in n and therefore the matrix **P** is invertible. Pre-multiplying both sides in Eq. (3.44) by $\mathbf{P}^{-1}$, we obtain

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^{-1}\mathbf{PD} = \mathbf{D} \tag{3.45}$$

which implies that **A** is similar to **D**. Therefore, the matrix of eigenvectors **P** reduces a matrix **A** to its diagonal form.

Post-multiplying both sides in Eq. (3.44) by $\mathbf{P}^{-1}$, we obtain

$$\mathbf{A} = \mathbf{PDP}^{-1}. \tag{3.46}$$

Remark 23

(a) A square matrix **A** of order n has always n linearly independent eigenvectors when its eigenvalues are distinct. The matrix may also have n linearly independent eigenvectors even when some eigenvalues are repeated (see Example 3.36 (iii)). Therefore, there is no restriction imposed on the eigenvalues of the matrix **A** in Theorem 3.11.

(b) From Eq. (3.46), we obtain

$$\mathbf{A}^2 = \mathbf{AA} = (\mathbf{PDP}^{-1})\,(\mathbf{PDP}^{-1}) = \mathbf{PD}^2\mathbf{P}^{-1}.$$

Repeating the pre-multiplication (post-multiplication) m times, we get

$$\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1} \text{ for any positive integer } m.$$

Therefore, if **A** is diagonalizable, so is $\mathbf{A}^m$.

(c) If **D** is a diagonal matrix of order n, and

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \lambda_2 & \\ \mathbf{0} & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \text{ then } \mathbf{D}^m = \begin{bmatrix} \lambda_1^m & & & \\ & \lambda_2^m & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{bmatrix}.$$

for any positive integer m. If $Q(\mathbf{D})$ is a polynomial in **D**, then we get

$$Q(\mathbf{D}) = \begin{bmatrix} Q(\lambda_1) & & \mathbf{0} \\ & Q(\lambda_2) & \\ & & \ddots & \\ \mathbf{0} & & & Q(\lambda_n) \end{bmatrix}.$$

Now, let a matrix **A** be diagonalizable. Then, we have

$$\mathbf{A} = \mathbf{PDP}^{-1} \quad \text{and} \quad \mathbf{A}^m = \mathbf{PD}^m\,\mathbf{P}^{-1}$$

for any positive integer m. Hence, we obtain

$$Q(\mathbf{A}) = \mathbf{P}Q(\mathbf{D})\mathbf{P}^{-1}$$

for any matrix polynomial $Q(\mathbf{A})$.

Example 3.38 Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

is diagonalizable. Hence, find **P** such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. Then, obtain the matrix $\mathbf{B} = \mathbf{A}^2 + 5\mathbf{A} + 3\mathbf{I}$.

Solution The characteristic equation of **A** is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0, \quad \text{or} \quad \lambda = 1, 2, 3.$$

Since the matrix **A** has three distinct eigenvalues, it has three linearly independent eigenvectors and hence it is diagonalizable.

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is the solution of the system

$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ The solution is } \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is the solution of the system

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ The solution is } \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is the solution of the system

$$(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ The solution is } \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Hence, the modal matrix is given by

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{P}^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

It can be verified that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \text{diag}\,(1, 2, 3)$.

We have $\mathbf{D} = \text{diag}\,(1, 2, 3)$, $\mathbf{D}^2 = \text{diag}\,(1, 4, 9)$.

nice Therefore, $\mathbf{A}^2 + 5\mathbf{A} + 3\mathbf{I} = \mathbf{P}(\mathbf{D}^2 + 5\mathbf{D} + 3\mathbf{I})\mathbf{P}^{-1}$.

Now,

$$\mathbf{D}^2 + 5\mathbf{D} + 3\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix}.$$

Hence, we obtain

$$\mathbf{A}^2 + 5\mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 25 & 8 & -8 \\ -18 & 9 & 18 \\ -2 & 8 & 19 \end{bmatrix}.$$

Example 3.39 Examine whether the matrix **A**, where **A** is given by

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}, \qquad \text{(ii) } \mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix},$$

is diagonalizable. If so, obtain the matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

Solution

(i) The characteristic equation of the matrix **A** is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(2-\lambda)(2-\lambda) - 2] - [2 - 2(2-\lambda)] = (1-\lambda)(2-\lambda)(2-\lambda) = 0,$$

or $\lambda = 1, 2, 2$. We first find the eigenvectors corresponding to the repeated eigenvalue $\lambda = 2$. We have the system

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the rank of the coefficient matrix is 2, it has one linearly independent eigenvector. We obtain another linearly independent eigenvector corresponding to the eigenvalue $\lambda = 1$. Since the matrix **A** has only two linearly independedent eigenvectors, the matrix is not diagonalizable.

(ii) The characteristic equation of the matrix **A** is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0 \text{ or } \lambda^3 + \lambda^2 - 21\lambda - 45 = 0, \text{ or } \lambda = 5, -3, -3.$$

Eigenvector corresponding to the eigenvalue $\lambda = 5$ is the solution of the system

$$(\mathbf{A} - 5\mathbf{I})\mathbf{x} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

A solution of this system is $[1, 2, -1]^T$.

Eigenvectors corresponding to $\lambda = -3$ are the solutions of the system

$$(\mathbf{A} + 3\mathbf{I})\mathbf{x} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } x_1 + 2x_2 - 3x_3 = 0.$$

The rank of the coefficient matrix is 1. Therefore, the system has two linearly independent eigenvectors. We use the equation $x_1 + 2x_2 - 3x_3 = 0$ to find two linearly independent eigenvectors. Taking $x_3 = 0$, $x_2 = 1$, we obtain the eigenvector $[-2, 1, 0]^T$ and taking $x_2 = 0$, $x_3 = 1$, we obtain the eigenvector $[3, 0, 1]^T$. The given 3×3 matrix has three linearly independent eigenvectors. Therefore, the matrix **A** is diagonalizable. The modal matrix **P** is given by

$$\mathbf{P} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{P}^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 2 & -3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix}.$$

It can be verified that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \text{diag}(5, -3, -3)$.

Example 3.40 The eigenvectors of a 3×3 matrix **A** corresponding to the eigenvalues 1, 1, 3 are $[1, 0, -1]^T$, $[0, 1, -1]^T$ and $[1, 1, 0]^T$ respectively. Find the matrix **A**.

Solution We have

$$\text{modal matrix } \mathbf{P} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \text{ and the spectral matrix } \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

We find that
$$\mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Therefore,
$$\mathbf{A} = \mathbf{P}\,\mathbf{D}\,\mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.5.3 Special Matrices

In this section, we define some special matrices and study the properties of the eigenvalues and eigenvectors of these matrices. These matrices have applications in many areas. We first give some definitions.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ be two vectors of dimension n in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then we define the following:

Inner Product (dot product) of vectors Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^n$. Then

$$\mathbf{x} \cdot \mathbf{y} = x^T y = \sum_{i=1}^{n} x_i y_i. \tag{3.47}$$

is called the *inner product* of the vectors $\mathbf{x}$ and $\mathbf{y}$ and is a scalar. The inner product is also denoted by $<\mathbf{x}, \mathbf{y}>$. In this case $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$. Note that $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{C}^n$, then the inner product of these vectors is defined as

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \bar{\mathbf{y}} = \sum_{i=1}^{n} x_i \bar{y}_i \quad \text{and} \quad \mathbf{y} \cdot \mathbf{x} = \mathbf{y}^T \bar{\mathbf{x}} = \sum_{i=1}^{n} y_i \bar{x}_i$$

where $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are complex conjugate vectors of $\mathbf{x}$ and $\mathbf{y}$ respectively. Note that $\mathbf{x} \cdot \mathbf{y} = \overline{\mathbf{y} \cdot \mathbf{x}}$.

It can be easily verified that

$$(\alpha \mathbf{x} + \beta \mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$$

for any vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and scalars α, β.

Length (norm of a vector) Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

is called the *length* or the *norm* of the vector $\mathbf{x}$.

Unit vector The vector $\mathbf{x}$ is called a *unit vector* if $\|\mathbf{x}\| = 1$. If $\mathbf{x} \neq \mathbf{0}$, then vector $\mathbf{x}/\|\mathbf{x}\|$ is always a unit vector.

Orthogonal vectors The vectors $\mathbf{x}$ and $\mathbf{y}$ for which $\mathbf{x} \cdot \mathbf{y} = 0$ are said to be *orthogonal vectors*.

Orthonormal vectors The vectors $\mathbf{x}$ and $\mathbf{y}$ for which

$$\mathbf{x} \cdot \mathbf{y} = 0 \quad \text{and} \quad \|\mathbf{x}\| = 1, \|\mathbf{y}\| = 1$$

are called orthonormal vectors. If $\mathbf{x}$, $\mathbf{y}$ are any vectors and $\mathbf{x} \cdot \mathbf{y} = 0$, then $\mathbf{x}/\|\mathbf{x}\|$, $\mathbf{y}/\|\mathbf{y}\|$ are orthonormal. For example, the set of vectors

(i) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form an orthonormal set in $\mathbb{R}^3$.

(ii) $\begin{pmatrix} 3i \\ 4i \\ 0 \end{pmatrix}, \begin{pmatrix} -4i \\ 3i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1+i \end{pmatrix}$ form an orthogonal set in $\mathbb{C}^3$ and $\begin{pmatrix} 3i/5 \\ 4i/5 \\ 0 \end{pmatrix}, \begin{pmatrix} -4i/5 \\ 3i/5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ (1+i)/\sqrt{2} \end{pmatrix}$ form an orthonormal set in $\mathbb{C}^3$.

Orthonormal and unitary system of vectors Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n vectors in IR^n. Then, this set of vectors forms an *orthonormal system* of vectors, if

$$\mathbf{x}_i \cdot \mathbf{x}_j = \mathbf{x}_i^T \mathbf{x}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n vectors in $\mathbb{C}^n$. Then, this set of vectors forms an *unitary system* of vectors, if

$$\mathbf{x}_i \cdot \mathbf{x}_j = \mathbf{x}_i^T \overline{\mathbf{x}}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

In section 3.2.2, we have defined symmetric, skew-symmetric, Hermitian and skew-Hermitian matrices. We now define a few more special matrices.

Orthogonal matrices A real matrix $\mathbf{A}$ is *orthogonal* if $\mathbf{A}^{-1} = \mathbf{A}^T$. A simple example is

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

A linear transformation in which the matrix of transformation is an orthogonal matrix is called an *orthogonal transformation*.

Unitary matrices A complex matrix $\mathbf{A}$ is *unitary* if $\mathbf{A}^{-1} = (\overline{\mathbf{A}})^T$, or $(\overline{\mathbf{A}})^{-1} = \mathbf{A}^T$. If $\mathbf{A}$ is real, then unitary matrix is same as orthogonal matrix. $AA^\theta = I$

A linear transformation in which the matrix of transformation is a unitary matrix is called a *unitary tranformation*.

We note the following:

1. If $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices, then $\alpha\mathbf{A} + \beta\mathbf{B}$ is also Hermitian for any real scalars α, β, since
$$(\overline{\alpha\mathbf{A} + \beta\mathbf{B}})^T = (\alpha\overline{\mathbf{A}} + \beta\overline{\mathbf{B}})^T = \alpha\overline{\mathbf{A}}^T + \beta\overline{\mathbf{B}}^T = \alpha\mathbf{A} + \beta\mathbf{B}.$$
2. Eigenvalues and eigenvectors of $\overline{\mathbf{A}}$ are the conjugates of the eigenvalues and eigenvectors of $\mathbf{A}$, since
$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \text{ gives } \overline{\mathbf{A}}\,\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}.$$
3. The inverse of a unitary (orthogonal) matrix is unitary (orthogonal). We have $\mathbf{A}^{-1} = \overline{\mathbf{A}}^T$. Let $\mathbf{B} = \mathbf{A}^{-1}$. Then
$$\mathbf{B}^{-1} = \mathbf{A} = (\overline{\mathbf{A}}^T)^{-1} = [(\overline{\mathbf{A}})^{-1}]^T = [\overline{(\mathbf{A}^{-1})}]^T = \overline{\mathbf{B}}^T.$$

Diagonally dominant matrix A matrix $\mathbf{A} = (a_{ij})$ is said to be diagonally dominant, if

$$|a_{ii}| \geq \sum_{j=1, i \neq j}^{n} |a_{ij}|, \text{ for all } i.$$

The system of equations $\mathbf{Ax} = \mathbf{b}$, is called a *diagonally dominant system*, if the above conditions are satisfied and the strict inequality is satisfied for at least one i. If the strict inequality is satisfied for all i, then it is called a *strictly diagonally dominant system*.

Permutation matrix A matrix $\mathbf{P}$ is called a *permutation matrix* if it has exactly one 1 in each row and column and all other elements are 0.

***Property A* of a matrix** Let $\mathbf{B}$ be a sparse matrix. Then, the matrix $\mathbf{B}$ is said to satisfy the *property A*, if and only if there exists a permutation matrix $\mathbf{P}$ such that

$$\mathbf{PBP}^T = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix},$$

where $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are diagonal matrices. The similarity transformation performs row interchanges followed by corresponding column interchanges in $\mathbf{B}$ such that $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ become diagonal matrices. The following procedure is a simple way of testing whether $\mathbf{B}$ can be reduced to the required form. It finds the locations of the non-zero elements and tests whether the interchanges of rows and corresponding interchanges of columns are possible to bring $\mathbf{B}$ to the required form. Let n be the order of the matrix $\mathbf{B}$ and $b_{ii} \neq 0$. Denote the set $U = \{1, 2, 3, \ldots, n\}$. Let there exist disjoint subsets U_1 and U_2 such that $U = U_1 \cup U_2$, where the suffixes of the non-zero off diagonal elements $b_{ik} \neq 0$, $i \neq k$, can be grouped as either $(i \in U_1, k \in U_2)$ or $(i \in U_2, k \in U_1)$. Then, the matrix $\mathbf{B}$ satisfies *property A*.

Consider, for example the matrix $\mathbf{B} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$.

Let the permutation matrix be taken as $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Then, $$\mathbf{PBP}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \begin{matrix} -2 & 0 \\ 0 & -2 \end{matrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \end{bmatrix} & (-2) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

where $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are diagonal matrices. Hence, **B** has *property A*. Note that the above similarity transformation is equivalent to interchanging rows 2 and 3, followed by an interchange of columns 2 and 3.

Now, $a_{ii} \neq 0$, $i = 1, 2, 3$. $a_{12} \neq 0$, $1 \in U_1$, $2 \in U_2$; $a_{21} \neq 0$, $2 \in U_2$, $1 \in U_1$; $a_{23} \neq 0$, $2 \in U_2$, $3 \in U_1$; $a_{32} \neq 0$, $3 \in U_1$, $2 \in U_2$. Subsets $U_1 = \{1, 3\}$, $U_2 = \{2\}$ exist such that $U = \{1, 2, 3\} = U_1 \cup U_2$. Hence, matrix **B** has *property A*.

We now establish some important results.

Therorem 3.12 An orthogonal set of vectors is linearly independent.

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be an orthogonal set of vectors, that is $\mathbf{x}_i \cdot \mathbf{x}_j = 0$, $i \neq j$. Consider the vector equation

$$\mathbf{x} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \ldots + \alpha_m \mathbf{x}_m = \mathbf{0} \tag{3.48}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are scalars. Taking the inner product of the vector $\mathbf{x}$ in Eq. (3.48) with $\mathbf{x}_1$, we get

$$\mathbf{x} \cdot \mathbf{x}_1 = (\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \ldots + \alpha_m \mathbf{x}_m) \cdot \mathbf{x}_1 = \mathbf{0} \cdot \mathbf{x}_1 = 0$$

or

$$\alpha_1(\mathbf{x}_1 \cdot \mathbf{x}_1) = 0 \quad \text{or} \quad \alpha_1 \, \|\mathbf{x}_1\|^2 = 0.$$

Since $\|\mathbf{x}_1\|^2 \neq 0$, we get $\alpha_1 = 0$. Similarly, taking the inner products of $\mathbf{x}$ with $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_m$ successively, we find that $\alpha_2 = \alpha_3 = \ldots = \alpha_m = 0$. Therefore, the set of orthogonal vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ is linearly independent.

Theorem 3.13 The eigenvalues of

(i) an Hermitian matrix are real.

(ii) a skew-Hermitian matrix are zero or pure imaginary.

(iii) an unitary matrix are of magnitude 1.

Proof Let λ be an eigenvalue and $\mathbf{x}$ be the corresponding eigenvector of the matrix **A**. We have $\mathbf{Ax} = \lambda \mathbf{x}$. Pre-multiplying both sides by $\bar{\mathbf{x}}^T$, we get

$$\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x} = \lambda \bar{\mathbf{x}}^T \mathbf{x} \quad \text{or} \quad \lambda = \frac{\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x}}{\bar{\mathbf{x}}^T \mathbf{x}}. \tag{3.49}$$

Note that $\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x}$ and $\bar{\mathbf{x}}^T \mathbf{x}$ are scalars. Also, the denominator $\bar{\mathbf{x}}^T \mathbf{x}$ is always real and positive. Therefore, the behavior of λ is governed by the scalar $\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x}$.

(i) Let **A** be an Hermitian matrix, that is $\bar{\mathbf{A}} = \mathbf{A}^T$. Now,

$$\overline{(\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x})} = \mathbf{x}^T \bar{\mathbf{A}} \bar{\mathbf{x}} = \mathbf{x}^T \mathbf{A}^T \bar{\mathbf{x}} = (\mathbf{x}^T \mathbf{A}^T \bar{\mathbf{x}})^T = \bar{\mathbf{x}}^T \mathbf{A}\mathbf{x}$$

since $\mathbf{x}^T \mathbf{A}^T \bar{\mathbf{x}}$ is a scalar. Therefore, $\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x}$ is real. From Eq. (3.49), we conclude that λ is real.

(ii) Let **A** be a skew-Hermitian matrix, that is $\mathbf{A}^T = -\bar{\mathbf{A}}$. Now,

$$\overline{(\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x})} = \mathbf{x}^T \bar{\mathbf{A}} \bar{\mathbf{x}} = -\mathbf{x}^T \mathbf{A}^T \bar{\mathbf{x}} = -(\mathbf{x}^T \mathbf{A}^T \bar{\mathbf{x}})^T = -\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x}$$

since $\mathbf{x}^T\mathbf{A}^T\bar{\mathbf{x}}$ is a scalar. Therefore, $\bar{\mathbf{x}}^T\mathbf{A}\mathbf{x}$ is zero or pure imaginary. From Eq. (3.49), we conclude that λ is zero or pure imaginary.

(iii) Let **A** be an unitary matrix, that is $\mathbf{A}^{-1} = (\bar{\mathbf{A}})^T$. Now, from

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \quad \text{or} \quad \bar{\mathbf{A}}\bar{\mathbf{x}} = \bar{\lambda}\,\bar{\mathbf{x}} \tag{3.50}$$

we get

$$(\bar{\mathbf{A}}\,\bar{\mathbf{x}})^T = \left(\bar{\lambda}\,\bar{\mathbf{x}}^T\right)^T \quad \text{or} \quad \bar{\mathbf{x}}^T\,\bar{\mathbf{A}}^T = \bar{\lambda}\,\bar{\mathbf{x}}^T.$$

or

$$\bar{\mathbf{x}}^T\mathbf{A}^{-1} = \bar{\lambda}\,\bar{\mathbf{x}}^T. \tag{3.51}$$

Using Eqs. (3.50) and (3.51), we can write

$$(\bar{\mathbf{x}}^T\,\mathbf{A}^{-1})(\mathbf{A}\mathbf{x}) = (\bar{\lambda}\,\bar{\mathbf{x}}^T)(\lambda\mathbf{x}) = |\lambda|^2\,\bar{\mathbf{x}}^T\mathbf{x}$$

or

$$\bar{\mathbf{x}}^T\mathbf{x} = |\lambda|^2\,\bar{\mathbf{x}}^T\mathbf{x}.$$

Since $\mathbf{x} \neq \mathbf{0}$, we have $\bar{\mathbf{x}}^T\mathbf{x} \neq \mathbf{0}$. Therefore, $|\lambda|^2 = 1$, or $|\lambda| = 1$. Hence, the result.

Remark 24

From Theorem 3.13, we conclude that the eigenvalues of

(i) a symmetric matrix are real.

(ii) a skew-symmetric matrix are zero or pure imaginary.

(iii) an orthogonal matrix are of magnitude 1 and are real or complex conjugate pairs.

Theorem 3.14 The column vectors (and also row vectors) of an unitary matrix form an unitary system of vectors.

Proof Let **A** be an unitary matrix of order n, with column vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Then

$$\mathbf{A}^{-1}\mathbf{A} = \bar{\mathbf{A}}^T\mathbf{A} = \begin{bmatrix} \bar{\mathbf{x}}_1^T \\ \bar{\mathbf{x}}_2^T \\ \vdots \\ \bar{\mathbf{x}}_n^T \end{bmatrix} [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = \begin{bmatrix} \bar{\mathbf{x}}_1^T\mathbf{x}_1 & \bar{\mathbf{x}}_1^T\mathbf{x}_2 & \cdots & \bar{\mathbf{x}}_1^T\mathbf{x}_n \\ \bar{\mathbf{x}}_2^T\mathbf{x}_1 & \bar{\mathbf{x}}_2^T\mathbf{x}_2 & \cdots & \bar{\mathbf{x}}_2^T\mathbf{x}_n \\ \vdots & & & \\ \bar{\mathbf{x}}_n^T\mathbf{x}_1 & \bar{\mathbf{x}}_n^T\mathbf{x}_2 & \cdots & \bar{\mathbf{x}}_n^T\mathbf{x}_n \end{bmatrix} = \mathbf{I}$$

Therefore,

$$\bar{\mathbf{x}}_i^T\mathbf{x}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

Hence, the column vectors of **A** form an unitary system. Since the inverse of an unitary matrix is also an unitary matrix and the columns of $\mathbf{A}^{-1}$ are the conjugate of the rows of **A**, we conclude that the row vectors of **A** also form an unitary system.

Remark 25

(a) From Theorem 3.14, we conclude that the column vectors (and also the row vectors) of an orthogonal matrix form an orthonormal system of vectors.

(b) A symmetric matrix of order n has n linearly independent eigenvectors and hence is diagonlizable.

Example 3.41 Show that the matrices $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues and for distinct eigenvalues the eigenvectors corresponding to $\mathbf{A}$ and $\mathbf{A}^T$ are mutually orthogonal.

Solution We have

$$|\mathbf{A} - \lambda\mathbf{I}| = |(\mathbf{A}^T)^T - \lambda\mathbf{I}^T| = |[\mathbf{A}^T - \lambda\mathbf{I}]^T| = |\mathbf{A}^T - \lambda\mathbf{I}|.$$

Since $\mathbf{A}$ and $\mathbf{A}^T$ have the same characteristic equation, they have the same eigenvalues.

Let λ and μ be two distinct eigenvalues of $\mathbf{A}$. Let $\mathbf{x}$ be the eigenvector corresponding to the eigenvalue λ for $\mathbf{A}$ and $\mathbf{y}$ be the eigenvector corresponding to the eigenvalue μ for $\mathbf{A}^T$. We have $\mathbf{Ax} = \lambda\mathbf{x}$. Pre-multiplying by $\mathbf{y}^T$, we get

$$\mathbf{y}^T\mathbf{A}\mathbf{x} = \lambda\mathbf{y}^T\mathbf{x}. \tag{3.52}$$

We also have $\mathbf{A}^T\mathbf{y} = \mu\mathbf{y}$, or $(\mathbf{A}^T\mathbf{y})^T = (\mu\mathbf{y})^T$ or $\mathbf{y}^T\mathbf{A} = \mu\mathbf{y}^T$.
Post-multiplying by $\mathbf{x}$, we get

$$\mathbf{y}^T\mathbf{A}\mathbf{x} = \mu\mathbf{y}^T\mathbf{x} \tag{3.53}$$

Subtracting Eqs. (3.52) and (3.53), we obtain

$$(\lambda - \mu)\mathbf{y}^T\mathbf{x} = 0.$$

Since $\lambda \neq \mu$, we obtain $\mathbf{y}^T\mathbf{x} = 0$. Therefore, the vectors $\mathbf{x}$ and $\mathbf{y}$ are mutually orthogonal.

3.6 Quadratic Forms

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be an arbitrary vector in IR^n. A real *quadratic form* is an homogeneous expression of the form

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, x_i\, x_j \tag{3.54}$$

in which the total power in each term is 2. Expanding, we can write

$$\begin{aligned} Q &= a_{11}x^2 + (a_{12} + a_{21})\, x_1 x_2 + \ldots + (a_{1n} + a_{n1})\, x_1 x_n \\ &\quad + a_{22}\, x_2^2 + (a_{23} + a_{32})\, x_2 x_3 + \ldots + (a_{2n} + a_{n2})\, x_2 x_n \\ &\quad + \ldots + a_{nn}\, x_n^2 \\ &= \mathbf{x}^T\mathbf{A}\mathbf{x} \end{aligned} \tag{3.55}$$

using the definition of matrix multiplication. Now, set $b_{ij} = (a_{ij} + a_{ji})/2$. The matrix $\mathbf{B} = (b_{ij})$ is symmetric since $b_{ij} = b_{ji}$. Further, $b_{ij} + b_{ji} = a_{ij} + a_{ji}$. Hence, Eq. (3.55) can be wtitten as

$$Q = \mathbf{x}^T\mathbf{B}\mathbf{x}$$

where $\mathbf{B}$ is a symmetric matrix and $b_{ij} = (a_{ij} + a_{ji})/2$.

Chapter 3

Matrices and Eigenvalue Problems

3.1 Introduction

In modern mathematics, matrix theory occupies an important place and has applications in almost all branches of engineering and physical sciences. Matrices of order $m \times n$ form a vector space and they define linear transformations which map vector spaces consisting of vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ into another vector space consisting of vectors in $\mathbb{R}^m$ or $\mathbb{C}^m$ under a given set of rules of rules of vector addition and scalar multiplication. A matrix does not denote a number and no value can be assigned to it. The usual rules of arithmetic operations do not hold for matrices. The rules defining the operations on matrices are usually called its algebra. In this chapter we shall discuss the matrix algebra and its use in solving linear system of algebraic equation $\mathbf{Ax} = \mathbf{b}$ and solving the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$.

3.2 Matrices

An $m \times n$ matrix is an arrangement of mn objects (not necessarily distinct) in m rows and n columns in the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \tag{3.1}$$

We say that the matrix is of *order* $m \times n$ (m by n). The objects $a_{11}, a_{12}, \ldots, a_{mn}$ are called the *elements* of the matrix. Each element of the matrix can be a real or complex number or a function of one or more variables or any other object. The element a_{ij} which is common to the ith row and the jth column is called its *general element*. The matrices are usually denoted by boldface uppercase letters **A**, **B**, **C**, ... etc. When the order of the matrix is understood, we can simply write $\mathbf{A} = (a_{ij})$. If all the element of a matrix are real, it is called a *real matrix*, whereas if one or more elements of a matrix are complex it is called a *complex matrix*. We define the following type of matrices.

Row Vector A matrix of order $1 \times n$, that is, it has one row and n columns is called a *row vector* or a row matrix of order n and is written as

$$[a_{11}\ a_{12}\ \dots\ a_{1n}],\ \text{or}\ [a_1\ a_2\ \dots\ a_n]$$

in which a_{1j} (or a_j) is the jth element.

Column vector A matrix of order $m \times 1$, that is, it has m rows and one column is called a *column vector* or a *column matrix* of order m and is written as

$$\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix},\ \text{or}\ \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in which b_{j1} (or b_j) is the jth element.

The number of elements in a row/column vector is called its *order*. The vectors are usually denoted by boldface lower case letters **a**, **b**, **c**, ... etc. If a vector has n elements and all its elements are real numbers, then it is called an *ordered n-tuple* in IR^n, whereas if one or more elements are complex numbers, then it is called an ordered *n-tuple* in $\mathbb{C}^n$.

Rectangular matrix A matrix **A** of order $m \times n$, $m \neq n$ is called a *rectangular matrix*.

Square matrices A matrix **A** of order $m \times n$ in which $m = n$, that is number of rows is equal to the number of columns is called a square matrix of order n. The elements a_{ii}, that is the elements $a_{11}, a_{22}, \dots, a_{nn}$ are called the *diagonal elements* and the line on which these elements lie is called the *principal diagonal* or the *main diagonal* of the matrix. The elements a_{ij}, when $i \neq j$ are called the *off-diagonal elements*. The sum of the diagonal elements of a square matrix is called the *trace* of the matrix.

Null matrix A matrix **A** of order $m \times n$ in which all the elements are zero is called a *null matrix* or a *zero matrix* and is denoted by **0**.

Diagonal matrix A square matrix **A** in which all the off-diagonal elements a_{ij}, $i \neq j$ are zero is called a diagonal matrix. For example

$$\mathbf{A} = \begin{bmatrix} a_{11} & & & \mathbf{0} \\ & a_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & a_{nn} \end{bmatrix} \text{ is a } \textit{diagonal matrix} \text{ of order } n.$$

A diagonal matrix is denoted by D. It is also written as diag $[a_{11}\ a_{22}\ \dots\ a_{nn}]$.

If all the elements of a diagonal matrix of order n are equal, that is $a_{ii} = \alpha$ for all i, then the matrix is called a *scalar matrix* of order n.

If all the elements of a diagonal matrix of order n are 1, then the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & & & \mathbf{0} \\ & 1 & & \\ & & \ddots & \\ \mathbf{0} & & & 1 \end{bmatrix} \text{ is called an } \textit{unit matrix} \text{ or an } \textit{identity matrix} \text{ of order } n.$$

An identity matrix is denoted by I.

Equal matrices Two matrices $\mathbf{A} = (a_{ij})_{m \times n}$ and $\mathbf{B} = (b_{ij})_{p \times q}$ are said to be equal, when

(i) they are of the same order, that is $m = p$, $n = q$ and

(ii) their corresponding elements are equal, that is $a_{ij} = b_{ij}$ for all i, j.

Submatrix A matrix obtained by omitting some rows and or columns from a given matrix **A** is called a *submartix* of **A**. As a convention, the given matrix **A** is also taken as the submatrix of **A**.

3.2.1 Matrix Algebra

The basic operations allowed on matrices are

(i) multiplication of a matrix by a scalar,

(ii) addition/subtraction of two matrices,

(iii) multiplication of two matrices.

Note that there is no concept of dividing a matrix by a matrix. Therefore, the operation **A/B** where **A** and **B** are matrices is not defined.

Multiplication of a matrix by a scalar

Let α be a scalar (real or complex) and $\mathbf{A} = (a_{ij})$ be a given matrix of order $m \times n$. Then

$$\mathbf{B} = \alpha\mathbf{A} = \alpha(a_{ij}) = (\alpha a_{ij}) \quad \text{for all } i \text{ and } j. \tag{3.2}$$

The order of the new matrix **B** is same as that of the matrix **A**.

Addition/subtraction of two matrices

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be two matrices of the same order. Then

$$\mathbf{C} = (c_{ij}) = \mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}), \quad \text{for all } i \text{ and } j \tag{3.3a}$$

and

$$\mathbf{D} = (d_{ij}) = \mathbf{A} - \mathbf{B} = (a_{ij}) - (b_{ij}) = (a_{ij} - b_{ij}), \quad \text{for all } i \text{ and } j. \tag{3.3b}$$

The order of the new matrix **C** or **D** is the same as that of the matrices **A** and **B**. Matrices of the same order are said to be *conformable* for addition/subtraction.

If $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p$ are p matrices which are conformable for addition and $\alpha_1, \alpha_2, \dots, \alpha_p$ are any scalars, then

$$\mathbf{C} = \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \dots + \alpha_p\mathbf{A}_p \tag{3.4}$$

is called a linear combination of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p$. The order of the matrix **C** is same as that of $\mathbf{A}_i$, $i = 1, 2, \dots, p$.

Properties of the matrix addition and scalar multiplication

Let **A**, **B**, **C** be the matrices which are conformable for addition and α, β be scalars. Then

1. $A + B = B + A$. (commutative law)
2. $(A + B) + C = A + (B + C)$ (associative law)
3. $A + 0 = A$ (0 is the null matrix of the same order as A)
4. $A + (-A) = 0$.
5. $\alpha(A + B) = \alpha A + \alpha B$
6. $(\alpha + \beta)A = \alpha A + \beta A$.
7. $\alpha(\beta A) = \alpha\beta A$
8. $1 \times A = A$ and $0 \times A = 0$.

Multiplication of two matrices

The product **AB** of two matrices **A** and **B** is defined only when the number of columns in **A** is equal to the number of rows in **B**. Such matrices are said to be *conformable* for multiplication. Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be an $n \times p$ matrix. Then the product matrix

$$C = (c_{ij}) = AB = \underset{m \times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{n \times p}{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & & & & \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}}$$

is a matrix of order $m \times p$. The general element of the product matrix **C** is given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}. \tag{3.5}$$

In the product **AB**, **B** is said to be pre-multiplied by **A** or **A** is said to be post-multiplied by **B**. If **A** is a row matrix of order $1 \times n$ and **B** is a column matrix of order $n \times 1$, then **AB** is a matrix of order 1×1, that is a single element, and **BA** is a matrix of order $n \times n$.

Remark 1

(a) It is possible that for two given matrices **A** and **B**, the product matrix **AB** is defined but the product matrix **BA** may not be defined. For example, if **A** is a 2×3 matrix and **B** is a 3×4 matrix, then the product matrix **AB** is defined and is a matrix of order 2×4, whereas the product matrix **BA** is not defined.

(b) If both the product matrices **AB** and **BA** are defined, then both the matrices **AB** and **BA** are square matrices. In general $AB \neq BA$. Thus, the matrix product is not commutative. If $AB = BA$, then the matrices **A** and **B** are said to *commute* with each other.

(c) If $AB = 0$, then it does not always imply that either $A = 0$ or $B = 0$. For example, let

$$A = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$$

then
$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 0 & 0 \\ ax + by & 0 \end{bmatrix} \neq AB.$$

(d) If $AB = AC$, it does not always imply that $B = C$.

(e) Define $A^k = A \times A \ldots \times A$ (k times). Then, a matrix **A** such that $A^k = 0$ for some positive integer k is said to be *nilpotent*. The smallest value of k for which $A^k = 0$ is called the *index of nilpotency* of the matrix **A**.

(f) If $A^2 = A$, then **A** is called an *idempotent matrix*.

Properties of matrix multiplication

1. If **A**, **B**, **C** are matrices of order $m \times n$, $n \times p$ and $p \times q$ respectively, then

   $(AB)C = A(BC)$ (associative law)

   is a matrix of order $m \times q$.

2. If **A** is a matrix of order $m \times n$ and **B**, **C** are matrices of order $n \times p$, then

   $A(B + C) = AB + AC$ (left distributive law).

3. If **A**, **B** are matrices of order $m \times n$ and **C** is a matrix of order $n \times p$, then

   $(A + B)C = AC + BC$ (right distributive law).

4. If **A** is a matrix of order $m \times n$ and **B** is a matrix of order $n \times p$, then

   $\alpha(AB) = A(\alpha B) = (\alpha A)B$

   for any scalar α.

3.2.2 Some Special Matrices

We now define some special matrices.

Transpose of a matrix The matrix obtained by interchanging the corresponding rows and columns of a given matrix **A** is called the *transpose matrix* of **A** and is denoted by A^T or A', that is, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{then} \quad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}.$$

If **A** is an $m \times n$ matrix, then A^T is an $n \times m$ matrix. Also, both the product matrices A^TA and AA^T are defined, and

$$A^TA = (n \times m)(m \times n) \text{ is an } n \times n \text{ square matrix}$$

and $$AA^T = (m \times n)(n \times m) \text{ is an } m \times m \text{ square matrix}$$

A column vector **b** can also be written as $[b_1\ b_2\ \dots\ b_n]^T$.

The following results can be easily verified

1. The transpose of a row matrix is a column matrix and the transpose of a column matrix is a row matrix.
2. $(\mathbf{A}^T)^T = \mathbf{A}$.
3. $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$, when the matrices **A** and **B** are conformable for addition.
4. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, when the matrices **A** and **B** are conformable for multiplication.
   If the product $\mathbf{A}_1\ \mathbf{A}_2\ \dots\ \mathbf{A}_p$ is defined, then
   $$[\mathbf{A}_1\ \mathbf{A}_2\ \dots\ \mathbf{A}_p]^T = \mathbf{A}_p^T\ \mathbf{A}_{p-1}^T\ \dots\ \mathbf{A}_1^T.$$

Remark 2

The product of a row vector $\mathbf{a}_i = (a_{i1}\ a_{i2}\ \dots\ a_{in})$ of order $1 \times n$ and a column vector $\mathbf{b}_j = (b_{1j}\ b_{2j}\ \dots\ b_{nj})^T$ of order $n \times 1$ is called the *dot product* or the *inner product* of the vectors $\mathbf{a}_i$ and $\mathbf{b}_j$, that is

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

which is a scalar. In terms of the inner products, the product matrix **C** in Eq. (3.5) can be written as

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_p \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_2 \cdot \mathbf{b}_p \\ \dots & \dots & \dots & \\ \mathbf{a}_m \cdot \mathbf{b}_1 & \mathbf{a}_m \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_m \cdot \mathbf{b}_p \end{bmatrix}. \tag{3.6}$$

Symmetric and skew-symmetric matrices A real square matrix $\mathbf{A} = (a_{ij})$ is said to be

symmetric, if $a_{ij} = a_{ji}$ for all i and j, that is $\mathbf{A} = \mathbf{A}^T$

skew-symmetric, if $a_{ij} = -a_{ji}$ for all i and j, that is $\mathbf{A} = -\mathbf{A}^T$.

Remark 3

(a) In a skew-symmetric matrix $\mathbf{A} = (a_{ij})$, all its diagonal elements are zero.

(b) The matrix which is both symmetric and skew-symmetric must be a null matrix.

(c) For any real square matrix **A**, the matrix $\mathbf{A} + \mathbf{A}^T$ is always symmetric and the matrix $\mathbf{A} - \mathbf{A}^T$ is always skew-symmetric. Therefore, a real square matrix **A** can be written as the sum of a symmetric matrix and a skew-symmetric matrix. That is

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T).$$

Triangular matrices A square matrix $\mathbf{A} = (a_{ij})$ is called a *lower triangular matrix* if $a_{ij} = 0$, whenever $i < j$, that is all elements above the principal diagonal are zero and an *upper triangular matrix* if $a_{ij} = 0$, whenever $i > j$, that is all the elements below the principal diagonal are zero.

Conjugate matrix Let $\mathbf{A} = (a_{ij})$ be a complex matrix. Let $\bar{a}_{ij}$ denote the complex conjugate of a_{ij}. Then, the matrix $\overline{\mathbf{A}} = (\bar{a}_{ij})$ is called the *conjugate matrix* of *A*.

Hermitian and skew-Hermitian matrices A complex matrix **A** is called an *Hermitian matrix* if $\overline{\mathbf{A}} = \mathbf{A}^T$ or $\mathbf{A} = (\overline{\mathbf{A}})^T$ and a *skew-Hermitian* matrix if $\overline{\mathbf{A}} = -\mathbf{A}^T$ or $\mathbf{A} = -(\overline{\mathbf{A}})^T$. Sometimes, a Hermitian matrix is denoted by $\mathbf{A}^H$ or $\mathbf{A}^*$.

Remark 4

(a) If **A** is a real matrix, then an Hermitian matrix is same as a symmetric matrix and a skew-Hermitian matrix is same as a skew-symmetric matrix.

(b) In an Hermitian matrix, all the diagonal elements are real (let $a_{jj} = x_j + iy_j$; then $a_{jj} = \bar{a}_{jj}$ gives $x_j + iy_j = x_j - iy_j$ or $y_j = 0$ for all j).

(c) In a skew-Hermitian matrix, all the diagonal elements are either 0 or pure imaginary (let $a_{jj} = x_j + iy_j$; then $a_{jj} = -\bar{a}_{jj}$ gives $x_j + iy_j = -(x_j - iy_j)$ or $x_j = 0$ for all j).

(d) For any complex square matrix **A**, the matrix $\mathbf{A} + \overline{\mathbf{A}}^T$ is always an Hermitian matrix and the matrix $\mathbf{A} - \overline{\mathbf{A}}^T$ is always a skew-Hermitian matrix. Therefore, a complex square matrix **A** can be written as the sum of an Hermitian matrix and a skew-Hermitian matrix, that is

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \overline{\mathbf{A}}^T) + \frac{1}{2}(\mathbf{A} - \overline{\mathbf{A}}^T).$$

Example 3.1 Let **A** and **B** be two symmetric matrices of the same order. Show that the matrix **AB** is symmetric if and only if **AB** = **BA**, that is the matrices **A** and **B** commute.

Solution Since the matrices **A** and **B** are symmetric, we have

$$\mathbf{A}^T = \mathbf{A} \quad \text{and} \quad \mathbf{B}^T = \mathbf{B}.$$

Let **AB** be symmetric. Then

$$(\mathbf{AB})^T = \mathbf{AB}, \quad \text{or} \quad \mathbf{B}^T\mathbf{A}^T = \mathbf{AB}, \quad \text{or} \quad \mathbf{BA} = \mathbf{AB}.$$

Now, let **AB** = **BA**. Taking transpose on both sides, we get

$$(\mathbf{AB})^T = (\mathbf{BA})^T = \mathbf{A}^T\mathbf{B}^T = \mathbf{AB}.$$

Hence, the result.

3.2.3 Determinants

With every square matrix **A** of order n, we associate a determinant of order n which is denoted by det (**A**) or |**A**|. The determinant has a value and this value is real if the matrix **A** is real and may be real or complex, if the matrix is complex. A determinant of order n is defined as

$$det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} a_{ij} A_{ij}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} A_{ij} \tag{3.7}$$

where M_{ij} and A_{ij} are the *minors* and *cofactors* of a_{ij} respectively.

We give now some important properties of determinants.

1. If all the elements of a row (or column) are zero then the value of the determinant is zero.
2. $|\mathbf{A}| = |\mathbf{A}^T|$.
3. If any two rows (or columns) are interchanged, then the value of the determinant is multiplied by (–1).
4. If the corresponding elements of two rows (or columns) are proportional to each other, then the value of the determinant is zero.
5. If each element of a row (or column) is multiplied by a scalar α then the value of the determinant is multiplied by the scalar α. Therefore, if β is a factor of each element of a row (or column), then this factor β can be taken out of the determinant.

   Note that when we multiply a matrix by a scalar α, then every element of the matrix is multiplied by α. Therefore, $|\alpha\mathbf{A}| = \alpha^n |\mathbf{A}|$ where $\mathbf{A}$ is a matrix of order n.
6. If a non-zero constant multiple of the elements of some row (or column) is added to the corresponding elements of some other row (or column), then the value of the determinant remains unchanged.
7. $|\mathbf{A} + \mathbf{B}| \neq |\mathbf{A}| + |\mathbf{B}|$, in general.

Remark 5

When the elements of the jth row are multiplied by a non-zero constant k and added to the corresponding elements of the ith row, we denote this operation as $R_i \leftarrow R_i + kR_j$, where R_i is the ith row of $|\mathbf{A}|$. The elements of the jth row remain unchanged whereas the elements of the ith row get changed. This operation is called an *elementary row operation*. Similarly, the operation $C_i \leftarrow C_i + kC_j$, where C_i is the ith column of $|\mathbf{A}|$, is called the *elementary column operation*. Therefore, under elementary row (or column) operations, the value of a determinant is unchanged.

Product of two determinants

If $\mathbf{A}$ and $\mathbf{B}$ are two square matrices of the same order, then

$$|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|.$$

Since $|\mathbf{A}| = |\mathbf{A}^T|$, we can multiply two determinants in any one of the following ways

(i) row by row, (ii) column by column,

(iii) row by column, (iv) column by row.

The value of the determinant is same in each case.

Rank of a matrix

The rank of a matrix **A**, denoted by r or $r(\mathbf{A})$ is the order of the largest non-zero minor of $|\mathbf{A}|$. Therefore, the rank of a matrix is the largest value of r, for which there exists at least one $r \times r$ submartix of **A** whose determinant is not zero. Thus, for an $m \times n$ matrix $r \leq \min(m, n)$. For a square matrix **A** of order n, the rank $r = n$ if $|\mathbf{A}| \neq 0$, otherwise $r < n$. The rank of a null matrix is zero and if the rank of matrix is 0, then it must be a null matrix.

Example 3.2 Find all values of μ for which rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & \mu \end{bmatrix}$$

is equal to 3.

Solution Since the matrix **A** is of order 4, $r(\mathbf{A}) \leq 4$. Now, $r(\mathbf{A}) = 3$, if $|\mathbf{A}| = 0$ and there is at least one submatrix of order 3 whose determinant is not zero. Expanding the determinant through the elements of first row, we get

$$|\mathbf{A}| = \mu \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 11 & -6 & 1 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 \\ 0 & \mu & -1 \\ -6 & -6 & 1 \end{vmatrix} = \mu\left[\mu(\mu - 6) + 11\right] - 6$$

$$= \mu^3 - 6\mu^2 + 11\mu - 6 = (\mu - 1)(\mu - 2)(\mu - 3).$$

Setting $|\mathbf{A}| = 0$, we obtain $\mu = 1, 2, 3$. For $\mu = 1, 2, 3$, the determinant of the leading third order submatrix

$$|\mathbf{A}_1| = \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 0 & 0 & \mu \end{vmatrix} = \mu^3 \neq 0.$$

Hence, $r(\mathbf{A}) = 3$, when $\mu = 1$ or 2 or 3. For other values of μ, $r(\mathbf{A}) = 4$.

3.2.4 Inverse of a Square Matrix

Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n. Then, **A** is called a

(i) *singular matrix* if $|\mathbf{A}| = 0$,

(ii) *non-singular matrix* if $|\mathbf{A}| \neq 0$.

In other words, a square matrix of order n is singular if its rank $r(\mathbf{A}) < n$ and non-singular if its rank $r(\mathbf{A}) = n$. A square non-singular matrix **A** of order n is said to be *invertible*, if there exists a non-singular square matrix **B** of order n such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \tag{3.8}$$

where $\mathbf{I}$ is an identity matrix of order n. The matrix $\mathbf{B}$ is called the *inverse matrix* of $\mathbf{A}$ and we write $\mathbf{B} = \mathbf{A}^{-1}$ or $\mathbf{A} = \mathbf{B}^{-1}$. Hence, we say that $\mathbf{A}^{-1}$ is the inverse of the matrix $\mathbf{A}$, if

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}. \tag{3.9}$$

The inverse, $\mathbf{A}^{-1}$ of the matrix $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\, adj(\mathbf{A}) \tag{3.10}$$

where $adj(\mathbf{A})$ = adjoint matrix of $\mathbf{A}$
= transpose of the matrix of cofactors of $\mathbf{A}$.

Remark 6

(a) $$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

We have

$$(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{I}.$$

Pre-multiplying both sides first by $\mathbf{A}^{-1}$ and then by $\mathbf{B}^{-1}$ we obtain

$$\mathbf{B}^{-1}\mathbf{A}^{-1}(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\,\mathbf{B}(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \text{ or } (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

In general, we have $(\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_p)^{-1} = \mathbf{A}_p^{-1}\mathbf{A}_{p-1}^{-1} \ldots \mathbf{A}_1^{-1}$.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are non-singular matrices, then $\mathbf{AB}$ is also a non-singular matrix.

(c) If $\mathbf{AB} = \mathbf{0}$ and $\mathbf{A}$ is a non-singular matrix, then $\mathbf{B}$ must be null matrix, since $\mathbf{AB} = \mathbf{0}$ can be pre-multiplied by $\mathbf{A}^{-1}$. If $\mathbf{B}$ is non-singular matrix, then $\mathbf{A}$ must be a null matrix, since $\mathbf{AB} = \mathbf{0}$ can be post-multiplied by $\mathbf{B}^{-1}$.

(d) If $\mathbf{AB} = \mathbf{AC}$ and $\mathbf{A}$ is a non-singular matrix, then $\mathbf{B} = \mathbf{C}$ (see Remark 1(d)).

(e) $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$, in general.

Properties of inverse martices

1. If $\mathbf{A}^{-1}$ exists, then it is unique.
2. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
3. $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$. (From $(\mathbf{AA}^{-1})^T = \mathbf{I}^T = \mathbf{I}$, we get $(\mathbf{A}^{-1})^T\mathbf{A}^T = \mathbf{I}$. Hence, the result).
4. Let $\mathbf{D}$ = diag $(d_{11}, d_{22}, \ldots, d_{nn})$, $d_{ii} \neq 0$. Then, $\mathbf{D}^{-1}$ = diag $(1/d_{11}, 1/d_{22}, \ldots 1/d_{nn})$.
5. The inverse of a non-singular upper or lower triangular matrix is respectively an upper or a lower triangular matrix.
6. The inverse of a non-singular symmetric matrix is a symmetric matrix.
7. $(\mathbf{A}^{-1})^n = \mathbf{A}^{-n}$ for any positive integer n.

Example 3.3 Show that the matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I} = \mathbf{0}$ where $\mathbf{I}$ is an identity matrix of order 3. Hence, find the matrix (i) $\mathbf{A}^{-1}$ and (ii) $\mathbf{A}^{-2}$.

Solution We have

$$\mathbf{A}^2 = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}.$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix}.$$

Substituting in $\mathbf{B} = \mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I}$, we get

$$\mathbf{B} = \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix} - \begin{bmatrix} 24 & -6 & -30 \\ 90 & 6 & -30 \\ 30 & 24 & 54 \end{bmatrix} + \begin{bmatrix} 22 & 0 & -11 \\ 55 & 11 & 0 \\ 0 & 11 & 33 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$$

(i) Premultiplying $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I} = \mathbf{0}$ by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{A}^3 - 6\mathbf{A}^{-1}\mathbf{A}^2 + 11\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1} = 0$$

or $$\mathbf{A}^{-1} = \mathbf{A}^2 - 6\mathbf{A} + 11\,\mathbf{I}$$

$$= \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix} - \begin{bmatrix} 12 & 0 & -6 \\ 30 & 6 & 0 \\ 0 & 6 & 18 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}.$$

(ii) $$\mathbf{A}^{-2} = (\mathbf{A}^{-1})^2 = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -11 & 10 \\ -160 & 61 & -55 \\ 55 & -21 & 19 \end{bmatrix}.$$

We can also write

$$\mathbf{A}^{-2} = (\mathbf{A}^{-1})(\mathbf{A}^{-1}) = \mathbf{A} - 6\,\mathbf{I} + 11(\mathbf{A}^{-1}).$$

3.2.5 Solution of $n \times n$ Linear System of Equations

Consider the system of n equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n. \end{aligned} \tag{3.11}$$

In matrix form, we can write the system of equations (3.11) as

$$\mathbf{Ax} = \mathbf{b} \tag{3.12}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and **A**, **b**, **x** are respectively called the *coefficient matrix*, the right hand side column vector and the solution vector. If $\mathbf{b} \neq \mathbf{0}$, that is, at least one of the elements $b_1, b_2, \dots, b_n$ is not zero, then the system of equations is called *non-homogeneous*. If $\mathbf{b} = \mathbf{0}$, then the system of equations is called *homogeneous*. The system of equations is called *consistent* if it has at least one solution and *inconsistent* if it has no solution.

Non-homogeneous system of equations

The non-homogeneous system of equations $\mathbf{Ax} = \mathbf{b}$ can be solved by the following methods.

Matrix method

Let **A** be non-singular. Pre-multiplying $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^{-1}$, we obtain

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{3.13}$$

The system of equations is consistent and has a unique solution. If $\mathbf{b} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$ (trivial solution) is the only solution.

Cramer's rule

Let **A** be non-singular. The Cramer's rule for the solution of $\mathbf{Ax} = \mathbf{b}$ is given by

$$x_i = \frac{|\mathbf{A}_i|}{|\mathbf{A}|}, \quad i = 1, 2, \dots, n \tag{3.14}$$

where $|\mathbf{A}_i|$ is the determinant of the matrix $\mathbf{A}_i$ obtained by replacing the ith column of **A** by the right hand side column vector **b**.

We discuss the following cases.

Case 1 When $|\mathbf{A}| \neq 0$, the system of equations is consistent and the unique solution is obtained by using Eq. (3.14).

Case 2 When $|\mathbf{A}| = 0$ and one or more of $|\mathbf{A}_i|$, $i = 1, 2, \dots, n$, are not zero, then the system of equations has no solution, that is the system is inconsistent.

Case 3 When $|\mathbf{A}| = 0$ and all $|\mathbf{A}_i| = 0$, $i = 1, 2, \dots, n$, then the system of equations is consistent and has infinite number of solutions. The system of equations has at least a one-parameter family of solutions.

Homogeneous system of equations

Consider the homogeneous system of equations

$$\mathbf{Ax} = \mathbf{0}. \tag{3.15}$$

Trivial solution $\mathbf{x} = \mathbf{0}$ is always a solution of this system.

If **A** is non-singular, then again $\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$ is the solution.

Therefore, a homogeneous system of equations is always consistent. We conclude that non-trivial solutions for $\mathbf{Ax} = \mathbf{0}$ exist if and only if **A** is singular. In this case, the homogeneous system of equations has infinite number of solutions.

Example 3.4 Show that the system of equations

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

has a unique solution. Solve this system using (i) matrix method, (ii) Cramer's rule.

Solution We find that

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1 + 3) - 2(-1 - 1) + 1\,(3 - 1) = 10 \neq 0.$$

Therefore, the coefficient matrix **A** is non-singular and the given system of equations has a unique solution. Let $\mathbf{x} = [x, y, z]^T$.

(i) We obtain

$$\mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

Therefore,

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Hence, $x = 2$, $y = -1$ and $z = 1$.

(ii) We have

$$|A_1| = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 4(1+3) - 0 + 2(3-1) = 20.$$

$$|A_2| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 1(0+6) - 2(4-2) + 1(-12-0) = -10.$$

$$|A_3| = \begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-0) - 2(-2-4) + 1(0-4) = 10.$$

Therefore, $$x = \frac{|A_1|}{|A|} = 2,\ y = \frac{|A_2|}{|A|} = -1,\ z = \frac{|A_3|}{|A|} = 1.$$

Example 3.5 Show that the system of equations

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

has infinite number of solutions. Hence, find the solutions.

Solutions We find that

$$|A| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix} = 0,\quad |A_1| = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{vmatrix} = 0,$$

$$|A_2| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 0,\quad |A_3| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix} = 0.$$

Therefore, the system of equations has infinite number of solutions. Using the first two equations

$$x_1 - x_2 = 3 - 3x_3$$
$$2x_1 + 3x_2 = 2 - x_3$$

and solving, we obtain $x_1 = (11 - 10x_3)/5$ and $x_2 = (5x_3 - 4)/5$ where x_3 is arbitrary. This solution satisfies the third equation.

Example 3.6 Show that the system of equations

$$\begin{bmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix}$$

is inconsistent.

Solution We find that

$$|A| = \begin{vmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{vmatrix} = 0,\quad |A_1| = \begin{vmatrix} 6 & 9 & 3 \\ 2 & 3 & 1 \\ 7 & 6 & 2 \end{vmatrix} = 0,\quad |A_2| = \begin{vmatrix} 4 & 6 & 3 \\ 2 & 7 & 1 \\ 2 & 7 & 2 \end{vmatrix} = 6.$$

Since $|A| = 0$ and $|A_2| \neq 0$, the system of equations is inconsistent.

Example 3.7 Solve the homogeneous system of equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -2 \\ 4 & 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution We find that $|A| = 0$. Hence, the given system has infinite number of solutions. Solving the first two equations

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3z \\ 2z \end{bmatrix}$$

we obtain $x = 13z$, $y = -8z$ where z is arbitrary. This solution satisfies the third equation.

Exercise 3.1

1. Given the matrices $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, verify that

   (i) $|AB| = |A||B|$. (ii) $|A + B| \neq |A| + |B|$.

2. If $A^T = [1, -5, 7]$, $B = [3, 1, 2]$, verify that $(AB)^T = B^T A^T$.

3. Show that the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the matrix equation $A^2 - 4A - 5I = 0$. Hence, find A^{-1}.

4. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ satisfies the matrix equation $A^3 - 6A^2 + 5A + 11I = 0$. Hence, find A^{-1}.

5. For the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$, verify that

(i) $[adj\,(A)]^T = adj\,(A^T)$, (ii) $[adj\,(A)]^{-1} = adj\,(A^{-1})$.

6. For the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that

(i) $(A^{-1})^T = (A^T)^{-1}$, (ii) $(A^{-1})^{-1} = A$.

7. For the matrices $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 0 & 9 \end{bmatrix}$, verify that

(i) $adj\,(AB) = adj\,(A)\,adj\,(B)$, (ii) $(A + B)^{-1} \neq A^{-1} + B^{-1}$.

8. For any non-singular matrix $A = (a_{ij})$ of order n, show that

(i) $|adj\,(A)| = |A|^{n-1}$, (ii) $adj\,(adj(A)) = |A|^{n-2} A$.

9. For any non-singular matrix A, show that $|A^{-1}| = 1/|A|$.

10. For any symmetric matrix A, show that BAB^T is symmetric, where B is any matrix for which the product matrix BAB^T is defined.

11. If A is a symmetric matrix, prove that $(BA^{-1})^T (A^{-1}B^T)^{-1} = I$ where B is any matrix for which the product matrices are defined.

12. If A and B are symmetric matrices, then prove that

(i) $A + B$ is symmetric, (ii) AA^T and A^TA are both symmetric,

(iii) $AB - BA$ is skew-symmetric.

13. If A and B are non-singular commutative and symmetric matrices, then prove that

(i) AB^{-1}, (ii) $A^{-1}B$, (iii) $A^{-1}B^{-1}$

are symmetric.

14. Let A be a non-singular matrix. Show that

(i) if $I + A + A^2 + \ldots + A^n = 0$, then $A^{-1} = A^n$,

(ii) if $I - A + A^2 - \ldots + (-1)^n A^n = 0$, then $A^{-1} = (-1)^{n-1} A^n$.

15. Let P, Q and A be non-singular square matrices of order n and $PAQ = I$, then show that $A^{-1} = QP$.

16. If $I - A$ is a non-singular matrix, then show that

$$(I - A)^{-1} = I + A + A^2 + \ldots$$

assuming that the series on the right hand side converges.

17. For any three non-singular matrices A, B, C, each of order n, show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Solve the following system of equation:

18. $x - y + z = 2,\quad 2x + 3y - z = 5,\quad x + y - z = 0.$

19. $x + 2y + 3z = 6,\quad 2x + 4y + z = 7,\quad 3x + 2y + 9z = 14.$

20. $-x + y + 2z = 2,\quad 3x - y + z = 3,\quad -x + 3y + 4z = 6.$

21. $2x - z = 1,\quad 5x + y = 7,\quad y + 3z = 5.$

22. Determine the values of k for which the system of equations

$$x - ky + z = 0,\quad kx + 3y - kz = 0,\quad 3x + y - z = 0$$

has (i) only trivial solution, (ii) non-trivial solution.

23. Find the value of θ for which the system of equations

$2\,(\sin\theta)\,x + y - 2z = 0,\quad 3x + 2\,(\cos 2\theta)\,y + 3z = 0,\quad 5x + 3y - z = 0$ has a non-trivial solution.

24. If the system of equations $x + ay + az = 0$, $bx + y + bz = 0$, $cx + cy + z = 0$, where a, b, c, are non-zero and non-unity, has a non-trivial solution, then show that

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} = -1.$$

25. Find the values of λ and μ for which the system of equations

$$x + 2y + z = 6,\quad x + 4y + 3z = 10,\quad x + 4y + \lambda z = \mu$$

has (i) a unique solution, (ii) infinite number of solution, (iii) no solution.

Find the rank of the matrix A, where A is given by

26. $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ 27. $\begin{bmatrix} 1 & 3 & -4 \\ -1 & -3 & 4 \\ 2 & 6 & -8 \end{bmatrix}$ 28. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ p^3 & q^3 & r^3 \end{bmatrix}$. 30. (a) $\begin{bmatrix} 2 & 1 & 5 & -1 \\ -1 & 2 & 5 & 3 \\ 3 & 2 & 9 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & c_1 & -b_1 & a_2 \\ -c_1 & 0 & a_1 & b_2 \\ b_1 & -a_1 & 0 & c_2 \\ -a_2 & -b_2 & -c_2 & 0 \end{bmatrix}$

31. Prove that if A is an Hermitian matrix then iA is a Skew-Hermitian matrix and if A is a Ske Hermitian matrix, then iA is a Hermitian martix.

32. Prove that if A is a real matrix and $A^n \to 0$ as $n \to \infty$, then $I + A$ is invertible.

33. Let A, B be $n \times n$ real matrices. Then, show that

(i) Trace $(\alpha A + \beta B) = \alpha$ Trace (A) + β Trace (B) for any scalars α and β.

(ii) Trace (AB) = Trace (BA), (iii) $AB - BA = I$ is never true.

34. If B, C are $n \times n$ matrices, $A = B + C$, $BC = CB$ and $C^2 = 0$, then show that $A^{p+1} = B^p\,[B + (p + 1)\,C]$ for any positive integer p.

35. Let $A = (a_{ij})$ be a square matrix of order n, such that $a_{ij} = d$, $i \neq j$ and $a_{ij} = c$, $i = j$. Then, show that $|A| = (c - d)^{n-1}\,[c + (n - 1)\,d]$.

Identity the following matrices as symmetric, skew-symmetric, Hermitian, skew-Hermitian or none of these:

36. $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & 4 \\ -3 & -4 & 6 \end{bmatrix}$. **37.** $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$. **38.** $\begin{bmatrix} 0 & b & c \\ -b & 0 & e \\ -c & -e & 0 \end{bmatrix}$.

39. $\begin{bmatrix} 1 & 2+4i & 1-i \\ 2-4i & -5 & 3-5i \\ 1+i & 3+5i & 6 \end{bmatrix}$. **40.** $\begin{bmatrix} 1 & 2+4i & 1-i \\ -2+4i & -5 & 3-5i \\ -1-i & 3-5i & 6 \end{bmatrix}$. **41.** $\begin{bmatrix} 0 & 2+4i & 1-i \\ -2+4i & 0 & 3-5i \\ -1-i & -3-5i & 0 \end{bmatrix}$.

42. $\begin{bmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{bmatrix}$. **43.** $\begin{bmatrix} 0 & -i & 1+i \\ -i & -2i & 0 \\ -1+i & 0 & i \end{bmatrix}$. **44.** $\begin{bmatrix} 1 & -1 & i \\ -1 & 0 & 1-i \\ -i & 1+i & 2 \end{bmatrix}$.

45. $\begin{bmatrix} 1 & 2i & -i \\ -2i & i & 1 \\ i & 1 & 2 \end{bmatrix}$.

3.3 Vector Spaces

Let V be a non-empty set of certain objects, which may be vectors, matrices, functions or some other objects. Each object is an element of V and is called a **vector**. The elements of V are denoted by **a, b, c, u, v**, etc. Assume that the two algebraic operations

(i) vector addition and (ii) scalar multiplication

are defined on elements of V.

If the vector addition is defined as the usual addition of vectors, then

$$\mathbf{a} + \mathbf{b} = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

If the scalar multiplication is defined as the usual scalar multiplication of a vector by the scaler α,then

$$\alpha\mathbf{a} = \alpha(a_1, a_2, \ldots, a_n) = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n).$$

The set V defines a vector space if for any elements **a, b, c** in V and any scalars α, β the following properties (axioms) are satisfied.

Properties (axioms) with respect to vector addition

1. $\mathbf{a} + \mathbf{b}$ is in V.
2. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. (commutative law)
3. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$. (associative law)
4. $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$. (existence of a unique zero element in V)
5. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$. (existence of additive inverse or negative vector in V)

Properties (axioms) with respect to scalar multiplication

6. $\alpha\mathbf{a}$ is in V.
7. $(\alpha + \beta)\,\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$. (left distributive law)
8. $(\alpha\beta)\mathbf{a} = \alpha(\beta\mathbf{a})$.
9. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$. (right distributive law)
10. $1\mathbf{a} = \mathbf{a}$. (existence of multiplicative identity)

The properties defined in **1** and **6** are called the *closure* properties. When these two properties are satisfied, we say that the vector space is closed under the vector addition and scalar multiplication. The vector addition and scalar multiplication defined above need not always be the usual addition and multiplication operators. Thus, *the vector space depends not only on the set V of vectors, but also on the definition of vector addition and scalar multiplication on V.*

If the elements of V are real, then it is called a *real vector space* when the scalars α, β are real numbers, whereas V is called a *complex vector space*, if the elements of V are complex and the scalars α, β may be real or complex numbers or if the elements of V are real and the scalars α, β are complex numbers.

Remark 7

(a) If even one of the above properties is not satisfied, then V is not a vector space. We usually check the closure properties first before checking the other properties.

(b) The concepts of length, dot product, vector product etc. are not part of the properties to be satisfied.

(c) The set of real numbers and complex numbers are called *fields* of scalars. We shall consider vector space only on the fields of scalars. In an advanced course on linear algebra, vector spaces over arbitrary fields are considered.

(d) The vector space $V = \{\mathbf{0}\}$ is called a trivial vector space.

The following are some examples of vector spaces under the usual operations of vector addition and scalar multiplication.

1. The set V of real or complex numbers.
2. The set of real valued continuous functions f on any closed interval $[a, b]$. The **0** vector defined in property **4** is the zero function.
3. The set of polynomials P_n of degree less than or equal to n.
4. The set V of n-tuples in IR^n or $\mathbb{C}^n$.
5. The set V of all $m \times n$ matrices. The element **0** defined in property **4** is the null matrix of order $m \times n$.

The following are some examples which are not vector spaces. Assume that usual operations of vector addition and scalar multiplication are being used.

1. The set V of all polynomials of degree n. Let P_n and Q_n be two polynomials of degree n in V. Then, $\alpha P_n + \beta Q_n$ need not be a polynomial of degree n and thus may not be in V. For example, if $P_n = x^n + a_1x^{n-1} + \ldots + a_n$ and $Q_n = -x^n + b_1x^{n-1} + \ldots + b_n$, then $P_n + Q_n$ is a polynomial of degree $(n - 1)$.

2. The set V of all real-valued functions of one varible x, defined and continuous on the closed interval $[a, b]$ such that the value of the function at b is some non-zero constant p. For example, let $f(x)$ and $g(x)$ be two elements in V. Now, $f(b) = g(b) = p$. Since $f(b) + g(b) = 2p$, $f(x) + g(x)$ is not in V. Note that if $p = 0$, then V forms a vector space.

Example 3.8 Let V be the set of all polynomials, with real coefficients, of degree n, where addition is defined by $\mathbf{a} + \mathbf{b} = \mathbf{ab}$ and under usual scalar multiplication. Show that V is not a vector space.

Solution Let P_n and Q_n be two elements in V. Now, $P_n + Q_n = (P_n)(Q_n)$ is a polynomial of degree $2n$, which is not in V. Therefore, V does not define a vector space.

Example 3.9 Let V be the set of all ordered pairs (x, y), where x, y are real numbers. Let $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ be two elements in V. Define the addition as

$$\mathbf{a} + \mathbf{b} = (x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$$

and the scalar multiplication as

$$\alpha(x_1, y_1) = (\alpha x_1/3, \alpha y_1/3).$$

Show that V is not a vector space.

Solution We illustrate the properties that are not satisfied.

(i) $(x_2, y_2) + (x_1, y_1) = (2x_2 - 3x_1, y_2 - y_1) \neq (x_1, y_1) + (x_2, y_2)$.

Therefore, property 2 (commutative law) does not hold.

(ii)
$$\begin{aligned}((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (2x_1 - 3x_2, y_1 - y_2) + (x_3, y_3)\\ &= (4x_1 - 6x_2 - 3x_3, y_1 - y_2 - y_3)\\ (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (2x_2 - 3x_3, y_2 - y_3)\\ &= (2x_1 - 6x_2 + 9x_3, y_1 - y_2 + y_3).\end{aligned}$$

Therefore, property 3 (associative law) is not satisfied.

Hence, V is not a vector space.

Example 3.10 Let V be the set of all ordered pairs (x, y), where x, y are real numbers. Let $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ be two elements in V. Define the addition as

$$\mathbf{a} + \mathbf{b} = (x_1, y_1) + (x_2, y_2) = (x_1x_2, y_1y_2)$$

and the scalar multiplication as

$$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1).$$

Show that V is not a vector space.

Solution Note that (1, 1) is an element of V. From the given definition of vector addition, we find that

$$(x_1, y_1) + (1, 1) = (x_1, y_1).$$

This is true only for the element (1, 1). Therefore, the element (1, 1) plays the role of **0** element as defined in property **4**.

Now, there exists the element $(1/x_1, 1/y_1)$ such that $(x_1, y_1) + (1/x_1 + 1/y_1) = (1, 1)$. The element $(1/x_1, 1/y_1)$ plays the role of additive inverse.

Therefore, property **5** is satisfied.

Now, let $\alpha = 1$, $\beta = 2$ be any two scalars. We have

$$(\alpha + \beta)(x_1, y_1) = 3(x_1, y_1) = (3x_1, 3y_1)$$

and $\quad \alpha(x_1, y_1) + \beta(x_1, y_1) = 1(x_1, y_1) + 2(x_1, y_1) = (x_1, y_1) + (2x_1, 2y_1) = (2x_1^2, 2y_1^2).$

Therefore, $(\alpha + \beta)(x_1, y_1) \neq \alpha(x_1, y_1) + \beta(x_1, y_1)$ and property **7** is not satisfied.

Similarly, it can be shown that property **9** is not satisfied. Hence, V is not a vector space.

3.3.1 Subspaces

Let V be an arbitrary vector space defined under a given vector addition and scalar multiplication. A non-empty subset W of V, such that W is also a vector space under the same two operations of vector addition and scalar multiplication, is called a *subspace* of V. Thus, W is also closed under the two given algebraic operations on V. As a convention, the vector space V is also taken as a subspace of V.

Remark 8

To show that W is a subspace of a vector space V, it is not necessary to verify all the 10 properties as given in section 3.3. If it is shown that W is closed under the given definition of vector addition and scalar multiplication, then the properties **2, 3, 7, 8, 9** and **10** are automatically satisfied because these properties are valid for all elements in V and hence are also valid for all elements in W. Thus, we need to verify the remaining properties, that is, the existence of the zero element and the additive inverse in W.

Consider the following examples:

1. Let V be the set of n-tuples $(x_1, x_2, \ldots x_n)$ in IR^n with usual addition and scalar multiplication. Then

   (i) W consisting of n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_1 = 0$ is a subspace of V.

   (ii) W consisting of n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_1 \geq 0$ is not a subspace of V, since W is not closed under scalar multiplication ($\alpha\mathbf{x}$, when α is a negative real number, is not in W).

   (iii) W consisting of n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_2 = x_1 + 1$ is not a subspace of V, since W is not closed under addition.

   (Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with $x_2 = x_1 + 1$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ with $y_2 = y_1 + 1$ be two elements in W. Then

   $$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

   is not in W as $x_2 + y_2 = x_1 + y_1 + 2 \neq x_1 + y_1 + 1$).

2. Let V be the set of all real polynomials P of degree $\leq m$ with usual addition and scalar multiplication. Then

   (i) W consisting of all real polynomials of degree $\leq m$ with $P(0) = 0$ is a subspace of V.

(ii) W consisting of all real polynomials of degree $\le m$ with $P(0) = 1$ is not a subspace of V, since W is not closed under addition (if P and $Q \in W$, then $P + Q \notin W$).

(iii) W consisting of all polynomials of degree $\le m$ with real positive coefficients is not a subspace of V since W is not closed under scalar multiplication (if P is an element of W, then $-P \notin W$).

3. Let V be the set of all $n \times n$ real square matrices with usual matrix addition and scalar multiplication. Then

(i) W consisting of all symmetric/skew-symmetric matrices of order n is a subspace of V.

(ii) W consisting of all upper/lower triangular matrices of order n is a subspace of V.

(iii) W consisting of all $n \times n$ matrices having real positive elements is not a subspace of V since W is not closed under scalar multiplication (if $\mathbf{A}$ is an element of W, then $-\mathbf{A} \notin W$).

4. Let V be the set of all $n \times n$ complex matrices with usual matrix addition and scalar multiplication. Then

(i) W consisting of all Hermitian matrices of order n forms a vector space when scalars are real numbers and does not form a vector space when scalars are complex numbers (W is not closed under scalar multiplication).

Let
$$\mathbf{A} = \begin{pmatrix} a & x+iy \\ x-iy & b \end{pmatrix} \in W.$$

Let $\alpha = i$. We get
$$\alpha\mathbf{A} = i\mathbf{A} = \begin{pmatrix} ai & xi-y \\ xi+y & bi \end{pmatrix} \notin W.$$

(ii) W consisting of all skew-Hermitian matrices of order n forms a vector space when scalars are real numbers and does not form a vector space when scalars are complex numbers.

Let
$$\mathbf{A} = \begin{pmatrix} i & x+iy \\ -x+iy & 2i \end{pmatrix} \in W.$$

Let $\alpha = i$. We get
$$i\mathbf{A} = \begin{pmatrix} -1 & ix-y \\ -ix-y & -2 \end{pmatrix} \notin W.$$

Example 3.11 Let F and G be subspaces of a vector space V such that $F \cap G = \{\mathbf{0}\}$. The sum of F and G is written as $F + G$ and is defined by

$$F + G = \{\mathbf{f} + \mathbf{g}: \mathbf{f} \in F, \mathbf{g} \in G\}.$$

Show that $F + G$ is a subspace of V assuming the usual definition of vector addition and scalar multiplication.

Solution Let $W = F + G$ and $\mathbf{f} \in F$, $\mathbf{g} \in G$. Since $\mathbf{0} \in F$, and $\mathbf{0} \in G$ we have $\mathbf{0} + \mathbf{0} = \mathbf{0} \in W$. Let $\mathbf{f}_1 + \mathbf{g}_1$ and $\mathbf{f}_2 + \mathbf{g}_2$ belong to W where $\mathbf{f}_1, \mathbf{f}_2 \in F$ and $\mathbf{g}_1, \mathbf{g}_2 \in G$. Then

$$(\mathbf{f}_1 + \mathbf{g}_1) + (\mathbf{f}_2 + \mathbf{g}_2) = (\mathbf{f}_1 + \mathbf{f}_2) + (\mathbf{g}_1 + \mathbf{g}_2) \in F + G = W.$$

Also, for any scalar α, $\alpha(\mathbf{f} + \mathbf{g}) = \alpha\mathbf{f} + \alpha\mathbf{g} \in F + G = W$.
Therefore, $W = F + G$ is a subspace of V.

We now state an important result on subspaces.

Theorem 3.1 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be any r elements of a vector space V under usual vector addition and scalar multiplication. Then, the set of all linear combinations of these elements, that is the set of all elements of the form

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_r\mathbf{v}_r \tag{3.16}$$

is a subspace of V, where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are scalars.

Spanning set Let S be a subset of a vector space V and suppose that every element in V can be obtained as a linear combination of the elements taken from S. Then S is said to be the *spanning set* for V. We also say that S spans V.

Example 3.12 Let V be the vector space of all 2×2 real matrices. Show that the sets

(i)
$$S = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

(ii)
$$S = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}$$

span V.

Solution Let $\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary element of V.

(i) We write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since every element of V can be written as a linear combination of the elements of S, the set S spans the vector space V.

(ii) We need to determine the scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Equating the corresponding elements, we obtain the system of equations

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = a, \quad \alpha_2 + \alpha_3 + \alpha_4 = b,$$
$$\alpha_3 + \alpha_4 = c, \quad \alpha_4 = d.$$

The solution of this system of equations is

$$\alpha_4 = d, \quad \alpha_3 = c - d, \quad \alpha_2 = b - c, \quad \alpha_1 = a - b.$$

Therefore, we can write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a-b)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (b-c)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (c-d)\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since every element of V can be written as a linear combination of the elements of S, the set S spans the vector space V.

Example 3.13 Let V be the vector space of all polynomials of degree ≤ 3. Determine whether or not the set

$$S = \{t^3, t^2 + t, t^3 + t + 1\}$$

spans V?

Solution Let $P(t) = \alpha t^3 + \beta t^2 + \gamma t + \delta$ be an arbitrary element in V. We need to find whether or not there exist scalars a_1, a_2, a_3 such that

$$\alpha t^3 + \beta t^2 + \gamma t + \delta = a_1 t^3 + a_2(t^2 + t) + a_3(t^3 + t + 1)$$

$$\alpha t^3 + \beta t^2 + \gamma t + \delta = (a_1 + a_3)t^3 + a_2 t^2 + (a_2 + a_3)\, t + a_3.$$

Comparing the coefficients of various powers of t, we get

$$a_1 + a_3 = \alpha, \quad a_2 = \beta, \quad a_2 + a_3 = \gamma, \quad a_3 = \delta.$$

The solution of the first three equations is given by

$$a_1 = \alpha + \beta - \gamma, \quad a_2 = \beta, \quad a_3 = \gamma - \beta.$$

Substituting in the last equation, we obtain $\gamma - \beta = \delta$, which may not be true for all elements in V. For example, the polynomial $t^3 + 2t^2 + t + 3$ does not satisfy this condition and therefore, it cannot be written as a linear combination of the elements of S. Therefore, S does not span the vector space V.

3.3.2 Linear Independence of Vectors

Let V be a vector space. A finite set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of the elements of V is said to be *linearly dependent* if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{0}. \tag{3.17}$$

If Eq. (3.17) is satisfied only for $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, then the set of vectors is said to be *linearly independent*.

The above definition of linear dependence of $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ can be written alternately as follows.

Theorem 3.2 The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent if and only if at least one element of the set is a linear combination of the remaining elements.

Remark 9

Eq. (3.17) gives a homogeneous system of algebraic equations. Non-trivial solutions exist if *det*(coefficient matrix) = 0, that is the vectors are linearly dependent in this case. If the *det*(coefficient matrix) $\neq 0$, then by Cramer's rule, $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ and the vectors are linearly independent.

Example 3.14 Let $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (0, 1, -1)$ and $\mathbf{v}_3 = (0, 0, 1)$ be elements of IR^3. Show that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Solution We consider the vector equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \mathbf{0}.$$

Substituting for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we obtain

$$\alpha_1(1, -1, 0) + \alpha_2(0, 1, -1) + \alpha_3(0, 0, 1) = \mathbf{0}$$

or

$$(\alpha_1, -\alpha_1 + \alpha_2, -\alpha_2 + \alpha_3) = \mathbf{0}.$$

Comparing, we obtain $\alpha_1 = 0$, $-\alpha_1 + \alpha_2 = 0$ and $-\alpha_2 + \alpha_3 = 0$. The solution of these equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, the given set of vectors is linearly independent.

Alternative

$$det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1 \neq 0.$$

Therefore, the given vectors are linearly independent.

Example 3.15 Let $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (0, 1, -1)$, $\mathbf{v}_3 = (0, 2, 1)$ and $\mathbf{v}_4 = (1, 0, 3)$ be elements of IR^3. Show that the set of vectors $\{\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\}$ is linearly dependent.

Solution The given set of elements will be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4 = \mathbf{0}. \tag{3.18}$$

Substituting for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and comparing, we obtain

$$\alpha_1 + \alpha_4 = 0, \quad -\alpha_1 + \alpha_2 + 2\alpha_3 = 0, \quad -\alpha_2 + \alpha_3 + 3\alpha_4 = 0.$$

The solution of this system of equations is

$$\alpha_1 = -\alpha_4, \quad \alpha_2 = 5\alpha_4/3, \quad \alpha_3 = -4\alpha_4/3, \quad \alpha_4 \text{ arbitrary}.$$

Substituting in Eq. (3.18) and cancelling α_4, we obtain

$$-\mathbf{v}_1 + \frac{5}{3}\mathbf{v}_2 - \frac{4}{3}\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}.$$

Hence, there exist scalars not all zero, such that Eq. (3.18) is satisfied. Therefore, the set of vectors is linearly dependent.

3.3.3 Dimension and Basis

Let V be a vector space. If for some positive integer n, there exists a set S of n linearly independent elements of V and if every set of $n + 1$ or more elements in V is linearly dependent, then V is said to have *dimension n*. Then, we write $dim\,(V) = n$. Thus, the maximum number of linearly independent elements of V is the dimension of V. The set S of n linearly independent vectors is called the *basis* of V. Note that a vector space whose only element is zero has dimension zero.

Theorem 3.3 Let V be a vector space of dimension n. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the linearly independent elements of V. Then, every other element of V can be written as a linear combination of these elements. Further, this representation is unique.

Proof Let $\mathbf{v}$ be an element of V. Then, the set $\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent as it has $n + 1$ elements. Therefore, there exist scalars $\alpha_0, \alpha_1, \ldots, \alpha_n$, not all zero, such that

$$\alpha_0\mathbf{v} + \alpha_1\mathbf{v}_1 + \ldots + \alpha_n\mathbf{v}_n = 0. \tag{3.19}$$

Now, $\alpha_0 \neq 0$. Because, if $\alpha_0 = 0$, we get $\alpha_1\mathbf{v}_1 + \ldots + \alpha_n\mathbf{v}_n = 0$ and since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, we get $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$. This implies that the set of $n + 1$ elements $\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly independent, which is not possible as the dimension of V is n. Therefore, we obtain from Eq. (3.19)

$$\mathbf{v} = \sum_{i=1}^{n} (-\alpha_i/\alpha_0)\mathbf{v}_i. \tag{3.20}$$

Hence, $\mathbf{v}$ is a linear combination of n linearly independent vectors of V.

Now, let there be two representations of $\mathbf{v}$ given by

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n \quad \text{and} \quad \mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_n\mathbf{v}_n$$

where $b_i \neq a_i$ for at least one i. Subtracting these two equations, we get

$$\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \ldots + (a_n - b_n)\mathbf{v}_n.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ are linearly independent, we get

$$a_i - b_i = 0 \quad \text{or} \quad a_i = b_i,\ i = 1, 2, \ldots, n.$$

Therefore, both the representations of $\mathbf{v}$ are same and the representation of $\mathbf{v}$ given by Eq. (3.20) is unique.

Remark 10

(a) A set of $(n + 1)$ vectors in IR^n is linearly dependent.

(b) A set of vectors containing $\mathbf{0}$ as one of its elements is linearly dependent as $\mathbf{0}$ is the linear combination of any set of vectors.

Theorem 3.4 Let V be an n-dimensional vector space. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, $k < n$ are linearly independent elements of V, then there exist elements $\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \ldots, \mathbf{v}_n$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V.

Proof There exists an element $\mathbf{v}_{k+1}$ such that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly independent. Otherwise, every element of V can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and therefore V has dimension $k < n$. This argument can be continued. If $n > k + 1$, we keep adding elements $\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \ldots, \mathbf{v}_n$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V.

Since all the elements of a vector space V of dimension n can be represented as linear combinations of the n elements in the basis of V, the basis of V spans V. However, there can be many basis for the same vector space. For example, consider the vector space IR^3. Each of the following set of vectors

(i) [1, –1, 0], [0, 1, –1], [0, 0, 1]

(ii) [1, –1, 0], [0, 0, 1], [1, 2, 3]

(iii) [1, 0, 0], [0, 1, 0], [0, 0, 1]

are linearly independent and therefore forms a basis in IR^3. Some of the standard basis are the following.

1. If V consists of n-tuples in IR^n, then

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \mathbf{e}_n = (0, 0, \ldots, 0, 1)$$

is called a standard basis in IR^n.

2. If V consists of all $m \times n$ matrices, then

$$\mathbf{E}_{rs} = \begin{bmatrix} 0 & \ldots & 0 & \ldots & 0 \\ \vdots & & & & \\ 0 & \ldots & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & \ldots & 0 & \ldots & 0 \end{bmatrix},\ r = 1, 2, \ldots, m \text{ and } s = 1, 2, \ldots, n$$

where 1 is located in the (r, s) location, that is the rth row and the sth column, is called its standard basis.

For example, if V consists of all 2×3 matrices, then any matrix $\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$ in V can be written as

$$\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = a\mathbf{E}_{11} + b\mathbf{E}_{12} + c\mathbf{E}_{13} + x\mathbf{E}_{21} + y\mathbf{E}_{22} + z\mathbf{E}_{23}$$

where $\quad \mathbf{E}_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{E}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ etc.

3. If V consists of all polynomials $P(t)$ of degree $\leq n$, then $\{1, t, t^2, \ldots, t^n\}$ is taken as its standard basis.

Example 3.16 Determine whether the following set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis in IR^3, where

(i) $\mathbf{u} = (2, 2, 0)$, $\mathbf{v} = (3, 0, 2)$, $\mathbf{w} = (2, -2, 2)$

(ii) $\mathbf{u} = (0, 1, -1)$, $\mathbf{v} = (-1, 0, -1)$, $\mathbf{w} = (3, 1, 3)$.

Solution If the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis in IR^3, then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ must be linearly independent. Let α_1, α_2, α_3 be scalars. Then, the only solution of the equation

$$\alpha_1\mathbf{u} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w} = \mathbf{0} \tag{3.21}$$

must be $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

(i) Using Eq. (3.21), we obtain the system of equations

$$2\alpha_1 + 3\alpha_2 + 2\alpha_3 = 0,\quad 2\alpha_1 - 2\alpha_3 = 0 \quad \text{and} \quad 2\alpha_2 + 2\alpha_3 = 0.$$

The solution of this system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent and they form a basis in IR^3.

(ii) Using Eq. (3.21), we obtain the system of equations

$$-\alpha_2 + 3\alpha_3 = 0,\quad \alpha_1 + \alpha_3 = 0,\quad \text{and} \quad -\alpha_1 - \alpha_2 + 3\alpha_3 = 0.$$

The solution of this system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent and they form a basis in IR^3.

Example 3.17 Find the dimension of the subspace of IR^4 spanned by the set $\{(1\ 0\ 0\ 0), (0\ 1\ 0\ 0), (1\ 2\ 0\ 1), (0\ 0\ 0\ 1)\}$. Hence find its basis.

Solution The dimension of the subspace is ≤ 4. If it is 4, then the only solution of the vector equation

$$\alpha_1(1\ 0\ 0\ 0) + \alpha_2(0\ 1\ 0\ 0) + \alpha_3(1\ 2\ 0\ 1) + \alpha_4(0\ 0\ 0\ 1) = \mathbf{0} \tag{3.22}$$

should be $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Comparing, we obtain the system of equations

$$\alpha_1 + \alpha_3 = 0,\quad \alpha_2 + 2\alpha_3 = 0,\quad \alpha_3 + \alpha_4 = 0.$$

The solution of this system of equations is given by

$$\alpha_1 = \alpha_4,\quad \alpha_2 = 2\alpha_4,\quad \alpha_3 = -\alpha_4,\quad \text{where} \quad \alpha_4 \text{ is arbitrary.}$$

Hence, the vector equation (3.22) is satisfied for non-zero values of α_1, α_2, α_3, and α_4. Therefore, the dimension of the set is less than 4.

Now, consider any three elements of the set, say $(1\ 0\ 0\ 0)$, $(0\ 1\ 0\ 0)$ and $(1\ 2\ 0\ 1)$. Consider the vector equation

$$\alpha_1(1\ 0\ 0\ 0) + \alpha_2(0\ 1\ 0\ 0) + \alpha_3(1\ 2\ 0\ 1) = \mathbf{0}. \tag{3.23}$$

Comparing, we obtain the system of equations

$$\alpha_1 + \alpha_3 = 0,\ \alpha_2 + 2\alpha_3 = 0 \quad \text{and} \quad \alpha_3 = 0.$$

The solution of this system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Hence, these three elements are linearly independent. Therefore, the dimension of the given subspace is 3 and the basis is the set of vectors $\{(1\ 0\ 0\ 0), (0\ 1\ 0\ 0), (1\ 2\ 0\ 1)\}$. We find that the fourth vector can be written as

$$(0\ 0\ 0\ 1) = (1\ 0\ 0\ 0) - 2(0\ 1\ 0\ 0) + 1(1\ 2\ 0\ 1).$$

Example 3.18 Let $\mathbf{u} = \{(a, b, c, d)$, such that $a + c + d = 0$, $b + d = 0\}$ be a subspace of IR^4. Find the dimension and the basis of the subspace.

Solution $\mathbf{u}$ satisfies the closure properties. From the given equations, we have

$$a + c + d = 0 \text{ and } b + d = 0 \quad \text{or} \quad a = -c - d \text{ and } b = -d.$$

We have two free parameters, say, c and d. Therefore, the dimension of the given subspace is 2. Choosing $c = 0$, $d = 1$ and $c = 1$, $d = 0$, we may write a basis as $\{(-1\ -1\ 0\ 1), (-1\ 0\ 1\ 0)\}$.

3.3.4 Linear Tranformations

Let A and B be two arbitrary sets. A rule that assigns to elements of A exactly one element of B is called a *function* or a *mapping* or a *transformation*. Thus, a transformation maps the elements of A into the elements of B. The set A is called the *domain* of the transformation. We use capital letters T, S etc. to denote a transformation. If T is a transformation from A into B, we write

$$T : A \to B. \tag{3.24}$$

For each element $\mathbf{a} \in A$, we get a unique element $\mathbf{b} \in B$. We write $\mathbf{b} = T(\mathbf{a})$ or $\mathbf{b} = T\mathbf{a}$ and $\mathbf{b}$ is called the image of $\mathbf{a}$ under the mapping T. The collection of all such images in B is called the *range* or the image set of the transformation T.

In this section, we shall discuss mapping from a vector space into a vector space. Let V and W be two vector spaces, both real or complex, over the same field F of scalars. Let T be a mapping from V into W. The mapping T is said to be a *linear transformation* or a *linear mapping*, if it satisfies the following two properties:

(i) For every scalar α and every element $\mathbf{v}$ in V

$$T(\alpha\mathbf{v}) = \alpha T(\mathbf{v}). \tag{3.25}$$

(ii) For any two elements $\mathbf{v}_1$, $\mathbf{v}_2$ in V

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2). \tag{3.26}$$

Since V is a vector space, the product $\alpha\,\mathbf{v}$ and the sum $\mathbf{v}_1 + \mathbf{v}_2$ are defined and are elements in V. Then, T defines a mapping from V into W. Since $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are in W, the product $\alpha T(\mathbf{v})$ and the sum $T(\mathbf{v}_1) + T(\mathbf{v}_2)$ are in W. The conditions given in Eqs. (3.25) and (3.26) are equivalent to

$$T(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = T(\alpha\mathbf{v}_1) + T(\alpha\mathbf{v}_2) = \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2)$$

for $\mathbf{v}_1$ and $\mathbf{v}_2$ in V and any scalars α, β.

Let V be a vector space of dimension n and let the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be its basis. Then, any element $\mathbf{v}$ in V can be written as a linear combination of the elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Remark 11

A linear transformation is completely determined by its action on basis vectors of a vector space.

Letting $\alpha = 0$ in Eq. (3.25), we find that for every element $\mathbf{v}$ in V

$$T(0\mathbf{v}) = T(\mathbf{0}) = 0\,T(\mathbf{v}) = \mathbf{0}.$$

Therefore, the zero element in V is mapped into zero element in W by the linear transformation T. The collection of all elements $w = T(v)$ is called the *range* of T and is written as $ran(T)$. The set of all elements of V that are mapped into the zero element by the linear transformation T is called the *kernel* or the *null-space* of T and is denoted by $ker(T)$. Therefore, we have

$$ker(T) = \{v \mid T(v) = 0\} \quad \text{and} \quad ran(T) = \{T(v) \mid v \in V\}.$$

Thus, the null space of T is a subspace of V and the range of T is a subspace of W.

The dimension of $ran(T)$ is called the rank(T) and the dimension of $ker(T)$ is called the *nullity* of T. We have the following result.

Theorem 3.5 If T has rank r and the dimension of V is n, then the nullity of T is $n - r$, that is

$$\text{rank } (T) + \text{nullity} = n = dim\ (V).$$

We shall discuss the linear transformation only in the context of matrices.

Let $\mathbf{A}$ be an $m \times n$ real (or complex) matrix. Let the rows of $\mathbf{A}$ represent the elements in IR^n (or $\mathbf{C}^n$) and the columns of $\mathbf{A}$ represent the elements in IR^m (or $\mathbf{C}^m$). If $\mathbf{x}$ is in IR^n, then $\mathbf{Ax}$ is in IR^m. Thus, an $m \times n$ matrix maps the elements in IR^n into the elements in IR^m. We write

$$T = \mathbf{A} : \mathrm{IR}^n \to \mathrm{IR}^m, \quad \text{and} \quad T\mathbf{x} = \mathbf{Ax}.$$

The mapping $\mathbf{A}$ is a linear transformation. The range of T is a linear subspace of IR^m and the kernel of T is a linear subspace of IR^n.

Remark 12

Let T_1 and T_2 be linear transformations from V into W. We define the sum $T_1 + T_2$ to be the transformation S such that

$$S\mathbf{v} = T_1\mathbf{v} + T_2\mathbf{v}, \quad \mathbf{v} \in V.$$

Then, $T_1 + T_2$ is a linear transformation and $T_1 + T_2 = T_2 + T_1$.

Example 3.19 Let T be a a linear transformation defined by

$$T\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \quad T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

Find $T\left[\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix}\right]$.

Solution The matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are linearly independent and hence form a basis in the space of 2×2 matrices. We write for any scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, not all zero

$$\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \end{bmatrix}.$$

Comparing the elements and solving the resulting system of equations, we get $\alpha_1 = 4$, $\alpha_2 = 1$, $\alpha_3 = -2$, $\alpha_4 = 5$. Since T is a linear transformation, we get

$$T\left[\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix}\right] = \alpha_1 T\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] + \alpha_2 T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] + \alpha_3 T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] + \alpha_4 T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= 4\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 20 \\ 36 \end{pmatrix}.$$

Example 3.20 For the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = (1, 3)^T$, $\mathbf{x}_2 = (4, 6)^T$, are in IR^2, find the matrix of linear transformation $T : \mathrm{IR}^2 \to \mathrm{IR}^3$, such that

$$T\mathbf{x}_1 = (-2 \;\; 2 \;\; -7)^T \quad \text{and} \quad T\mathbf{x}_2 = (-2 \;\; -4 \;\; -10)^T.$$

Solution The transformation T maps column vector in IR^2 into column vectors in IR^3. Therefore, T must be a matrix $\mathbf{A}$ of order 3×2. Let

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

Therefore, we have

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -10 \end{bmatrix}.$$

Multiplying and comparing the corresponding elements, we get

$$a_1 + 3b_1 = -2, \qquad 4a_1 + 6b_1 = -2,$$
$$a_2 + 3b_2 = 2, \qquad 4a_2 + 6b_2 = -4,$$
$$a_3 + 3b_3 = -7, \qquad 4a_3 + 6b_3 = -10$$

Solving these equations, we obtain

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -4 & 2 \\ 2 & -3 \end{bmatrix}.$$

Example 3.21 Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, where

$\mathbf{Tx} = \mathbf{Ax}$, $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = (x\ y\ z)^T$. Find $ker(T)$, $ran(T)$ and their dimensions.

Solution To find $ker(T)$, we need to determine all $\mathbf{v} = (v_1\ v_2\ v_3)^T$ such that $T\mathbf{v} = \mathbf{0}$. Now, $T\mathbf{v} = \mathbf{Av} = \mathbf{0}$ gives the equations

$$v_1 + v_2 = 0, \quad -v_1 + v_3 = 0$$

whose solution is $v_1 = -v_2 = v_3$. Therefore $\mathbf{v} = v_1[1\ \ -1\ \ 1]^T$.
Hence, dimension of $ker(T)$ is 1.
Now, $ran(T)$ is defined as $\{T(\mathbf{v}) \mid \mathbf{v} \in V\}$. We have

$$T(\mathbf{v}) = \mathbf{Av} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} v_1 + v_2 \\ -v_1 + v_3 \end{bmatrix}$$

$$= v_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + v_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the dimension of $ran(T)$ is 2.

Example 3.22 Find the matrix of a linear transformation T from $\mathbb{R}^3$ into $\mathbb{R}^3$ such that

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix},\ T\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix},\ T\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 5 \end{pmatrix}.$$

Solution The transformation T maps elements in $\mathbb{R}^3$ into $\mathbb{R}^3$. Therefore, the transformation is a matrix of order 3×3. Let this matrix be written as

$$T = \mathbf{A} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

We determine the elements of the matrix $\mathbf{A}$ such that

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}.$$

Equating the elements and solving the resulting equations, we obtain

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ -15/2 & 3 & 13/2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Example 3.23 Let T be a transformation from $\mathbb{R}^3$ into $\mathbb{R}^1$ defined by

$$T(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2.$$

Show that T is not a linear transformation.

Solution Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be any two elements in $\mathbb{R}^3$. Then

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3).$$

We have

$$T(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2, \quad T(\mathbf{y}) = y_1^2 + y_2^2 + y_3^2$$

$$T(\mathbf{x} + \mathbf{y}) = (x_1 + y_1)^2 + (x_2 + y_2)^2 + (x_3 + y_3)^2 \neq T(\mathbf{x}) + T(\mathbf{y}).$$

Therefore, T is not a linear transformation.

Matrix representation of a linear transformation

We observe from the earlier discussion that a matrix $\mathbf{A}$ of order $m \times n$ is a linear transformation which maps the elements in $\mathbb{R}^n$ into the elements in $\mathbb{R}^m$. Now, let T be a linear transformation from a finite dimensional vector space into another finite dimensional vector space over the same field F. We shall now show that with this linear transformation, we may associate a matrix $\mathbf{A}$.

Let V and W be respectively, n-dimensional and m-dimensional vector spaces over the same field F. Let T be a linear transformation such that $T: V \to W$. Let

$$\mathbf{x} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\},\ \mathbf{y} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$$

be the ordered basis of V and W respectively. Let $\mathbf{v}$ be an arbitrary element in V and $\mathbf{w}$ be an arbitrary element in W. Then, there exist scalars, $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_m$, not all zero, such that

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n \tag{3.27i}$$

$$\mathbf{w} = \beta_1\mathbf{w}_1 + \beta_2\mathbf{w}_2 + \ldots + \beta_m\mathbf{w}_m \tag{3.27 ii}$$

and

$$\mathbf{w} = T\mathbf{v} = T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n)$$

$$= \alpha_1 T\mathbf{v}_1 + \alpha_2 T\mathbf{v}_2 + \ldots + \alpha_n T\mathbf{v}_n \tag{3.27 iii}$$

Since every element $T\mathbf{v}_i$, $i = 1, 2, \ldots, n$ is in W, it can be written as a linear combination of the basis vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ in W. That is, there exist scalars a_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ not all zero, such that

$$T\mathbf{v}_i = a_{1i}\mathbf{w}_1 + a_{2i}\mathbf{w}_2 + \ldots + a_{mi}\mathbf{w}_m$$

$$= [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m]\,[a_{1i}, a_{2i}, \ldots, a_{mi}]^T,\ i = 1, 2, \ldots, n. \tag{3.27 iv}$$

Hence, we can write

$$T\,[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n] = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m]\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \quad (3.27\text{ v})$$

or
$$T\mathbf{x} = \mathbf{y}\mathbf{A}$$

where A is the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}. \quad (3.27\text{ vi})$$

The $m \times n$ matrix **A** is called the matrix representation of T or the matrix of T with respect to the ordered basis **x** and **y**. It may be observed that **x** is a basis of the vector space V, on which T acts and **y** is the basis of the vector space W that contains the range of T. Therefore, the matrix representation of T depends not only on T but also on the basis **x** and **y**. For a given linear transformation T, the elements a_{ij} of the matrix $\mathbf{A} = (a_{ij})$ are determined from (3.27 v), using the given basis vectors in **x** and **y**. From (3. 27 iii), we have (using 3.27iv)

$$\begin{aligned}\mathbf{w} &= \alpha_1(a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \ldots + a_{m1}\mathbf{w}_m) + \alpha_2(a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \ldots + a_{m2}\mathbf{w}_m) \\ &\quad + \ldots + \alpha_n(a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \ldots + a_{mn}\mathbf{w}_m) \\ &= (\alpha_1 a_{11} + \alpha_2 a_{12} + \ldots + \alpha_n a_{1n})\,\mathbf{w}_1 + (\alpha_1 a_{21} + \alpha_2 a_{22} + \ldots + \alpha_n a_{2n})\mathbf{w}_2 \\ &\quad + \ldots + (\alpha_1 a_{m1} + \alpha_2 a_{m2} + \ldots + \alpha_n a_{mn})\mathbf{w}_m \\ &= \beta_1\mathbf{w}_1 + \beta_2\mathbf{w}_2 + \ldots + \beta_m\mathbf{w}_m\end{aligned}$$

where
$$\beta_i = \alpha_1 a_{i1} + \alpha_2 a_{i2} + \ldots + \alpha_n a_{in}, \quad i = 1, 2, \ldots, m.$$

Hence,
$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

or
$$\boldsymbol{\beta} = \mathbf{A}\boldsymbol{\alpha}$$

where the matrix **A** is as defined in (3.27 vi) and

$$\boldsymbol{\beta} = [\beta_1, \beta_2, \ldots, \beta_m]^T, \ \boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T.$$

For a given ordered basis vectors **x** and **y** of vector spaces V and W respectively, and a linear transformation $T : V \to W$, the matrix **A** obtained from (3.27 v) is unique. We prove this result as follows:

Let $\mathbf{A} = (a_{ij})$ and $B = (b_{ij})$ be two matrices each of order $m \times n$ such that

$$T\mathbf{x} = \mathbf{y}\mathbf{A} \quad \text{and} \quad T\mathbf{x} = \mathbf{y}B.$$

Therefore, we have
$$\mathbf{y}\mathbf{A} = \mathbf{y}B$$

or
$$\sum_{i=1}^{m} \mathbf{w}_i a_{ij} = \sum_{i=1}^{m} \mathbf{w}_i b_{ij}, \quad j = 1, 2, \ldots, n.$$

Since $Y = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is a given basis, we obtain $a_{ij} = b_{ij}$ for all i and j and hence $\mathbf{A} \equiv \mathbf{B}$.

Example 3.24 Let $T : \mathrm{IR}^3 \to \mathrm{IR}^2$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ y-z \end{pmatrix}.$$

Determine the matrix of the linear transformation T, with respect to the ordered basis

(i) $\mathbf{x} = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ in IR^3 and $\mathbf{y} = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ in IR^2

(standard basis e_1, e_2, e_3 in IR^3 and e_1, e_2 in IR^2).

(ii) $\mathbf{x} = \left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ in IR^3 and $\mathbf{y} = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ in IR^2.

Solution Let $V = \mathrm{IR}^3$, $W = \mathrm{IR}^2$. Let $\mathbf{x} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\mathbf{y} = \{\mathbf{w}_1, \mathbf{w}_2\}$.

(i) We have
$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \ \mathbf{w}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \ \mathbf{w}_2 = \begin{pmatrix}0\\1\end{pmatrix}.$$

We obtain
$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}(0) + \begin{pmatrix}0\\1\end{pmatrix}(0), \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}(1) + \begin{pmatrix}0\\1\end{pmatrix}(1),$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}(1) + \begin{pmatrix}0\\1\end{pmatrix}(-1)$$

Using the notation given in (3.27 v), that is $T\mathbf{x} = \mathbf{y}A$, we write

$$T\,[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = [\mathbf{w}_1, \mathbf{w}_2]\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

or
$$T\left[\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Therefore, the matrix of the linear transformation T with respect to the given basis vectors is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

(ii) We have
$$\mathbf{v}_1 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix}1\\1\end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix}1\\-1\end{pmatrix}.$$

We obtain
$$T\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}(1) + \begin{pmatrix}1\\-1\end{pmatrix}(1), \quad T\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}(0) + \begin{pmatrix}1\\-1\end{pmatrix}(1)$$

$$T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}(1) + \begin{pmatrix}1\\-1\end{pmatrix}(0).$$

Using (3.27 v), that is $T\mathbf{x} = \mathbf{y}\mathbf{A}$, we write

$$T\left[\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right]\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Therefore, the matrix of the linear transformation T with respect to the given basis vectors is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Exercise 3.2

Discuss whether V defined in problems **1** to **10** is a vector space. If V is not a vector space, state which of the properties are not satisfied.

1. Let V be the set of the real polynomials of degree $\le m$ and having 2 as a root with the usual addition and scalar multiplication.
2. Let V be the set of all real polynomials of degree 4 or 6 with the usual addition and scalar multiplication.
3. Let V be the set of all real polynomials of degree ≥ 4 with the usual addition and scalar multiplication.
4. Let V be the set of all rational numbers with the usual addition and scalar multiplication.
5. Let V be the set of all positive real numbers with addition defined as $x + y = xy$ and usual scalar multiplication.
6. Let V be the set of all ordered pairs (x, y) in IR^2 with vector addition defined as $(x, y) + (u, v) = (x + u, y + v)$ and scalar multiplication defined as $\alpha(x, y) = (3\alpha x, y)$.
7. Let V be the set of all ordered triplets (x, y, z), $x, y, z \in \mathrm{IR}$, with vector addition defined as
$$(x, y, z) + (u, v, w) = (3x + 4u, y - 2v, z + w)$$
and scalar multiplication defined as
$$\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z/3).$$
8. Let V be the set of all positive real numbers with addition defined as $x + y = xy$ and scalar multiplication defined as $\alpha x = x^{\alpha}$.
9. Let V be the set of all positive real valued continuous functions f on $[a, b]$ such that
(i) $\int_a^b f(x)dx = 0$ and (ii) $\int_a^b f(x)\,dx = 2$ with usual addition and scalar multiplication.
10. Let V be the set of all solutions of the
(i) homogeneous linear differential equation $y'' - 3y' + 2y = 0$.
(ii) non-homogeneous linear differential equation $y'' - 3y' + 2y = x$.
under the usual addition and scalar multiplication.

Is W a subspace of V in problems 11 to 15? If not, state why?

11. Let V be the set of all 3×1 real matrices with usual matrix addition and scalar multiplication and W consisting of all 3×1 real matrices of the from
(i) $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$, (ii) $\begin{bmatrix} a \\ a \\ a^2 \end{bmatrix}$, (iii) $\begin{bmatrix} a \\ b \\ 2 \end{bmatrix}$, (iv) $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$
12. Let V be the set of all 3×3 real matrices with the usual matrix addition and scalar multiplication and W consisting of all 3×3 matrices $\mathbf{A}$ which
(i) have positive elements, (ii) are non-singular,
(iii) are symmetric, (iv) $\mathbf{A}^2 = \mathbf{A}$.
13. Let V be the set of all 2×2 complex matrices with the usual matrix addition and scalar multiplication and W consisting of all matricés with the usual addition and scalar multiplication and W consisting of all martices of the form $\begin{bmatrix} z & x + iy \\ x - iy & u \end{bmatrix}$, where x, y, z, u are real numbers and (i) scalars are real numbers, (ii) scalars are complex numbers.

14. Let V consist of all real polynomials of degree ≤ 4 with the usual polynomial addition and scalar multiplication and W consisting of polynomials of degree ≤ 4 having

(i) constant term 1, (ii) coefficient of t^2 as 0,
(iii) coefficient of t^3 as 1, (iv) only real roots.

15. Let V be the vector space of all triplets of the form (x_1, x_2, x_3) in $\mathbb{R}^3$ with the usual addition and scalar multiplication and W be the set of triplets of the form (x_1, x_2, x_3) such that

(i) $x_1 = 2x_2 = 3x_3$, (ii) $x_1 = x_2 = x_3 + 1$,
(iii) $x_1 \ge 0$, x_2, x_3 arbitrary, (iv) $x_1^2 + x_2^2 + x_3^2 \le 4$. (v) x_3 is an integer.

16. Let $\mathbf{u} = (1, 2, -1)$, $\mathbf{v} = (2, 3, 4)$ and $\mathbf{w} = (1, 5, -3)$. Determine whether or not $\mathbf{x}$ is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$, where $\mathbf{x}$ is given by

(i) (4, 3, 10), (ii) (3, 2, 5) (iii) (−2, 1, −5).

17. Let $\mathbf{u} = (1, -2, 1, 3)$, $\mathbf{v} = (1, 2, -1, 1)$ and $\mathbf{w} = (2, 3, 1, -1)$. Determine whether or not $\mathbf{x}$ is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$, where $\mathbf{x}$ is given by

(i) (3, 0, 5, −1), (ii) (2, −7, 1, 11), (iii) (4, 3, 0, 3).

18. Let $P_1(t) = t^2 - 4t - 6$, $P_2(t) = 2t^2 - 7t - 8$, $P_3(t) = 2t - 3$. Write $P(t)$ as a linear combination of $P_1(t)$, $P_2(t)$, $P_3(t)$, when

(i) $P(t) = -t^2 + 1$, (ii) $P(t) = 2t^2 - 3t - 25$.

19. Let V be the set of all 3 × 1 real matrices. Show that the set

$$S = \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \text{ spans } V.$$

20. Let V be the set of all 2 × 2 real matrices. Show that the set

$$S = \left\{\begin{pmatrix}2 & 1\\1 & -2\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}0 & 2\\0 & -1\end{pmatrix}\right\} \text{ spans } V.$$

21. Examine whether the following vectors in $\mathbb{R}^3/\mathbb{C}^3$ are linearly independent.

(i) (2, 2, 1), (1, −1, 1), (1, 0, 1), (ii) (1, 2, 3), (3, 4, 5), (6, 7, 8),
(iii) (0, 0, 0), (1, 2, 3), (3, 4, 5), (iv) (2, i, −1), (1, −3, i), (2i, −1, 5),
(v) (1, 3, 4), (1, 1, 0), (1, 4, 2), (1, −2, 1).

22. Examine whether the following vectors in $\mathbb{R}^4$ are linearly independent.

(i) (4, 1, 2, −6), (1, 1, 0, 3), (1, −1, 0, 2), (−2, 1, 0, 3),
(ii) (1, 2, 3, 1), (2, 1, −1, 1), (4, 5, 5, 3), (5, 4, 1, 3),
(iii) (1, 2, 3, 4), (2, 0, 1, −2), (3, 2, 4, 2),
(iv) (1, 1, 0, 1), (1, 1, 1, 1), (−1, −1, 1, 1), (1, 0, 0, 1),
(v) (1, 2, 3, −1), (0, 1, −1, 2), (1, 5, 1, 8), (−1, 7, 8, 3).

23. If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly independent vectors in $\mathbb{R}^3$, then show that

(i) $\mathbf{x} + \mathbf{y}, \mathbf{y} + \mathbf{z}, \mathbf{z} + \mathbf{x}$; (ii) $\mathbf{x}, \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} + \mathbf{z}$

are also linearly independent in $\mathbb{R}^3$.

24. Write (− 4, 7, 9) as a linear combination of the elements of the set S: {(1, 2, 3), (− 1, 3, 4), (3, 1, 2)}. Show that S is not a spanning set in $\mathbb{R}^3$.

25. Write $t^2 + t + 1$ as a linear combination of the elements of the set S: $\{3t, t^2 - 1, t^2 + 2t + 2\}$. Show that S is the spanning set for all polynomials of degree 2 and can be taken as its basis.

26. Let V be the set of all vectors in $\mathbb{R}^4$ and S be a subset of V consisting of all vectors of the form

(i) $(x, y, -y, -x)$, (ii) (x, y, z, w) such that $x + y + z - w = 0$,
(iii) $(x, 0, z, w)$, (iv) (x, x, x, x).

Find the dimension and the basis of S.

27. For what values of k do the following set of vectors form a basis in $\mathbb{R}^3$?

(i) {(k, 1 − k, k), (0, 3k − 1, 2), (− k, 1, 0)},
(ii) {(k, 1, 1), (0, 1, 1), (k, 0, k)},
(iii) {(k, k, k), (0, k, k), (k, 0, k)},
(iv) {(1, k, 5), (1, −3, 2), (2, −1, 1)}.

28. Find the dimension and the basis for the vector space V, when V is the set of all 2 × 2 (i) real matrices (ii) symmetric matrices, (iii) skew-symmetric matrices, (iv) skew-Hermitian matrices, (v) real matrices $\mathbf{A} = (a_{ij})$ with $a_{11} + a_{22} = 0$, (vi) real matrices $\mathbf{A} = (a_{ij})$ with $a_{11} + a_{12} = 0$.

29. Find the dimension and the basis for the vector space V, when V is the set of all 3 × 3 (i) diagonal matrices (ii) upper triangular matrices, (iii) lower triangular matrices.

30. Find the dimension of the vector space V, when V is the set of all $n \times n$ (i) real matrices, (ii) diagonal matrices, (iii) symmetric matrices (iv) skew-symmetric matrices.

Examine whether the transformation T given in problems 31 to 35 is linear or not. If not linear, state why?

31. $T : \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix}x\\y\end{pmatrix} = x + y + a$, $a \ne 0$, a real constant.

32. $T : \mathbb{R}^3 \to \mathbb{R}^2$; $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}y\\x+z\end{pmatrix}$.

33. $T : \mathbb{R}^1 \to \mathbb{R}^2$; $T(x) = \begin{pmatrix}x^2\\3x\end{pmatrix}$.

34. $T : \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{cases}0 & x \ne 0, y \ne 0\\2y, & x = 0\\3x, & y = 0.\end{cases}$

35. $T : \mathbb{R}^3 \to \mathbb{R}^1$; $T\begin{pmatrix}x\\y\\z\end{pmatrix} = xy + x + z$.

Find $ker(T)$ and $ran(T)$ and their dimensions in problems 36 to 42.

36. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+y\\z\\x-y\end{pmatrix}$.

37. $T : \mathbb{R}^2 \to \mathbb{R}^3$; $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x+y\\y-x\\3x+4y\end{pmatrix}$.

38. $T : \mathbb{R}^4 \to \mathbb{R}^3$; $T\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}x+y+w\\z\\y+2w\end{pmatrix}$.

39. $T : \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix}x\\y\end{pmatrix} = x + 3y$.

40. $T : \mathbb{R}^3 \to \mathbb{R}^1$; $T\begin{pmatrix}x\\y\\z\end{pmatrix} = x + 3y$.

41. $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x-y\\x-y\end{pmatrix}$.

42. $T : \mathbb{R}^3 \to \mathbb{R}^2;\ T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y \\ 3x + z \end{pmatrix}$.

43. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ x - z \end{pmatrix}$.

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } \mathbb{R}^3 \text{ and } \mathbf{y} = \left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right\} \text{ in } \mathbb{R}^2.$$

44. Let V and W be two vector spaces in $\mathbb{R}^3$. Let $T : V \to W$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ x + y \\ x + y + z \end{pmatrix}.$$

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} \text{ in } V \text{ and } \mathbf{y} = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } W.$$

45. Let V and W be two vector spaces in $\mathbb{R}^3$. Let $T : V \to W$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} x + z \\ x + y \\ x + y + z \end{pmatrix}.$$

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\} \text{ in } V \text{ and } \mathbf{y} = \left\{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\} \text{ in } W$$

46. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ y + z \\ x + z \\ x + y + z \end{pmatrix}$.

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } \mathbb{R}^3 \text{ and } \mathbf{y} = \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\} \text{ in } \mathbb{R}^4$$

47. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$ be the matrix representation of the linear transformation T with respect to the ordered basis vectors $\mathbf{v}_1 = [1, 2]^T$, $\mathbf{v}_2 = [3, 4]^T$ in $\mathbb{R}^2$ and $\mathbf{w}_1 = [-1, 1, 1]^T$, $\mathbf{w}_2 = [1, -1, 1]^T$, $\mathbf{w}_3 = [1, 1, -1]^T$ in $\mathbb{R}^3$. Then, determine the linear transformation T.

48. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & -4 \end{bmatrix}$ be the matrix representation of the linear transformation with respect to the ordered basis vectors $\mathbf{v}_1 = [1, -1, 1]^T$, $\mathbf{v}_2 = [2, 3, -1]^T$, $\mathbf{v}_3 = [1, 1, -1]^T$ in $\mathbb{R}^3$ and $\mathbf{w}_1 = [1, 1]^T$, $\mathbf{w}_2 = [2, 3]^T$ in $\mathbb{R}^2$. Then, determine the linear transformation T.

49. Let $T : P_1(t) \to P_2(t)$ be a linear transformation. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 1 \end{bmatrix}$ be the matrix representation of the linear transformation with respect to the ordered basis $[1 + t, t]$ in $P_1(t)$ and $[1 - t, 2t, 2 + 3t - t^2]$ in $P_2(t)$. Then, determine the linear transformation T.

50. Let V be the set of all vectors of the form (x_1, x_2, x_3) in $\mathbb{R}^3$ satisfying (i) $x_1 - 3x_2 + 2x_3 = 0$; (ii) $3x_1 - 2x_2 + x_3 = 0$ and $4x_1 + 5x_2 = 0$. Find the dimension and basis for V.

3.4 Solution of General linear System of Equations

In section 3.2.5, we have discussed the matrix method and the Cramer's rule for solving a system of n equations in n unknowns, $\mathbf{A}\mathbf{x} = \mathbf{b}$. We assumed that the coefficient matrix $\mathbf{A}$ is non-singular, that is $|\mathbf{A}| \neq 0$, or the rank of the matrix $\mathbf{A}$ is n. The matrix method requires evaluation of n^2 determinants each of order $(n - 1)$, to generate the cofactor matrix, and one determinant of order n, whereas the Cramer's rule requires evaluation of $(n + 1)$ determinants each of order n. Since the evaluation of high order determinants is very time consuming, these methods are not used for large values of n, say $n > 4$. In this section, we discuss a method for solving a general system of m equations in n unknowns, given by

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.28}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

are respectively called the ***coefficient matrix***, ***right hand side column vector*** and the ***solution vector***. The order of the matrices $\mathbf{A}$, $\mathbf{b}$, $\mathbf{x}$ are respectively $m \times n$, $m \times 1$ and $n \times 1$.

The matrix

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \tag{3.29}$$

is called the *augmented matrix* and has m rows and $(n + 1)$ columns. The augmented matrix describes completely the system of equations. The solution vector of the system of equations (3.28) is an n-tuple $(x_1, x_2, \ldots, x_n)$ that satisfies all the equations. There are three possibilities:

(i) the system has a unique solution,

(ii) the system has no solution,

(iii) the system has infinite number of solutions.

The system of equations is said to be *consistent*, if it has atleast one solution and *inconsistent*, if it has no solution. Using the concepts of ranks and vector spaces, we now obtain the necessary and sufficient conditions for the existence and uniqueness of the solution of the linear system of equations.

3.4.1 Existence and Uniqueness of the Solution

Let V_n be a vector space consisting of n-tuples in IR^n(or $\mathbf{C}^n$). The row vectors $R_1, R_2, \ldots, R_m$ of the $m \times n$ matrix **A** are n-tuples which belong to V_n. Let S be the subspace of V_n generated by the rows of **A**. Then, S is called the *row-space* of the matrix **A** and its dimension is called the *row-rank* of **A** and is denoted by $rr(\mathbf{A})$. Therefore,

$$\textit{row-rank of } \mathbf{A} = rr(\mathbf{A}) = \dim(S). \tag{3.30}$$

Similarly, we define the *column-space* of **A** and the *column-rank* of **A** denoted by $cr(\mathbf{A})$.

Since the row-space of $m \times n$ matrix **A** is generated by m row vectors of **A**, we have $\dim(S) \le m$ and since S is a subspace of V_n, we have $\dim(S) \le n$. Therefore, we have

$$rr(\mathbf{A}) \le \min(m, n) \quad \text{and} \quad \text{similarly} \quad cr(\mathbf{A}) \le \min(m, n). \tag{3.31}$$

Theorem 3.6 Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix. Then the row-rank and column-rank of **A** are same.

Now, we state an important result which is known as the *fundamental theorem of linear algebra*.

Theorem 3.7 The non-homogeneous system of equations $\mathbf{Ax} = \mathbf{b}$, where **A** is an $m \times n$ matrix, has a solution if and only if the matrix **A** and the augmented matrix $(\mathbf{A} \mid \mathbf{b})$ have the same rank.

In section 3.2.3, we defined the rank of $m \times n$ matrix **A** in terms of the determinants of the submatrices of **A**. An $m \times n$ matrix has rank r if it has at least one square submatrix of order r which is non-singular and all square submatrices of order greater than r are singular. This approach is very time consuming when n is large. Now, we discuss an alternative procedure to obtain the rank of a matrix.

3.4.2 Elementary Row and Column Operation

The following three operations on a matrix **A** are called the *elementary row operations*:

(i) Interchange of any two rows (written as $R_i \sim R_j$).

(ii) Multiplication/division of any row by a non-zero scalar (written as αR_i)

(iii) Adding/subtracting a scalar multiple of any row to another row (written as $R_i \leftarrow R_i + \alpha R_j$, that is α multiples of the elements of the jth row are added to the corresponding elements of the ith row. The elements of the jth row remain unchanged, whereas, the elements of the ith row get changed).

These operations change the form of **A** but do not change the row-rank of **A** as they do not change the row-space of **A**. A matrix **B** is said to be *row equivalent* to a matrix **A**, if the matrix **B** can be obtained from the matrix **A** by a finite sequence of elementary row operations. Then, we usually write $\mathbf{B} \approx \mathbf{A}$. We observe that

(i) every matrix is row equivalent to itself.

(ii) if **A** is row equivalent to **B**, then **B** is row equivalent to **A**.

(iii) if **A** is row equivalent to **B** and **B** is row equivalent to **C**, then **A** is row equivalent to **C**.

The above operations performed on columns (that is column in place of row) are called *elementary column operations*.

3.4.3 Echelon Form of a Matrix

An $m \times n$ matrix is called a *row echelon* matrix or in *row echelon* form if the number of zeros preceeding the first non-zero entry of a row increases row by row until a row having all zero entries (or no other elimination is possible) is obtained. Therefore, a matrix is in row echelon form if the following are satisfied.

(i) If the ith row contains all zeros, it is true for all subsequent rows.

(ii) If a column contains a non-zero entry of any row, then every subsequent entry in this column is zero, that is, if the ith and $(i + 1)$th rows are both non-zero rows, then the initial non-zero entry of the $(i + 1)$th row appears in a later column than that of the ith row.

(iii) Rows containing all zeros occur only after all non-zero rows.

For example, the following matrices are in row echelon form.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathbf{A} = (a_{ij})$ be a given $m \times n$ matrix. Assume that $a_{11} \ne 0$. If $a_{11} = 0$, we interchange the first row with some other row to make the element in the (1, 1) position as non-zero. Using elementary row operations, we reduce the matrix **A** to its row echelon form (elements of first column below a_{11} are made zero, then elements in the second column below a_{22} are made zero and so on). Similarly, we define the column echelon form of a matrix.

Rank of A The number of non-zero rows in the row echelon form of a matrix **A** gives the rank of the matrix **A** (that is, the dimension of the row-space of the matrix **A**) and the set of the non-zero rows in the row echelon form gives the basis of the row-space.

Similar results hold for column echelon matrices.

Remark 13

(i) If **A** is a square matrix, then the row-echelon form is an upper triangular matrix and the column echelon form is a lower triangular matrix.

(ii) This approach can be used to examine whether a given set of vectors are linearly independent or not. We form the matrix with each vector as its row (or column) and reduce it to the row (column) echelon form. The given vectors are linearly independent, if the row echelon form has no row with all its elements as zeros. The number of non-zero rows is the dimension of the given set of vectors and the set of vectors consisting of the non-zero rows is the basis.

Example 3.25 Reduce the following matrices to row echelon form and find their ranks.

(i) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$.

Solution Let the given matrix be denoted by **A**. We have

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 + 2R_1 \end{matrix} \approx \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix} R_3 + 2R_2 \approx \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is the row echelon form of **A**. Since the number of non-zero rows in the row echelon form is 2, we get rank (**A**) = 2.

$$\text{(ii) } \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - 8R_1 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \begin{matrix} R_3 + R_2 \\ R_4 - 5R_2 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the number of non-zero rows in the echelon form of **A** is 2, we get rank (**A**) = 2.

Example 3.26 Reduce the following matrices to column echelon form and find their ranks.

(i) $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 4 & -1 & 5 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix}$.

Solution Let the given matrix be denoted by **A**. We have

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix} \begin{matrix} C_2 - C_1/3 \\ C_3 - 7C_1/3 \end{matrix} \approx \begin{bmatrix} 3 & 0 & 0 \\ 1 & 5/3 & 5/3 \\ 4 & -7/3 & -7/3 \\ 2 & 1/3 & 1/3 \end{bmatrix} C_3 - C_2 \approx \begin{bmatrix} 3 & 0 & 0 \\ 1 & 5/3 & 0 \\ 4 & -7/3 & 0 \\ 2 & 1/3 & 0 \end{bmatrix}.$$

Since the column echelon form of **A** has two non-zero columns, rank (**A**) = 2.

$$\text{(ii) } \mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix} \begin{matrix} C_2 - C_1 \\ C_3 + C_1 \\ C_4 - C_1 \end{matrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & -2 \\ 1 & -1 & 2 & 1 \\ 1 & -2 & 4 & 2 \end{bmatrix} \begin{matrix} C_3 + 2C_2 \\ C_4 + C_2 \end{matrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}.$$

Since the column echelon form of **A** has 2 non-zero columns, rank (**A**) = 2.

Example 3.27 Examine whether the following set of vectors is linearly independent. Find the dimension and the basis of the given set of vectors.

(i) (1, 2, 3, 4), (2, 0, 1, – 2), (3, 2, 4, 2),

(ii) (1, 1, 0, 1), (1, 1, 1, 1), (–1, 1, 1, 1), (1, 0, 0, 1),

(iii) (2, 3, 6, –3, 4), (4, 2, 12, –3, 6), (4, 10, 12, –9, 10).

Solution Let each given vector represent a row of a matrix **A**. We reduce **A** to row echelon form. If all the rows of the echelon form have some non-zero elements, then the given set of vectors are linearly independent.

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & 2 & 4 & 2 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & -4 & -5 & -10 \end{bmatrix} R_3 - R_2 \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since all the rows in the row echelon form of **A** are not non-zero, the given set of vectors are linearly dependent. Since the number of non-zero rows is 2, the dimension of the given set of vectors is 2. The basis can be taken as the set of vectors {(1 2 3 4), (0, –4, –5, –10)}.

$$\text{(ii) } \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 + R_1 \\ R_4 - R_1 \end{matrix} \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix} R_2 \sim R_3 \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} R_4 + R_2/2$$

$$\approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{bmatrix} R_4 - R_3/2 \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since all the rows in the row echelon form of **A** are non-zero, the given set of vectors are linearly independent and the dimension of the given set of vectors is 4. The set of vectors {(1, 1, 0, 1), (0, 2, 1, 2), (0, 0, 1, 0), (0, 0, 0, 1)} or the given set itself forms the basis.

(iii) $\mathbf{A} = \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 4 & 2 & 12 & -3 & 6 \\ 4 & 10 & 12 & -9 & 10 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix} \approx \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 4 & 0 & -3 & 2 \end{bmatrix} R_3 + R_4 \approx \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

Since all the rows in the echelon form of **A** are not non-zero, the given set of vectors are linearly dependent. Since the number of non-zero rows is 2, the dimension of the given set of vectors is 2 and its basis can be taken as the set $\{(2, 3, 6, -3, 4), (0, -4, 0, 3, -2)\}$.

3.4.4 Gauss Elimination Method for Non-homogeneous Systems

Consider a non-homogeneous system of m equations in n unknowns

$$\mathbf{Ax} = \mathbf{b} \tag{3.32}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},\ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

We assume that at least one element of **b** is not zero. We write the augmented matrix of order $m \times (n+1)$ as

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

and reduce it to the row echelon form by using elementary row operations. We need a maximum of $(m-1)$ stages of eliminations to reduce the given augmented matrix to the equivalent row echelon form. This process may terminate at an earlier stage. We then have an equivalent system of the form

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccccccc|c} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1n} & & b_1 \\ 0 & \bar{a}_{22} & \cdots & \bar{a}_{2r} & \cdots & \bar{a}_{2n} & & \bar{b}_2 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & \cdots & a^*_{rr} & \cdots & a^*_{rn} & & b^*_r \\ 0 & 0 & \cdots & 0 & \cdots & 0 & & b^*_{r+1} \\ \vdots & & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & & b^*_m \end{array}\right]. \tag{3.33}$$

where $r \le m$ and $a_{11} \ne 0$, $\bar{a}_{22} \ne 0, \ldots, a^*_{rr} \ne 0$ are called *pivots*. We have the following cases:

(a) Let $r < m$ and one or more of the elements $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are not zero. Then, rank $(\mathbf{A}) \ne$ rank $(\mathbf{A} \mid \mathbf{b})$ and the system of equations has no solution.

(b) Let $m \ge n$ and $r = n$ (the number of columns in **A**) and $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are all zeros. In this case, rank $(\mathbf{A})$ = rank $(\mathbf{A} \mid \mathbf{b}) = n$ and the system of equations has a unique solution. We solve the nth equation for x_n, the $(n-1)$th equation for x_{n-1} and so on. This procedure is called the *back substitution method*.

For example, if we have 10 equations in 5 variables, then the augmented matrix is of order 10×6. When rank $(\mathbf{A})$ = rank $(\mathbf{A} \mid \mathbf{b}) = 5$, the system has a unique solution.

(c) Let $r < n$ and $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are all zeros. In this case, r unknowns, $x_1, x_2, \ldots, x_r$ can be determined in terms of the remaining $(n-r)$ unknowns $x_{r+1}, x_{r+2}, \ldots, x_n$ by solving the rth equation for x_r, $(r-1)$th equation for x_{r-1} and so on. In this case, we obtain an $(n-r)$ parameter family of solutions, that is infinitely many solutions.

Remark 14

(a) We do not, normally use column elementary operations in solving the linear system of equations. When we interchange two columns, the order of the unknowns in the given system of equations is also changed. Keeping track of the order of unknowns is quite difficult.

(b) Gauss elimination method may be written as

$$(\mathbf{A} \mid \mathbf{b}) \xrightarrow[\text{row operations}]{\text{Elementary}} (\mathbf{B} \mid \mathbf{c}).$$

The matrix **B** is the row echelon form of the matrix **A** and **c** is the new right hand side column vector. We obtain the solution vector (if it exists) using the back substitution method.

(c) If **A** is a square matrix of order n, then **B** is an upper triangular matrix of order n.

(d) Gauss elimination method can be used to solve p systems of the form $\mathbf{Ax} = \mathbf{b}_1$, $\mathbf{Ax} = \mathbf{b}_2, \ldots, \mathbf{Ax} = \mathbf{b}_p$ which have the same coefficient matrix but different right hand side column vectors. We form the augmented matrix as $(\mathbf{A} \mid \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p)$, which has m rows and $(n+p)$ columns. Using the elementary row operations, we obtain the row equivalent system $(\mathbf{B} \mid \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p)$, where **B** is the row echelon form of **A**. Now, we solve the systems $\mathbf{Bx} = \mathbf{c}_1$, $\mathbf{Bx} = \mathbf{c}_2, \ldots, \mathbf{Bx} = \mathbf{c}_p$, using the back substitution method.

Remark 15

(a) If at any stage of elimination, the pivot element becomes zero, then we interchange this row with any other row below it such that we obtain a non-zero pivot element. We normally choose the row such that the pivot element becomes largest in magnitude.

(b) For an $n \times n$ system, we require $(n-1)$ stages of elimination. It is possible to compute the total number of additions, subtractions, multiplications and divisions. This number is called the *operation count* of the method. The operation count of the Gauss elimination method for solving an $n \times n$ system is $n(n^2 + 3n - 1)/3$. For large n, the operation count is approximately $n^3/3$.

Example 3.28 Solve the following systems of equations (if possible) using Gauss elimination method.

(i) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$,

(iii) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

Solution We write the augmented matrix and reduce it to row echelon form by applying elementary row operations.

$$\textbf{(i) } (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{array}\right] \begin{matrix} R_2 - R_1/2 \\ R_3 + R_1/2 \end{matrix} \approx \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 5/2 & -3/2 & 4 \end{array}\right] R_3 + 5R_2/3$$

$$\approx \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 0 & 8/3 & -8/3 \end{array}\right].$$

Using the back substitution method, we obtain the solution as

$$\frac{8}{3}z = -\frac{8}{3}, \text{ or } z = -1,$$

$$-\frac{3}{2}y + \frac{5}{2}z = -4, \text{ or } y = 1,$$

$$2x + y - z = 4, \text{ or } x = 1.$$

Therefore, the system of equations has the unique solution $x = 1, y = 1, z = -1$.

$$\textbf{(ii) } (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 4 & -2 & 3 & 3 \end{array}\right] \begin{matrix} R_2 - R_1/2 \\ R_3 - 2R_1 \end{matrix} \approx \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & -1 & 1/2 & -1/2 \\ 0 & -2 & 1 & -3 \end{array}\right] R_3 - 2R_2 \approx \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & -1 & 1/2 & -1/2 \\ 0 & 0 & 0 & -2 \end{array}\right].$$

We find that rank (**A**) = 2 and rank (**A** | **b**) = 3. Therefore, the system of equations has no solution.

$$\textbf{(iii) } (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 5 & -2 & 2 & 5 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix} \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] R_3 - R_2 \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system is consistent and has infinite number of solutions. We find that the last equation is satisfied for all values of x, y, z. From the second equation, we get $3y - 3z = 0$, or $y = z$. From the first equation, we get $x - y + z = 1$, or $x = 1$. Therefore, we obtain the solution $x = 1$, $y = z$ and z is arbitrary.

Example 3.29 Solve the following system of equations using Gauss elimination method.

(i) $4x - 3y - 9z + 6w = 0$
$2x + 3y + 3z + 6w = 6$
$4x - 21y - 39z - 6w = -24$,

(ii) $x + 2y - 2z = 1$
$2x - 3y + z = 0$
$5x + y - 5z = 1$
$3x + 14y - 12z = 5$.

Solution We have

$$\textbf{(i) } (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 2 & 3 & 3 & 6 & 6 \\ 4 & -21 & -39 & -6 & -24 \end{array}\right] \begin{matrix} R_2 - R_1/2 \\ R_3 - R_1 \end{matrix} \approx \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 0 & 9/2 & 15/2 & 3 & 6 \\ 0 & -18 & -30 & -12 & -24 \end{array}\right] R_3 + 4R_2$$

$$= \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 0 & 9/2 & 15/2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system of equations is consistent and has infinite number of solutions. Choose w as arbitrary. From the second equation, we obtain

$$\frac{9}{2}y + \frac{15}{2}z = 6 - 3w, \text{ or } y = \frac{2}{9}\left(6 - 3w - \frac{15}{2}z\right) = \frac{1}{3}(4 - 5z - 2w).$$

From the first equation, we obtain

$$4x = 3y + 9z - 6w = 4 - 5z - 2w + 9z - 6w = 4 + 4z - 8w$$

or

$$x = 1 + z - 2w.$$

Thus, we obtain a two parameter family of solutions

$$x = 1 + z - 2w \text{ and } y = (4 - 5z - 2w)/3$$

where z and w are arbitrary.

$$\textbf{(ii) } (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & -3 & 1 & 0 \\ 5 & 1 & -5 & 1 \\ 3 & 14 & -12 & 5 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \\ R_4 - 3R_1 \end{matrix} \approx \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & -9 & 5 & -4 \\ 0 & 8 & -6 & 2 \end{array}\right] \begin{matrix} R_3 - 9R_2/7 \\ R_4 + 8R_2/7 \end{matrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & 0 & -10/7 & -10/7 \\ 0 & 0 & -2/7 & -2/7 \end{array}\right] R_4 - R_3/5 \approx \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & 0 & -10/7 & -10/7 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The last equation is satisfied for all values of x, y, z. From the third equation, we obtain $z = 1$. Back substitution gives $y = 1$, $x = 1$. Hence, the system of equation has a unique solution $x = 1$, $y = 1$ and $z = 1$. Since $R_4 = (24R_1 - 7R_2 + R_3)/5$, the last equation is redundant.

3.4.5 Gauss-Jordan Method

In this method, we perform elementary row transformations on the augmented matrix **[A | b]**, where **A** is a square matrix, and reduce it to the form

$$[\mathbf{A} \mid \mathbf{b}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \mathbf{c}]$$

Where **I** is the identity matrix and **c** is the solution vector. This reduction is equivalent to finding the solution as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. The first step is same as in the Gauss elimination method. From second step onwards, we make elements below and above the pivot as zeros, using elementary row transformations. Finally, we divide each row by its pivot to obtain the form **[I | c]**. Alternately, at every step, the pivot can be made as 1 before elimination. Then, **c** is the solution vector.

This method is more expensive (larger operation count) than the Gauss elimination. Hence, we do not normally use the Gauss-Jordan method form finding the solution of a system. However, this method is very useful for finding the inverse $(\mathbf{A}^{-1})$ of a matrix **A**. We consider the augmented matrix **[A | I]** and reduce it to the form

$$[\mathbf{A} \mid \mathbf{I}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \mathbf{A}^{-1}]$$

using elementary row transformations. If we are solving the system of equations (3.28), then we have $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, and the matrix multiplication in the right hand side gives the solution vector.

Remark 16

If any pivot element at any stage of elimination becomes zero, then we interchange rows as in the Gauss elimination method.

Example 3.30 Using the Gauss-Jordan method, solve the system of equation $\mathbf{A}\,\mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

Solution We perform elementary row transformations on the augmented matrix and reduce it the form **[I | C]**. We get

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & -3 & 4 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -5 & 4 \\ 0 & 2 & 0 & 1 \end{array}\right] R_2/3$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 2 & 0 & 1 \end{array}\right] \begin{matrix} R_1 + R_2 \\ R_3 - 2R_2 \end{matrix} \approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 10/3 & -5/3 \end{array}\right] R_3/(10/3)$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \begin{matrix} R_1 + 2R_3/3 \\ R_2 + 5R_3/3 \end{matrix} \approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right].$$

Hence, the solution vector is

$$\mathbf{x} = [1 \quad 1/2 \quad -1/2]^T.$$

Example 3.31 Using Gauss-Jordan method, find the inverse of the matix $\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

Solution We have

$$(\mathbf{A} \mid \mathbf{I}) = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right].$$

The pivot element a_{11} is -1. We make it 1 by multiplying the first row by -1. Therefore,

$$(\mathbf{A} \mid \mathbf{I}) \approx \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_2 - 3R_1 \\ R_3 + R_1 \end{matrix} \approx \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] R_2/2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{matrix} R_1 + R_2 \\ R_3 - 2R_2 \end{matrix} \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] (-R_3)/5$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 3/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right] \begin{matrix} R_1 - 3R_3/2 \\ R_3 - 7R_3/2 \end{matrix} \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/10 & 2/10 & 3/10 \\ 0 & 1 & 0 & -13/10 & -2/10 & 7/10 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right].$$

Hence, $$\mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}.$$

3.4.6 Homogeneous System of Linear Equations

Consider the homogeneous system of equations

$$\mathbf{A x} = \mathbf{0} \tag{3.34}$$

where **A** is an $m \times n$ matrix. The homogenous system is always consistent since $\mathbf{x} = \mathbf{0}$ (trivial solution) is always a solution. In this case, rank (**A**) = rank (**A** | **0**). Therefore, for the homogeneous system to have a non-trivial solution, we require that rank (**A**) < n. If rank (**A**) = $r < n$, we obtain an $(n - r)$ parameter family of solutions which form a vector space of dimension $(n - r)$ as $(n - r)$ parameters can be chosen arbitrarily.

The solution space of the homogeneous system is called the *null space* and its dimension is called the *nullity* of **A**. Therefore, we obtain the result

$$\text{rank }(\mathbf{A}) + \text{nullity }(\mathbf{A}) = n \text{ (see Theorem 3.5).}$$

Remark 17

(a) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of a linear homogeneous system, then $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ is also a solution of the homogenous system for any scalars α, β. This result does not hold for non-homogenous systems.

(b) A homogeneous system of m equations in n unknowns and $m < n$, always possesses a non-trivial solution.

Theorem 3.8 If a non-homogeneous system of linear equations $\mathbf{Ax} = \mathbf{b}$ has solutions, then all these solutions are of the form $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$ where $\mathbf{x}_0$ is any fixed solution of $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x}_h$ is any solution of the corresponding homogeneous system.

Proof Let **x** be any solution and $\mathbf{x}_0$ be any fixed solution of $\mathbf{Ax} = \mathbf{b}$. Therefore, we have

$$\mathbf{Ax} = \mathbf{b} \quad \text{and} \quad \mathbf{Ax}_0 = \mathbf{b}.$$

Subtracting, we get

$$\mathbf{Ax} - \mathbf{Ax}_0 = \mathbf{0}, \quad \text{or} \quad \mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{0}.$$

Thus, the difference $\mathbf{x} - \mathbf{x}_0$ between any solution **x** of $\mathbf{Ax} = \mathbf{b}$ and any fixed solution $\mathbf{x}_0$ of $\mathbf{Ax} = \mathbf{b}$ is a solution of the homogeneous system $\mathbf{Ax} = \mathbf{0}$, say $\mathbf{x}_h$. Hence, the result.

Remark 18

If the non-homogeneous system $\mathbf{Ax} = \mathbf{b}$ where **A** is an $m \times n$ matrix $(m \geq n)$ has a unique solution, that is rank (**A**) = n, then the corresponding homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution, that is $\mathbf{x}_h = \mathbf{0}$.

Example 3.32 Solve the following homogeneous system of equation $\mathbf{Ax} = \mathbf{0}$, where **A** is given by

(i) $\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, (iii) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & -6 & 1 \end{bmatrix}$.

Find the rank (**A**) and nullity (**A**).

Solution We write the augmented matrix (**A** | **0**) and reduce it to row echelon form.

(i) $(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 0 \end{array}\right] \begin{matrix} R_2 - R_1/2 \\ R_3 - 3R_1/2 \end{matrix} = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -3/2 & 0 \\ 0 & 1/2 & 0 \end{array}\right] R_3 + R_2/3 = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$.

Since, rank (**A**) = 2 = number of unknowns, the system has only a trivial solution. Hence, nullity (**A**) = 0.

(ii) $(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -3 & 4 & 0 \end{array}\right] R_3 - 3R_2 = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$.

Since rank (**A**) = 3 = number of unknowns, the homogeneous system has only a trivial solution. Therefore, nullity (**A**) = 0.

(iii) $(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 2 & 3 & 1 & 4 & 0 \\ 3 & 2 & -6 & 1 & 0 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & -1 & -3 & -2 & 0 \end{array}\right] R_3 + R_2 = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Therefore, rank (**A**) = 2 and the number of unknowns is 4. Hence, we obtain a two parameter family of solutions as $x_2 = -3x_3 - 2x_4$, $x_1 = -x_2 + x_3 - x_4 = 4x_3 + x_4$, where x_3 and x_4 are arbitrary. Therefore, nullity (**A**) = 2.

Exercise 3.3

Using the elementary row operations, determine the ranks of the following matrices.

1. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 2 \\ 5 & -5 & 11 \end{bmatrix}$.

3. $\begin{bmatrix} 2 & 1 & -2 \\ -1 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$.

4. $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & 4 & -13 & -5 \end{bmatrix}$.

5. $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 3 & -1 \\ 8 & 13 & 14 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 7 & 11 & 15 & 19 \\ 9 & 15 & 21 & 27 \end{bmatrix}$.

8. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

9. $\begin{bmatrix} 2 & 0 & -1 & 0 \\ 4 & 1 & 0 & 5 \\ 0 & 1 & 3 & 6 \\ 6 & 1 & -2 & 6 \end{bmatrix}$.

10. $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$

Using the elementary column operations, determine the rank of the following matrices

11. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix}$ **12.** $\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ **13.** $\begin{bmatrix} 1 & 2 & 3 & 2 \\ -1 & 1 & 3 & 5 \\ 2 & 3 & 4 & 5 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$ **15.** $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$

Determine whether the following set of vectors is linearly independent. Find also its dimension

16. {(3, 2, 4), (1, 0, 2), (1, −1, −1)}.

17. {(2, 2, 1), (1, −1, 1), (1,0,1)}

18. {(2, 1, 0), (1, −1, 1), (4, 1, 2), (2, −3, 3)}.

19. {(2, 2, 1), (2, i, −1), (1 + i, − i, 1)}

20. {(1, 1, 1), (i, i, i), (1 + i, −1 −i, i)}

21. {(1, 1, 1, 1), (−1, 1, 1, −1), (1, 0, 1, 1), (1, 1, 0,1)}.

22. {(1, 2, 3, 1), (2, 1, −1, 1), (4, 5, 5, 3), (5, 4, 1, 3)}.

23. {(1, 2, 3, 4), (0, 1, −1, 2), (1, 4, 1, 8), (3, 7, 8, 14)}.

24. {(1, 1, 0, 1), (1, 1, 1, 1), (4, 4, 1, 1), (1, 0, 0, 1)}.

25. {(2, 2, 0, 2), (4, 1, 4, 1), (3, 0, 4, 0)}.

Determine which of the following systems are consistent and find all the solutions for the consistent system

26. $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$. **27.** $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. **28.** $\begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 3 \end{bmatrix}$.

29. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & -9 & 2 \\ 5 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$. **30.** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \\ 22 \end{bmatrix}$. **31.** $\begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & -3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

32. $\begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$. **33.** $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. **34.** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$.

35. $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Find all the solutions of the following homogeneous systems **Ax** = **0**, where **A** is given as the following.

36. $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}$. **37.** $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & -7 \\ -1 & -2 & 11 \end{bmatrix}$. **38.** $\begin{bmatrix} 3 & -11 & 5 \\ 4 & 1 & -10 \\ 4 & 9 & -6 \end{bmatrix}$

39. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 6 & 12 \end{bmatrix}$. **40.** $\begin{bmatrix} 2 & -1 & -3 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & -7 & -13 & -1 \\ -1 & 5 & 9 & 1 \end{bmatrix}$. **41.** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$.

42. $\begin{bmatrix} 3 & 1 & 1 & 4 \\ 0 & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{bmatrix}$. **43.** $\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & -2 & -3 \\ 3 & 0 & -5 & -1 \\ 5 & -1 & -7 & -4 \end{bmatrix}$. **44.** $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$.

45. $\begin{bmatrix} 1 & 1 & -2 & -1 \\ 2 & 1 & 1 & -2 \\ 3 & 2 & -1 & -3 \\ 4 & 2 & 2 & -4 \end{bmatrix}$.

Using the Gauss-Jordan method, find the inverses of the following matrices.

46. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. **47.** $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix}$. **48.** $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

49. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$. **50.** $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 4 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

3.5 Eigenvalue Problems

Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n. The matrix **A** may be singular or non-singular. Consider the homogeneous system of equations

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \quad \text{or} \quad (\mathbf{A} - \lambda \mathbf{I})\, \mathbf{x} = \mathbf{0} \tag{3.35}$$

where λ is a scalar and **I** is an identity matrix of order n. The homogeneous system of equations (3.35) always has a trivial solution. We need to find values of λ for which the homogeneous system (3.35) has non-trivial solutions. The values of λ for which non-trivial solutions of the homogeneous system (3.35) exist, are called the *eigenvalues* or the *characteristic values* of A and the corresponding non-trivial solution vectors **x** are called the *eigenvectors* or the *characteristic vectors* of A. If **x** is a non-trivial solution of the homogeneous system (3.35), then α **x**, where α is any constant is also a solution of the homogeneous system. Hence, an eigenvector is unique only upto a constant multiple. The problem of determining the eigenvalues and the corresponding eigenvectors of a square matrix A is called an *eigenvalues problem*.

3.5.1 Eigenvalues and Eigenvectors

If the homogeneous system (3.35) has a non-trivial solution, then the rank of the coefficient matrix $(\mathbf{A} - \lambda\mathbf{I})$ is less than n, that is the coefficient matrix must be singular. Therefore,

$$det\ (\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0.. \tag{3.36}$$

Expanding the determinant given in Eq. (3.36), we obtain a polynomial of degree n in λ, which is of the form

$$P_n(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| = (-1)^n [\lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} - \dots + (-1)^n c_n] = 0\ .$$

or

$$\lambda^n - c_1\lambda^{n-1} + c_2 \lambda^{n-2} - \dots + (-1)^n c_n = 0. \tag{3.37}$$

where $c_1, c_2, \dots, c_n$ can be expressed in terms of the elements a_{ij} of the matrix **A**. This equation is called the *characteristic equation* of the matrix **A**. The polynomial equation $P_n(\lambda) = 0$ has n roots which can be real or complex, simple or repeated. The roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of the polynomial equation $P_n(\lambda) = 0$ are called the *eigenvalues*. By using the relation between the roots and the coefficients, we can write

$$\begin{aligned} &\lambda_1 + \lambda_2 + \dots + \lambda_n = c_1 = a_{11} + a_{22} + \dots + a_{nn} \\ &\lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{n-1}\lambda_n = c_2 \\ &\vdots \\ &\lambda_1\lambda_2 \dots \lambda_n = c_n. \end{aligned} \tag{3.38}$$

If we set $\lambda = 0$ in Eq. (3.36), then we get

$$|\mathbf{A}| = (-1)^{2n} c_n = c_n = \lambda_1\lambda_2 \dots \lambda_n.$$

Therefore, we get

sum of eigenvalues = trace (**A**), and product of eigenvalues = |**A**|.

The set of the eigenvalues is called the *spectrum* of **A** and the largest eigenvalue in magnitude is called the *spectral radius* of **A** and is denoted by $\rho(\mathbf{A})$. If $|\mathbf{A}| = 0$, that is the martix is singular, then from Eq. (3.38), we find that atleast one of the eigenvalues must be zero. Conversely, if one of the eigenvalues is zero, then $|\mathbf{A}| = 0$. Note that if **A** is a diagonal or an upper triangular or a lower triangular matrix, then the diagonal elements of the matrix **A** are the eigenvalues of **A**.

After determining the eigenvalues λ_i's, we solve the homogeneous system $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x} = \mathbf{0}$ for each λ_i, $i = 1, 2, \dots, n$ to obtain the corresponding eigenvectors.

Properties of eigenvalues and eigenvectors

Let λ be an eigenvalue of A and **x** be its corresponding eigenvector. Then, we have the following results.

1. α **A** has eigenvalue $\alpha\lambda$ and the corresponding eigenvector is **x**.

$$\mathbf{Ax} = \lambda\mathbf{x} \Rightarrow \alpha\,\mathbf{Ax} = (\alpha\lambda)\mathbf{x}.$$

2. $\mathbf{A}^m$ has eigenvalue λ^m and the corresponding eigenvector is **x** for any positive interger m. Pre-multiplying both sides of $\mathbf{Ax} = \lambda\mathbf{x}$ by **A**, we get

$$\mathbf{A\,Ax} = \mathbf{A}\lambda\mathbf{x} = \lambda\mathbf{Ax} = \lambda(\lambda\mathbf{x}) \text{ or } \mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}.$$

Therefore, $\mathbf{A}^2$ has the eigenvalue λ^2 and the corresponding eigenvector is **x**. Pre-multiplying successively m times, we obtain the result.

3. $\mathbf{A} - k\mathbf{I}$ has the eigenvalue $\lambda - k$, for any scalar k and the corresponding eigenvector is **x**.

$$\mathbf{Ax} = \lambda\mathbf{x} \Rightarrow \mathbf{Ax} - k\,\mathbf{Ix} = \lambda\mathbf{x} - k\mathbf{x}$$

or

$$(\mathbf{A} - k\mathbf{I})\mathbf{x} = (\lambda - k)\mathbf{x}.$$

4. $\mathbf{A}^{-1}$(if it exists) has the eigenvalue $1/\lambda$ and the corresponding eigenvector is **x**. Pre-multiplying both sides of $\mathbf{Ax} = \lambda\mathbf{x}$ by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{Ax} = \lambda\mathbf{A}^{-1}\mathbf{x} \quad \text{or} \quad \mathbf{A}^{-1}\mathbf{x} = (1/\lambda)\mathbf{x}.$$

5. $(\mathbf{A} - k\mathbf{I})^{-1}$ has the eigenvalue $1/(\lambda - k)$ and the corresponding eigenvector is **x** for any scalar k.
6. **A** and $\mathbf{A}^T$ have the same eigenvalues (since a determinant can be expanded by rows or by columns) but different eigenvectors, (see Example 3.41).
7. For a real matrix **A**, if $\alpha + i\beta$ is an eigenvalue, then its conjugate $\alpha - i\beta$ is also an eigenvalue (since the characteristic equation has real coefficients). When the matrix **A** is complex, this property does not hold.

We now present an important result which gives the relationship of a matrix **A** and its characteristic equation.

Theorem 3.9 (Cayley-Hamilton theorem) Every square matrix **A** satisfies its own characteristic equation

$$\mathbf{A}^n - c_1\mathbf{A}^{n-1} + \dots + (-1)^{n-1} c_{n-1}\mathbf{A} + (-1)^n c_n \mathbf{I} = \mathbf{0}. \tag{3.39}$$

Proof The cofactors of the elements of the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ are polynomials in λ of degree $(n-1)$ or less. Therefore, the elements of the adjoint matrix (transpose of the cofactor matrix) are also polynomials in λ of degree $(n-1)$ or less. Hence, we can express the adjoint matrix as a polynomial in λ whose coefficients $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$ are square matrices of order n having elements as functions of the elements of the matrix **A**. Thus, we can write

$$adj(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \ldots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n.$$

We also have

$$(\mathbf{A} - \lambda\mathbf{I})\, adj\,(\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}|\,\mathbf{I}.$$

Therefore, we can write for any λ

$$(\mathbf{A} - \lambda\mathbf{I})(\mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \ldots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n)$$
$$= \lambda^n\mathbf{I} - c_1\lambda^{n-1}\mathbf{I} + \ldots + (-1)^{n-1}c_{n-1}\lambda\mathbf{I} + (-1)^n c_n\mathbf{I}$$

Comparing the coefficients of various powers of λ, we obtain

$$-\mathbf{B}_1 = \mathbf{I}$$
$$\mathbf{A}\mathbf{B}_1 - \mathbf{B}_2 = c_1\mathbf{I}$$
$$\mathbf{A}\mathbf{B}_2 - \mathbf{B}_3 = c_2\mathbf{I}$$
$$\ldots$$
$$\mathbf{A}\mathbf{B}_{n-1} - \mathbf{B}_n = (-1)^{n-1}c_{n-1}\mathbf{I}$$
$$\mathbf{A}\mathbf{B}_n = (-1)^n c_n\mathbf{I}.$$

Pre-multiplying these equations by $\mathbf{A}^n, \mathbf{A}^{n-1}, \ldots, \mathbf{A}, \mathbf{I}$ respectively and adding, we get

$$\mathbf{A}^n - c_1\mathbf{A}^{n-1} + \ldots + (-1)^{n-1}c_{n-1}\mathbf{A} + (-1)^n c_n\mathbf{I} = \mathbf{0}$$

which proves the theorem.

Remark 19

(a) We can use Eq. (3.39) to find $\mathbf{A}^{-1}$ (if it exists) in terms of the powers of the matrix **A**.

Pre-multiplying both sides in Eq. (3.39) by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{A}^n - c_1\mathbf{A}^{-1}\mathbf{A}^{n-1} + \ldots + (-1)^{n-1}c_{n-1}\mathbf{A}^{-1}\mathbf{A} + (-1)^n c_n\mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$$

or
$$\mathbf{A}^{-1} = -\frac{(-1)^n}{c_n}[\mathbf{A}^{n-1} - c_1\mathbf{A}^{n-2} + \ldots + (-1)^{n-1}c_{n-1}\mathbf{I}] \tag{3.40}$$

(b) We can use Eq.(3.39) to obtain $\mathbf{A}^n$ in terms of lower powers of **A** as

$$\mathbf{A}^n = c_1\mathbf{A}^{n-1} - c_2\mathbf{A}^{n-2} + \ldots + (-1)^{n-1}c_n\mathbf{I}. \tag{3.41}$$

Example 3.33 Verify Cayley-Hamilton theorem for the martrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Also, (i) obtain $\mathbf{A}^{-1}$ and $\mathbf{A}^3$, (ii) find eigenvalues of **A**, $\mathbf{A}^2$ and verify that eigenvalues of $\mathbf{A}^2$ are squares of those of **A**, (iii) find the spectral radius of **A**.

Solution The characteristic equation of **A** is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\{(1-\lambda)^2 - 4\} - 2\{-(1-\lambda) - 2\}$$
$$= (1-\lambda)(\lambda^2 - 2\lambda - 3) - 2(\lambda - 3) = -\lambda^3 + 3\lambda^2 - \lambda + 3 = 0.$$

Now,
$$\mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}.$$

We have

$$-\mathbf{A}^3 + 3\mathbf{A}^2 - \mathbf{A} + 3\mathbf{I} = -\begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix} + 3\begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}. \tag{3.42}$$

Hence, **A** satisfies the characteristic equation $-\lambda^3 + 3\lambda^2 - \lambda + 3 = 0$.

(i) From Eq. (3.42), we get

$$\mathbf{A}^{-1} = \frac{1}{3}[\mathbf{A}^2 - 3\mathbf{A} + \mathbf{I}] = \frac{1}{3}\left[\begin{pmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 6 & 0 \\ -3 & 3 & 6 \\ 3 & 6 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}.$$

From Eq. (3.42), we get

$$\mathbf{A}^3 = 3\mathbf{A}^2 - \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} -3 & 12 & 12 \\ 0 & 9 & 12 \\ 0 & 18 & 15 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}.$$

(ii) Eigenvalues of $\mathbf{A}$ are the roots of

$$\lambda^3 - 3\lambda^2 + \lambda - 3 = 0 \text{ or } (\lambda - 3)(\lambda^2 + 1) = 0 \text{ or } \lambda = 3, i, -i.$$

The characteristic equation of $\mathbf{A}^2$ is given by

$$\begin{vmatrix} -1-\lambda & 4 & 4 \\ 0 & 3-\lambda & 4 \\ 0 & 6 & 5-\lambda \end{vmatrix} = (-1-\lambda)\,[(3-\lambda)(5-\lambda) - 24] = 0$$

or $\quad (\lambda + 1)(\lambda^2 - 8\lambda - 9) = 0$ or $(\lambda + 1)(\lambda - 9)(\lambda + 1) = 0.$

The eigenvalues of $\mathbf{A}^2$ are $9, -1, -1$ which are the squares of the eigenvalues of $\mathbf{A}$.

(iii) The spectral radius of $\mathbf{A}$ is given by

$$\rho(\mathbf{A}) = \text{largest eigenvalue in magnitude} = \max_i |\lambda_i| = 3.$$

Example 3.34 If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then show that $\mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}$ for $n \geq 3$. Hence, find $\mathbf{A}^{50}$.

Solution The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1) = 0, \quad \text{or} \quad \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

Using Cayley-Hamilton theorem, we get

$$\mathbf{A}^3 - \mathbf{A}^2 - \mathbf{A} + \mathbf{I} = \mathbf{0}, \quad \text{or} \quad \mathbf{A}^3 - \mathbf{A}^2 = \mathbf{A} - \mathbf{I}.$$

Pre-multiplying both sides successively by $\mathbf{A}$, we obtain

$$\mathbf{A}^3 - \mathbf{A}^2 = \mathbf{A} - \mathbf{I}$$
$$\mathbf{A}^4 - \mathbf{A}^3 = \mathbf{A}^2 - \mathbf{A}$$
$$\cdots$$
$$\mathbf{A}^{n-1} - \mathbf{A}^{n-2} = \mathbf{A}^{n-3} - \mathbf{A}^{n-4}$$
$$\mathbf{A}^n - \mathbf{A}^{n-1} = \mathbf{A}^{n-2} - \mathbf{A}^{n-3}.$$

Adding these equations, we get

$$\mathbf{A}^n - \mathbf{A}^2 = \mathbf{A}^{n-2} - \mathbf{I}, \quad \text{or} \quad \mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}, \quad n \geq 3.$$

Using this equation recursively, we get

$$\mathbf{A}^n = (\mathbf{A}^{n-4} + \mathbf{A}^2 - \mathbf{I}) + \mathbf{A}^2 - \mathbf{I} = \mathbf{A}^{n-4} + 2(\mathbf{A}^2 - \mathbf{I})$$
$$= (\mathbf{A}^{n-6} + \mathbf{A}^2 - \mathbf{I}) + 2(\mathbf{A}^2 - \mathbf{I}) = \mathbf{A}^{n-6} + 3(\mathbf{A}^2 - \mathbf{I})$$
$$\cdots$$
$$= \mathbf{A}^{n-(n-2)} + \frac{1}{2}(n-2)(\mathbf{A}^2 - \mathbf{I}) = \frac{n}{2}\mathbf{A}^2 - \frac{1}{2}(n-2)\mathbf{I}.$$

Substituting $n = 50$, we get

$$\mathbf{A}^{50} = 25\mathbf{A}^2 - 24\mathbf{I} = 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}.$$

Example 3. 35 Find the eigenvalues and the corresponding eigenvectors of the following matrices.

(i) $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, (ii) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, (iii) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$.

Solution

(i) The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 3\lambda - 10 = 0, \quad \text{or} \quad \lambda = -2, 5.$$

Corresponding to the eigenvalue $\lambda = -2$, we have

$$(\mathbf{A} + 2\mathbf{I})\,\mathbf{x} = \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } 3x_1 + 4x_2 = 0 \text{ or } x_1 = -\frac{4}{3}x_2.$$

Hence, the eigenvector $\mathbf{x}$ is given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_2/3 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}.$$

Since an eigenvector is unique upto a constant multiple, we can take the eigenvector as $[-4, 3]^T$.

Corresponding to the eigenvalue $\lambda = 5$, we have

$$(\mathbf{A} - 5\mathbf{I})\,\mathbf{x} = \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } x_1 - x_2 = 0, \text{ or } x_1 = x_2.$$

Therefore, the eigenvalue is given by $\mathbf{x} = (x_1, x_2)^T = x_1(1, 1)^T$ or $(1, 1)^T$.

(ii) The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 - 2\lambda + 2 = 0, \quad \text{or} \quad \lambda = 1 \pm i.$$

Corresponding to the eigenvalue $\lambda = 1 + i$, we have

$$[\mathbf{A} - (1 + i)\,\mathbf{I}]\,\mathbf{x} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or
$$-ix_1 + x_2 = 0 \text{ and } -x_1 - ix_2 = 0.$$

Both the equations reduce to $-x_1 - ix_2 = 0$. Choosing $x_2 = 1$, we get $x_1 = -i$. Therefore, the eigenvector is $\mathbf{x} = [-i, 1]^T$.

Corresponding to the eigenvalue $\lambda = 1 - i$, we have

$$[\mathbf{A} - (1 - i)\mathbf{I}]\,\mathbf{x} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or
$$ix_1 + x_2 = 0 \text{ and } -x_1 + ix_2 = 0.$$

Both the equations reduce to $-x_1 + ix_2 = 0$. Choosing $x_2 = 1$, we get $x_1 = i$. Therefore, the eigenvector is $x = [i, 1]^T$.

Remark 20

For a real matrix **A**, the eigenvalues and the corresponding eigenvectors can be complex.

(iii) The characteristic equation of **A** is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad (1-\lambda)(2-\lambda)(3-\lambda) = 0 \quad \text{or} \quad \lambda = 1, 2, 3.$$

Corresponding to the eigenvalue $\lambda = 1$, we have

$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} x_2 + x_3 = 0 \\ x_1 + x_3 = 0 \end{cases}$$

We obtain two equations in three unknowns. One of the variables x_1, x_2, x_3 can be chosen arbitrarily. Taking $x_3 = 1$, we obtain the eigenvector as $[-1, -1, 1]^T$.

Corresponding to the eigenvalue $\lambda = 2$, we have

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $x_1 = 0$, $x_3 = 0$ and x_2 arbitrary. Taking $x_2 = 1$, we obtain the eigenvector as $[0, 1, 0]^T$.

Corresponding to the eigenvalue $\lambda = 3$, we have

$$(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} x_1 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

Choosing $x_3 = 1$, we obtain the eigenvector as $[0, -1, 1]^T$.

Example 3.36 Find the eigenvalues and the corresponding eigenvectors of the following matrices.

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{(ii) } \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{(iii) } \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution In each of the above problems, we obtain the characteristic equation as $(1 - \lambda)^3 = 0$. Therefore, the eigenvalues are $\lambda = 1, 1, 1$, a repeated value. Since a 3×3 matrix has 3 eigenvalues, it is important to know, whether the given matrix has 3 linearly independent eigenvectors, or it has lesser number of linearly independent eigenvectors.

Corresponding to the eigenvalue $\lambda = 1$, we obtain the following eigenvectors.

(i)
$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_1 \text{ arbitrary.} \end{cases}$$

Choosing $x_1 = 1$, we obtain the solution as $[1, 0, 0]^T$.

Hence, **A** has only one independent eigenvector.

(ii)
$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{cases} x_2 = 0 \\ x_1, x_3 \text{ arbitrary.} \end{cases}$$

Taking $x_1 = 0$, $x_3 = 1$ and $x_1 = 1$, $x_3 = 0$, we obtain two linearly independent solutions

$$\mathbf{x}_1 = [0, 0, 1]^T, \quad \mathbf{x}_2 = [1, 0, 0]^T.$$

In this case **A** has two linearly independent eigenvectors.

(iii)
$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system is satisfied for arbitrary values of all the three variables. Hence, we obtain three linearly independent eigenvectors, which can be taken as

$$\mathbf{x}_1 = [1, 0, 0]^T, \quad \mathbf{x}_2 = [0, 1, 0]^T, \quad \mathbf{x}_3 = [0, 0, 1]^T.$$

We now state some important results regarding the relationship between the eigenvalues of a matrix and the corresponding linearly independent eigenvectors.

1. Eigenvectors corresponding to distinct eigenvalues are linearly independent.
2. If λ is an eigenvalue of multiplicity m of a square matix **A** of order n, then the number of linearly independent eigenvectors associated with λ is given by

$$p = n - r, \quad \text{where} \quad r = \text{rank } (\mathbf{A} - \lambda \mathbf{I}), \ 1 \le p \le m.$$

Remark 21

In Example 3.35, all the eigenvalues are distinct and therefore, the corresponding eigenvectors are linearly indenpendent. In Example 3.36, the eigenvalue $\lambda = 1$ is of multiplicity 3. We find that in

(i) Example 3.36(i), the rank of the matrix $\mathbf{A} - \mathbf{I}$ is 2 and we obtain one linearly independent eigenvector.

(ii) Example 3.36(ii), the rank of the matrix $\mathbf{A} - \mathbf{I}$ is 1 and we obtain two linearly independent eigenvectors.

(iii) Example 3.36(iii), the rank of the matrix $\mathbf{A} - \mathbf{I}$ is 0 and we obtain three linearly independent eigenvectors.

3.5.2 Similar and Diagonalizable Matrices

Similar matrices

Let **A** and **B** be square matrices of the same order. The matrix **A** is said to be similar to the matrix **B** if there exists an invertible matrix **P** such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P} \quad \text{or} \quad \mathbf{PA} = \mathbf{BP}. \tag{3.43}$$

Post-multiplying both sides in Eq. (3.43) by $\mathbf{P}^{-1}$, we get

$$\mathbf{PAP}^{-1} = \mathbf{B}.$$

Therefore, **A** is similar to **B** if and only if **B** is similar to **A**. The matrix **P** is called the *similarity matrix*. The transformation in Eq. (3.43) is called a *similarity transformation*. We now prove a result regarding eigenvalues of similar matrices.

Theorem 3.10 Similar matrices have the same characteristic equation (and hence the same eigenvalues). Further, if **x** is an eigenvector of **A** corresponding to the eigenvalue λ, then $\mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of **B** corresponding to the eigenvalue λ, where **P** is the similarity matrix.

Proof Let λ be an eigenvalue and **x** be the corresponding eigenvector of **A**. That is

$$\mathbf{Ax} = \lambda \mathbf{x}.$$

Pre-multiplying both sides by an invertible matrix $\mathbf{P}^{-1}$, we obtain

$$\mathbf{P}^{-1}\mathbf{Ax} = \lambda \mathbf{P}^{-1}\mathbf{x}.$$

Set $\mathbf{x} = \mathbf{Py}$. We get

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{y} = \lambda \mathbf{P}^{-1}\mathbf{P}\mathbf{y}, \quad \text{or} \quad (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\mathbf{y} = \lambda \mathbf{y} \quad \text{or} \quad \mathbf{B}\mathbf{y} = \lambda \mathbf{y}.$$

where $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. Therefore, **B** has the same eigenvalues as **A**, that is, the characterstic equation of **B** is same as the characteristic equation of **A**. Now, **A** and **B** are similar matrices. Therefore, similar matrices have the same characteristic equation (and hence the same eigenvalues). Also, $\mathbf{x} = \mathbf{Py}$, that is eigenvectors of **A** and **B** are related by $\mathbf{x} = \mathbf{Py}$ or $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$.

Remark 22

(a) Theorem 3.10 states that if two matrices are similar, then they have the same characterstic equation and hence the same eigenvalues. However, the converse of this theorem is not true. Two matrices which have the same characteristic equation need not always be similar.

(b) If **A** is similar to **B** and **B** is similar to **C**, then **A** is similar to **C**.

Let there be two invertible matrices **P** and **Q** such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P} \quad \text{and} \quad \mathbf{B} = \mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}.$$

Then $\quad \mathbf{A} = \mathbf{P}^{-1}\mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}\mathbf{P} = \mathbf{R}^{-1}\mathbf{C}\mathbf{R}, \quad \text{where} \quad \mathbf{R} = \mathbf{QP}.$

Example 3.37 Examine whether **A** is similar to **B**, where

$$\text{(i)} \ \mathbf{A} = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}, \quad \text{(ii)} \ \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Solution The given matrices are similar if there exists an invertible matrix **P** such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P} \quad \text{or} \quad \mathbf{PA} = \mathbf{BP}.$$

Let $\mathbf{P} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We shall determine a, b, c and d such that $\mathbf{PA} = \mathbf{BP}$ and then check whether **P** is non-singular.

$$\text{(i)} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 5a-2b & 5a \\ 5c-2d & 5c \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -3a+4c & -3b+4d \end{bmatrix}.$$

Equating the corresponding elements, we obtain the system of equations

$$\begin{aligned} 5a - 2b &= a + 2c, & \text{or} \quad & 4a - 2b - 2c = 0 \\ 5a &= b + 2d, & \text{or} \quad & 5a - b - 2d = 0 \\ 5c - 2d &= -3a + 4c, & \text{or} \quad & 3a + c - 2d = 0 \\ 5c &= -3b + 4d, & \text{or} \quad & 3b + 5c - 4d = 0. \end{aligned}$$

A solution to this system of equations is $a = 1$, $b = 1$, $c = 1$, $d = 2$.

Therefore, we get $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, which is a non-singular matrix. Hence, the matrices **A** and **B** are similar.

(ii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$

Equating the corresponding elements, we get

$$a = a + c, \quad b = b + d \quad \text{or} \quad c = d = 0$$

Therefore, we get $\mathbf{P} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, which is a singular matrix.

Since an invertible matrix **P** does not exist, the matrices **A** and **B** are not similar.

It can be verified that the eigenvalues of **A** are 1, 1 whereas the eigenvalues of **B** are 0, 2.

In practice, it is usually difficult to obtain a non-singular matrix **P** which satisfies the equation $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ for any two matrices **A** and **B**. However, it is possible to obtain the matrix **P** when **A** or **B** is a diagonal matrix. Thus, our interest is to find a similarity matrix **P** such that for a given matrix **A**, we have

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \quad \text{or} \quad \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{A}$$

where **D** is a diagonal matrix. If such a matrix exists, then we say that the matrix **A** is *diagonalizable*.

Diagonalizable matrices

A matrix **A** is diagonalizable, if it is similar to a diagonal matrix, that is there exists an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where **D** is a diagonal matrix. Since, similar matrices have the same eigenvalues, the diagonal elements of **D** are the eigenvalues of **A**. A necessary and sufficient condition for the existence of **P** is given in the following theorem.

Therorem 3.11 A square matrix **A** of order n is diagonalizable if and only if it has n linearly independent eigenvectors.

Proof We shall prove the case that if **A** has n linearly independent eigenvectors, then **A** is diagonalizable. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct) of the matrix **A** in the same order, that is the eigenvector $\mathbf{x}_j$ corresponds to the eigenvalue λ_j, $j = 1, 2, \ldots, n$. Let

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] \quad \text{and} \quad \mathbf{D} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

be the diagonal matrix with eigenvalues of **A** as its diagonal elements. The matrix **P** is called the *modal matrix* of **A** and **D** is called the *spectral matrix* of **A**. We have

$$\mathbf{AP} = \mathbf{A}[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = (\mathbf{Ax}_1, \mathbf{Ax}_2, \ldots, \mathbf{Ax}_n)$$
$$= (\lambda_1\mathbf{x}_1, \lambda_2\mathbf{x}_2, \ldots, \lambda_n\mathbf{x}_n) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)\,\mathbf{D} = \mathbf{PD}. \quad (3.44)$$

Since the columns of P are linearly independent, the rank of P is n and therefore the matrix P is invertible. Pre-multiplying both sides in Eq. (3.44) by $\mathbf{P}^{-1}$, we obtain

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^{-1}\mathbf{PD} = \mathbf{D} \quad (3.45)$$

which implies that **A** is similar to **D**. Therefore, the matrix of eigenvectors **P** reduces a matrix **A** to its diagonal form.

Post-multiplying both sides in Eq. (3.44) by $\mathbf{P}^{-1}$, we obtain

$$\mathbf{A} = \mathbf{PDP}^{-1}. \quad (3.46)$$

Remark 23

(a) A square matrix **A** of order n has always n linearly independent eigenvectors when its eigenvalues are distinct. The matrix may also have n linearly independent eigenvectors even when some eigenvalues are repeated (see Example 3.36(iii)). Therefore, there is no restriction imposed on the eigenvalues of the matrix **A** in Theorem 3.11.

(b) From Eq. (3.46), we obtain

$$\mathbf{A}^2 = \mathbf{AA} = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}^2\mathbf{P}^{-1}.$$

Repeating the pre-multiplication (post-multiplication) m times, we get

$$\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1} \text{ for any positive integer } m.$$

Therefore, if **A** is diagonalizable, so is $\mathbf{A}^m$.

(c) If **D** is a diagonal matrix of order n, and

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}, \text{ then } \mathbf{D}^m = \begin{bmatrix} \lambda_1^m & & & 0 \\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{bmatrix}$$

for any positive integer m. If $Q(\mathbf{D})$ is a polynomial in **D**, then we get

$$Q(\mathbf{D}) = \begin{bmatrix} Q(\lambda_1) & & & 0 \\ & Q(\lambda_2) & & \\ & & \ddots & \\ 0 & & & Q(\lambda_n) \end{bmatrix}.$$

Now, let a matrix **A** be diagonalizable. Then, we have

$$\mathbf{A} = \mathbf{PDP}^{-1} \quad \text{and} \quad \mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$$

for any positive integer m. Hence, we obtain

$$Q(\mathbf{A}) = \mathbf{P}Q(\mathbf{D})\mathbf{P}^{-1}$$

for any matrix polynomial $Q(\mathbf{A})$.

Example 3.38 Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

is diagonalizable. Hence, find $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. Then, obtain the matrix $\mathbf{B} = \mathbf{A}^2 + 5\mathbf{A} + 3\mathbf{I}$.

Solution The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0, \quad \text{or} \quad \lambda = 1, 2, 3.$$

Since the matrix $\mathbf{A}$ has three distinct eigenvalues, it has three linearly independent eigenvectors and hence it is diagonalizable.

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is the solution of the system

$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ The solution is } \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is the solution of the system

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ The solution is } \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is the solution of the system

$$(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ The solution is } \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Hence, the modal matrix is given by

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{P}^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

It can be verified that $\quad \mathbf{P}^{-1}\mathbf{AP} = \text{diag}(1, 2, 3)$.

We have $\mathbf{D} = \text{diag}(1, 2, 3)$, $\mathbf{D}^2 = \text{diag}(1, 4, 9)$.

Therefore, $\quad \mathbf{A}^2 + 5\mathbf{A} + 3\mathbf{I} = \mathbf{P}(\mathbf{D}^2 + 5\mathbf{D} + 3\mathbf{I})\mathbf{P}^{-1}$.

Now, $$\mathbf{D}^2 + 5\mathbf{D} + 3\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix}.$$

Hence, we obtain

$$\mathbf{A}^2 + 5\mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 25 & 8 & -8 \\ -18 & 9 & 18 \\ -2 & 8 & 19 \end{bmatrix}.$$

Example 3.39 Examine whether the matrix A, where A is given by

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}, \qquad \text{(ii) } \mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix},$$

is diagonalizable. If so, obtain the matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

Solution

(i) The characteristic equation of the matrix $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(2-\lambda)(2-\lambda) - 2] - [2 - 2(2-\lambda)] = (1-\lambda)(2-\lambda)(2-\lambda) = 0,$$

or $\lambda = 1, 2, 2$. We first find the eigenvectors corresponding to the repeated eigenvalue $\lambda = 2$. We have the system

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the rank of the coefficient matrix is 2, it has one linearly independent eigenvector. We obtain another linearly independent eigenvector corresponding to the eigenvalue $\lambda = 1$. Since the matrix $\mathbf{A}$ has only two linearly independedent eigenvectors, the matrix is not diagonalizable.

(ii) The characteristic equation of the matrix $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0 \text{ or } \lambda^3 + \lambda^2 - 21\lambda - 45 = 0, \text{ or } \lambda = 5, -3, -3.$$

Eigenvector corresponding to the eigenvalue $\lambda = 5$ is the solution of the system

$$(\mathbf{A} - 5\mathbf{I})\mathbf{x} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

A solution of this system is $[1, 2, -1]^T$.

Eigenvectors corresponding to $\lambda = -3$ are the solutions of the system

$$(\mathbf{A} + 3\mathbf{I})\mathbf{x} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } x_1 + 2x_2 - 3x_3 = 0.$$

The rank of the coefficient matrix is 1. Therefore, the system has two linearly independent eigenvectors. We use the equation $x_1 + 2x_2 - 3x_3 = 0$ to find two linearly independent eigenvectors. Taking $x_3 = 0$, $x_2 = 1$, we obtain the eigenvector $[-2, 1, 0]^T$ and taking $x_2 = 0$, $x_3 = 1$, we obtain the eigenvector $[3, 0, 1]^T$. The given 3×3 matrix has three linearly independent eigenvectors. Therefore, the matrix $\mathbf{A}$ is diagonalizable. The modal matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{P}^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 2 & -3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix}.$$

It can be verified that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \text{diag}(5, -3, -3)$.

Example 3.40 The eigenvectors of a 3×3 matrix $\mathbf{A}$ corresponding to the eigenvalues 1, 1, 3 are $[1, 0, -1]^T$, $[0, 1, -1]^T$ and $[1, 1, 0]^T$ respectively. Find the matrix $\mathbf{A}$.

Solution We have

$$\text{modal matrix } \mathbf{P} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \text{ and the spectral matrix } \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

We find that

$$\mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Therefore,

$$\mathbf{A} = \mathbf{P}\,\mathbf{D}\,\mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.5.3 Special Matrices

In this section, we define some special matrices and study the properties of the eigenvalues and eigenvectors of these matrices. These matrices have applications in many areas. We first give some definitions.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ be two vectors of dimension n in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then we define the following:

Inner Product (dot product) of vectors Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^n$. Then

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i \tag{3.47}$$

is called the *inner product* of the vectors $\mathbf{x}$ and $\mathbf{y}$ and is a scalar. The inner product is also denoted by $<\mathbf{x}, \mathbf{y}>$. In this case $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$. Note that $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{C}^n$, then the inner product of these vectors is defined as

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\,\bar{\mathbf{y}} = \sum_{i=1}^{n} x_i \bar{y}_i \quad \text{and} \quad \mathbf{y} \cdot \mathbf{x} = \mathbf{y}^T\bar{\mathbf{x}} = \sum_{i=1}^{n} y_i \bar{x}_i.$$

where $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are complex conjugate vectors of $\mathbf{x}$ and $\mathbf{y}$ respectively. Note that $\mathbf{x} \cdot \mathbf{y} = \overline{\mathbf{y} \cdot \mathbf{x}}$.

It can be easily verified that

$$(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$$

for any vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and scalars α, β.

Length (norm of a vector) Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

is called the *length* or the *norm* of the vector $\mathbf{x}$.

Unit vector The vector $\mathbf{x}$ is called a *unit vector* if $\|\mathbf{x}\| = 1$. If $\mathbf{x} \neq \mathbf{0}$, then vector $\mathbf{x}/\|\mathbf{x}\|$ is always a unit vector.

Orthogonal vectors The vectors $\mathbf{x}$ and $\mathbf{y}$ for which $\mathbf{x} \cdot \mathbf{y} = 0$ are said to be *orthogonal vectors.*

Orthonormal vectors The vectors $\mathbf{x}$ and $\mathbf{y}$ for which

$$\mathbf{x} \cdot \mathbf{y} = 0 \quad \text{and} \quad \|\mathbf{x}\| = 1, \|\mathbf{y}\| = 1$$

are called orthonormal vectors. If $\mathbf{x}$, $\mathbf{y}$ are any vectors and $\mathbf{x} \cdot \mathbf{y} = 0$, then $\mathbf{x}/\|\mathbf{x}\|$, $\mathbf{y}/\|\mathbf{y}\|$ are orthonormal. For example, the set of vectors

(i) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form an orthonormal set in $\mathbb{R}^3$.

(ii) $\begin{pmatrix} 3i \\ 4i \\ 0 \end{pmatrix}, \begin{pmatrix} -4i \\ 3i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1+i \end{pmatrix}$ form an orthogonal set in $\mathbb{C}^3$ and $\begin{pmatrix} 3i/5 \\ 4i/5 \\ 0 \end{pmatrix}, \begin{pmatrix} -4i/5 \\ 3i/5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ (1+i)/\sqrt{2} \end{pmatrix}$

form an orthonormal set in $\mathbb{C}^3$.

Orthonormal and unitary system of vectors Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n vectors in IR^n. Then, this set of vectors forms an *orthonormal system* of vectors, if

$$\mathbf{x}_i \cdot \mathbf{x}_j = \mathbf{x}_i^T \mathbf{x}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n vectors in $\mathbb{C}^n$. Then, this set of vectors forms an *unitary system* of vectors, if

$$\mathbf{x}_i \cdot \mathbf{x}_j = \mathbf{x}_i^T \bar{\mathbf{x}}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

In section 3.2.2, we have defined symmetric, skew-symmetric, Hermitian and skew-Hermitian matrices. We now define a few more special matrices.

Orthogonal matrices A real matrix **A** is *orthogonal* if $\mathbf{A}^{-1} = \mathbf{A}^T$. A simple example is

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

A linear transformation in which the matrix of transformation is an orthogonal matrix is called an *orthogonal transformation.*

Unitary matrices A complex matrix **A** is *unitary* if $\mathbf{A}^{-1} = (\bar{\mathbf{A}})^T$, or $(\bar{\mathbf{A}})^{-1} = \mathbf{A}^T$. If **A** is real, then unitary matrix is same as orthogonal matrix.

A linear transformation in which the matrix of transformation is a unitary matrix is called a *unitary tranformation.*

We note the following:

1. If **A** and **B** are Hermitian matrices, then $\alpha\mathbf{A} + \beta\mathbf{B}$ is also Hermitian for any real scalars α, β, since

$$\overline{(\alpha\mathbf{A} + \beta\mathbf{B})}^T = (\alpha\bar{\mathbf{A}} + \beta\bar{\mathbf{B}})^T = \alpha\bar{\mathbf{A}}^T + \beta\bar{\mathbf{B}}^T = \alpha\mathbf{A} + \beta\mathbf{B}.$$

2. Eigenvalues and eigenvectors of $\bar{\mathbf{A}}$ are the conjugates of the eigenvalues and eigenvectors of **A**, since

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \text{ gives } \bar{\mathbf{A}}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}.$$

3. The inverse of a unitary (orthogonal) matrix is unitary (orthogonal). We have $\mathbf{A}^{-1} = \bar{\mathbf{A}}^T$. Let $\mathbf{B} = \mathbf{A}^{-1}$. Then

$$\mathbf{B}^{-1} = \mathbf{A} = (\bar{\mathbf{A}}^T)^{-1} = [(\bar{\mathbf{A}})^{-1}]^T = [\overline{(\mathbf{A}^{-1})}]^T = \bar{\mathbf{B}}^T.$$

Diagonally dominant matrix A matrix $\mathbf{A} = (a_{ij})$ is said to be diagonally dominant, if

$$|a_{ii}| \geq \sum_{j=1, i \neq j}^{n} |a_{ij}|, \text{ for all } i.$$

The system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, is called a *diagonally dominant system*, if the above conditions are satisfied and the strict inequality is satisfied for at least one i. If the strict inequality is satisfied for all i, then it is called a *strictly diagonally dominant system.*

Permutation matrix A matrix **P** is called a *permutation matrix* if it has exactly one 1 in each row and column and all other elements are 0.

***Property A* of a matrix** Let **B** be a sparse matrix. Then, the matrix **B** is said to satisfy the *property A*, if and only if there exists a permutation matrix **P** such that

$$\mathbf{P}\mathbf{B}\mathbf{P}^T = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix},$$

where $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are diagonal matrices. The similarity transformation performs row interchanges followed by corresponding column interchanges in **B** such that $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ become diagonal matrices. The following procedure is a simple way of testing whether **B** can be reduced to the required form. It finds the locations of the non-zero elements and tests whether the interchanges of rows and corresponding interchanges of columns are possible to bring **B** to the required form. Let n be the order of the matrix **B** and $b_{ii} \neq 0$. Denote the set $U = \{1, 2, 3, \ldots, n\}$. Let there exist disjoint subsets U_1 and U_2 such that $U = U_1 \cup U_2$, where the suffixes of the non-zero off diagonal elements $b_{ik} \neq 0$, $i \neq k$, can be grouped as either ($i \in U_1$, $k \in U_2$) or ($i \in U_2$, $k \in U_1$). Then, the matrix **B** satisfies *property A.*

Consider, for example the matrix $\mathbf{B} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$

Let the permutation matrix be taken as $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

Then,

$$\mathbf{P}\mathbf{B}\mathbf{P}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \begin{matrix} -2 & 0 \\ 0 & -2 \end{matrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \end{bmatrix} & (-2) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

where A_{11} and A_{22} are diagonal matrices. Hence, **B** has *property A*. Note that the above similarity transformation is equivalent to interchanging rows 2 and 3, followed by an interchange of columns 2 and 3.

Now, $a_{ii} \neq 0$, $i = 1, 2, 3$. $a_{12} \neq 0$, $1 \in U_1$, $2 \in U_2$; $a_{21} \neq 0$, $2 \in U_2$, $1 \in U_1$; $a_{23} \neq 0$, $2 \in U_2$, $3 \in U_1$; $a_{32} \neq 0$, $3 \in U_1$, $2 \in U_2$. Subsets $U_1 = \{1, 3\}$, $U_2 = \{2\}$ exist such that $U = \{1, 2, 3\} = U_1 \cup U_2$. Hence, matrix **B** has *property A*.

We now establish some important results.

Therorem 3.12 An orthogonal set of vectors is linearly independent.

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be an orthogonal set of vectors, that is $\mathbf{x}_i \cdot \mathbf{x}_j = 0$, $i \neq j$. Consider the vector equation

$$\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_m\mathbf{x}_m = \mathbf{0} \tag{3.48}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are scalars. Taking the inner product of the vector **x** in Eq. (3.48) with $\mathbf{x}_1$, we get

$$\mathbf{x} \cdot \mathbf{x}_1 = (\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_m\mathbf{x}_m) \cdot \mathbf{x}_1 = \mathbf{0} \cdot \mathbf{x}_1 = 0$$

or

$$\alpha_1(\mathbf{x}_1 \cdot \mathbf{x}_1) = 0 \quad \text{or} \quad \alpha_1 \|\mathbf{x}_1\|^2 = 0.$$

Since $\|\mathbf{x}_1\|^2 \neq 0$, we get $\alpha_1 = 0$. Similarly, taking the inner products of **x** with $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_m$ successively, we find that $\alpha_2 = \alpha_3 = \ldots = \alpha_m = 0$. Therefore, the set of orthogonal vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ is linearly independent.

Theorem 3.13 The eigenvalues of

(i) an Hermitian matrix are real.

(ii) a skew-Hermitian matrix are zero or pure imaginary.

(iii) an unitary matrix are of magnitude 1.

Proof Let λ be an eigenvalue and **x** be the corresponding eigenvector of the matrix **A**. We have $\mathbf{Ax} = \lambda\mathbf{x}$. Pre-multiplying both sides by $\bar{\mathbf{x}}^T$, we get

$$\bar{\mathbf{x}}^T\mathbf{Ax} = \lambda\bar{\mathbf{x}}^T\mathbf{x} \quad \text{or} \quad \lambda = \frac{\bar{\mathbf{x}}^T\mathbf{Ax}}{\bar{\mathbf{x}}^T\mathbf{x}}. \tag{3.49}$$

Note that $\bar{\mathbf{x}}^T\mathbf{Ax}$ and $\bar{\mathbf{x}}^T\mathbf{x}$ are scalars. Also, the denominator $\bar{\mathbf{x}}^T\mathbf{x}$ is always real and positive. Therefore, the behavior of λ is governed by the scalar $\bar{\mathbf{x}}^T\mathbf{Ax}$.

(i) Let **A** be an Hermitian matrix, that is $\bar{\mathbf{A}} = \mathbf{A}^T$. Now,

$$\overline{(\bar{\mathbf{x}}^T\mathbf{Ax})} = \mathbf{x}^T\bar{\mathbf{A}}\bar{\mathbf{x}} = \mathbf{x}^T\mathbf{A}^T\bar{\mathbf{x}} = (\mathbf{x}^T\mathbf{A}^T\bar{\mathbf{x}})^T = \bar{\mathbf{x}}^T\mathbf{Ax}$$

since $\mathbf{x}^T\mathbf{A}^T\bar{\mathbf{x}}$ is a scalar. Therefore, $\bar{\mathbf{x}}^T\mathbf{Ax}$ is real. From Eq. (3.49), we conclude that λ is real.

(ii) Let **A** be a skew-Hermitian matrix, that is $\mathbf{A}^T = -\bar{\mathbf{A}}$. Now,

$$\overline{(\bar{\mathbf{x}}^T\mathbf{Ax})} = \mathbf{x}^T\bar{\mathbf{A}}\bar{\mathbf{x}} = -\mathbf{x}^T\mathbf{A}^T\bar{\mathbf{x}} = -(\mathbf{x}^T\mathbf{A}^T\bar{\mathbf{x}})^T = -\bar{\mathbf{x}}^T\mathbf{Ax}$$

since $\mathbf{x}^T\mathbf{A}^T\bar{\mathbf{x}}$ is a scalar. Therefore, $\bar{\mathbf{x}}^T\mathbf{Ax}$ is zero or pure imaginary. From Eq. (3.49), we conclude that λ is zero or pure imaginary.

(iii) Let **A** be an unitary matrix, that is $\mathbf{A}^{-1} = (\bar{\mathbf{A}})^T$. Now, from

$$\mathbf{Ax} = \lambda\mathbf{x} \quad \text{or} \quad \bar{\mathbf{A}}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}} \tag{3.50}$$

we get

$$(\bar{\mathbf{A}}\bar{\mathbf{x}})^T = \left(\bar{\lambda}\bar{\mathbf{x}}^T\right)^T \quad \text{or} \quad \bar{\mathbf{x}}^T\bar{\mathbf{A}}^T = \bar{\lambda}\bar{\mathbf{x}}^T.$$

or

$$\bar{\mathbf{x}}^T\mathbf{A}^{-1} = \bar{\lambda}\bar{\mathbf{x}}^T. \tag{3.51}$$

Using Eqs. (3.50) and (3.51), we can write

$$(\bar{\mathbf{x}}^T\mathbf{A}^{-1})(\mathbf{Ax}) = (\bar{\lambda}\bar{\mathbf{x}}^T)(\lambda\mathbf{x}) = |\lambda|^2\bar{\mathbf{x}}^T\mathbf{x}$$

or

$$\bar{\mathbf{x}}^T\mathbf{x} = |\lambda|^2\bar{\mathbf{x}}^T\mathbf{x}.$$

Since $\mathbf{x} \neq \mathbf{0}$, we have $\bar{\mathbf{x}}^T\mathbf{x} \neq \mathbf{0}$. Therefore, $|\lambda|^2 = 1$, or $|\lambda| = 1$. Hence, the result.

Remark 24

From Theorem 3.13, we conclude that the eigenvalues of

(i) a symmetric matrix are real.

(ii) a skew-symmetric matrix are zero or pure imaginary.

(iii) an orthogonal matrix are of magnitude 1 and are real or complex conjugate pairs.

Theorem 3.14 The column vectors (and also row vectors) of an unitary matrix form an unitary system of vectors.

Proof Let **A** be an unitary matrix of order n, with column vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Then

$$\mathbf{A}^{-1}\mathbf{A} = \bar{\mathbf{A}}^T\mathbf{A} = \begin{bmatrix} \bar{\mathbf{x}}_1^T \\ \bar{\mathbf{x}}_2^T \\ \vdots \\ \bar{\mathbf{x}}_n^T \end{bmatrix} [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = \begin{bmatrix} \bar{\mathbf{x}}_1^T\mathbf{x}_1 & \bar{\mathbf{x}}_1^T\mathbf{x}_2 & \ldots & \bar{\mathbf{x}}_1^T\mathbf{x}_n \\ \bar{\mathbf{x}}_2^T\mathbf{x}_1 & \bar{\mathbf{x}}_2^T\mathbf{x}_2 & \ldots & \bar{\mathbf{x}}_2^T\mathbf{x}_n \\ \vdots & & & \\ \bar{\mathbf{x}}_n^T\mathbf{x}_1 & \bar{\mathbf{x}}_n^T\mathbf{x}_2 & \ldots & \bar{\mathbf{x}}_n^T\mathbf{x}_n \end{bmatrix} = \mathbf{I}$$

Therefore,

$$\bar{\mathbf{x}}_i^T\mathbf{x}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Hence, the column vectors of **A** form an unitary system. Since the inverse of an unitary matrix is also an unitary matrix and the columns of $\mathbf{A}^{-1}$ are the conjugate of the rows of **A**, we conclude that the row vectors of **A** also form an unitary system.

Remark 25

(a) From Theorem 3.14, we conclude that the column vectors (and also the row vectors) of an orthogonal matrix form an orthonormal system of vectors.

(b) A symmetric matrix of order n has n linearly independent eigenvectors and hence is diagonlizable.

Example 3.41 Show that the matrices $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues and for distinct eigenvalues the eigenvectors corresponding to $\mathbf{A}$ and $\mathbf{A}^T$ are mutually orthogonal.

Solution We have

$$|\mathbf{A} - \lambda\mathbf{I}| = |(\mathbf{A}^T)^T - \lambda\mathbf{I}^T| = |[\mathbf{A}^T - \lambda\mathbf{I}]^T| = |\mathbf{A}^T - \lambda\mathbf{I}|.$$

Since $\mathbf{A}$ and $\mathbf{A}^T$ have the same characteristic equation, they have the same eigenvalues.

Let λ and μ be two distinct eigenvalues of $\mathbf{A}$. Let $\mathbf{x}$ be the eigenvector corresponding to the eigenvalue λ for $\mathbf{A}$ and $\mathbf{y}$ be the eigenvector corresponding to the eigenvalue μ for $\mathbf{A}^T$. We have $\mathbf{Ax} = \lambda\mathbf{x}$. Pre-multiplying by $\mathbf{y}^T$, we get

$$\mathbf{y}^T\mathbf{Ax} = \lambda\mathbf{y}^T\mathbf{x}. \tag{3.52}$$

We also have $\mathbf{A}^T\mathbf{y} = \mu\mathbf{y}$. or $(\mathbf{A}^T\mathbf{y})^T = (\mu\mathbf{y})^T$ or $\mathbf{y}^T\mathbf{A} = \mu\mathbf{y}^T$.

Post-multiplying by $\mathbf{x}$, we get

$$\mathbf{y}^T\mathbf{Ax} = \mu\mathbf{y}^T\mathbf{x} \tag{3.53}$$

Subtracting Eqs. (3.52) and (3.53), we obtain

$$(\lambda - \mu)\mathbf{y}^T\mathbf{x} = 0.$$

Since $\lambda \neq \mu$, we obtain $\mathbf{y}^T\mathbf{x} = 0$. Therefore, the vectors $\mathbf{x}$ and $\mathbf{y}$ are mutually orthogonal.

3.6 Quadratic Forms

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be an arbitrary vector in IR^n. A real *quadratic form* is an homogeneous expression of the form

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j \tag{3.54}$$

in which the total power in each term is 2. Expanding, we can write

$$\begin{aligned} Q &= a_{11}x^2 + (a_{12} + a_{21})\, x_1 x_2 + \ldots + (a_{1n} + a_{n1})\, x_1 x_n \\ &\quad + a_{22}\, x_2^2 + (a_{23} + a_{32})\, x_2 x_3 + \ldots + (a_{2n} + a_{n2})\, x_2 x_n \\ &\quad + \ldots + a_{nn}\, x_n^2 \\ &= \mathbf{x}^T\mathbf{Ax} \end{aligned} \tag{3.55}$$

using the definition of matrix multiplication. Now, set $b_{ij} = (a_{ij} + a_{ji})/2$. The matrix $\mathbf{B} = (b_{ij})$ is symmetric since $b_{ij} = b_{ji}$. Further, $b_{ij} + b_{ji} = a_{ij} + a_{ji}$. Hence, Eq. (3.55) can be wtitten as

$$Q = \mathbf{x}^T\mathbf{Bx}$$

where $\mathbf{B}$ is a symmetric matrix and $b_{ij} = (a_{ij} + a_{ji})/2$.

For example, for $n = 2$, we have

$$b_{11} = a_{11}, \quad b_{12} = b_{21} = (a_{12} + a_{21})/2 \quad \text{and} \quad b_{22} = a_{22}.$$

Example 3.42 Obtain the symmetric matrix $\mathbf{B}$ for the quadratic form

(i) $Q = 2x_1^2 + 3x_1x_2 + x_2^2$.

(ii) $Q = x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3 - 5\, x_2^2 + 4\, x_3^2$.

Solution

(i) $a_{11} = 2, \quad a_{12} + a_{21} = 3$ and $a_{22} = 1$. Therefore,

$$b_{11} = a_{11} = 2, \quad b_{12} = b_{21} = \frac{1}{2}\,(a_{12} + a_{21}) = \frac{3}{2} \quad \text{and} \quad b_{22} = a_{22} = 1.$$

Therefore, $\quad \mathbf{B} = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}.$

(ii) $a_{11} = 1,\ a_{12} + a_{21} = 2,\ a_{13} + a_{31} = -4,\ a_{23} + a_{32} = 6,\ a_{22} = -5,\ a_{33} = 4$. Therefore,

$$b_{11} = a_{11} = 1,\ b_{12} = b_{21} = \frac{1}{2}\,(a_{12} + a_{21}) = 1,\ b_{13} = b_{31} = \frac{1}{2}\,(a_{13} + a_{31}) = -2,$$

$$b_{23} = b_{32} = \frac{1}{2}\,(a_{23} + a_{32}) = 3,\ b_{22} = a_{22} = -5,\ b_{33} = a_{33} = 4.$$

Therefore, $\quad \mathbf{B} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}.$

If $\mathbf{A}$ is a complex matrix, then the quadratic form is defined as

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, \bar{x}_i x_j = \bar{\mathbf{x}}^T\mathbf{Ax} \tag{3.56}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an arbitrary vector in $\mathbb{C}^n$. However, this quadratic form is usually defined for an Hermitian matrix $\mathbf{A}$. Then, it is called a *Hermitian form* and is always real.

For example, consider the Hermitian matrix $\mathbf{A} = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$. The quadratic form becomes

$$\begin{aligned} Q = \bar{\mathbf{x}}^T\mathbf{Ax} &= \begin{bmatrix} \bar{x}_1, \bar{x}_2 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= |x_1|^2 + (1+i)\, \bar{x}_1x_2 + (1-i)x_1\bar{x}_2 + 2\,|x_2|^2. \\ &= |x_1|^2 + (\bar{x}_1x_2 + x_1\bar{x}_2) + i(\bar{x}_1x_2 - x_1\bar{x}_2) + 2|x_2|^2. \end{aligned}$$

Now, $\bar{x}_1 x_2 + x_1 \bar{x}_2$ is real and $\bar{x}_1 x_2 - x_1 \bar{x}_2$ is imaginary. For example if $x_1 = p_1 + iq_1$, $x_2 = p_2 + iq_2$, we obtain

$$\bar{x}_1 x_2 + x_1 \bar{x}_2 = 2(p_1 p_2 + q_1 q_2) \text{ and } \bar{x}_1 x_2 - x_1 \bar{x}_2 = 2i(p_1 q_2 - p_2 q_1).$$

We can also write

$$(\bar{x}_1 x_2 + \bar{x}_2 x_1) + i(\bar{x}_1 x_2 - x_1 \bar{x}_2) = 2[(p_1 p_2 + q_1 q_2) - (p_1 q_2 - p_2 q_1)] = 2 \operatorname{Re}[(1 + i)\bar{x}_1 x_2].$$

Therefore,

$$Q = |x_1|^2 + 2 \operatorname{Re}[(1 + i)\bar{x}_1 x_2] + |x_2|^2.$$

Positive definite matrices

Let $\mathbf{A} = (a_{ij})$ be a square matrix. Then, the matrix $\mathbf{A}$ is said to be *positive definite* if

$Q = \bar{\mathbf{x}}^T \mathbf{A}\mathbf{x} > 0$ for any vector $\mathbf{x} \neq \mathbf{0}$ and $\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x} = 0$, if and only if $\mathbf{x} = \mathbf{0}$.

If $\mathbf{A}$ is real, then $\mathbf{x}$ can be taken as real.

Positive definite matrices have the following properties.

1. The eigenvalues of a positive definite matrix are all real and positive.

This is easily proved when $\mathbf{A}$ is a real matrix. From Eq. (3.49), we have

$$\lambda = (\mathbf{x}^T \mathbf{A}\mathbf{x})/(\mathbf{x}^T \mathbf{x}).$$

Since $\mathbf{x}^T\mathbf{x} > 0$ and $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$, we obtain $\lambda > 0$. If $\mathbf{A}$ is Hermetian, then $\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x}$ is real and λ is real (see Theorem 3.13). Therefore, if the Hermitian form $Q > 0$, then the eigenvalues are real and positive.

2. All the leading minors of $\mathbf{A}$ are positive.

Remark 26

(a) If $\mathbf{A}$ is Hermitian and strictly diagonally dominant with positive real elements on the diagonal, then $\mathbf{A}$ is positive definite.

(b) If $\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x} \geq 0$, then the matrix $\mathbf{A}$ is called *semi-positive definite*.

(c) A matrix $\mathbf{A}$ is called *negative definite* if $(-\mathbf{A})$ is positive definite. All the eigenvalues of a negative definite matrix are real and negative.

Example 3.43 Examine which of the following matrices are positive definite.

(a) $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, (b) $\mathbf{A} = \begin{bmatrix} 3 & -2i \\ 2i & 4 \end{bmatrix}$, (c) $\mathbf{A} = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 3 \end{bmatrix}$.

Solution

(a) (i) $Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = [x_1, x_2]\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + 3x_1 x_2 + 4x_2^2$

$$= 3\left(x_1 + \frac{1}{2}x_2\right)^2 + \frac{13}{4}x_2^2 > 0 \text{ for all } \mathbf{x} \neq \mathbf{0}.$$

(ii) Eigenvalues of $\mathbf{A}$ are 2 and 5 which are both positive.

(iii) Leading minors $|3| = 3$, $\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10$ are both positive.

Hence, the matrix $\mathbf{A}$ is positive definite (it is not necessary to show all the three parts).

(b) $Q = \bar{\mathbf{x}}^T\mathbf{A}\mathbf{x} = [\bar{x}_1, \bar{x}_2]\begin{bmatrix} 3 & -2i \\ 2i & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\bar{x}_1, \bar{x}_2]\begin{bmatrix} 3x_1 - 2ix_2 \\ 2ix_1 + 4x_2 \end{bmatrix}$

$$= 3x_1\bar{x}_1 - 2i\bar{x}_1 x_2 + 2ix_1\bar{x}_2 + 4x_2\bar{x}_2.$$

Taking $x_1 = p_1 + iq_1$ and $x_2 = p_2 + iq_2$ and simplifying, we get

$$Q = 3(p_1^2 + q_1^2) + 4(p_2^2 + q_2^2) + 4(p_1 q_2 - p_2 q_1)$$
$$= p_1^2 + q_1^2 + 2p_2^2 + 2q_2^2 + 2(p_2 - q_1)^2 + 2(p_1 + q_2)^2 > 0.$$

Therefore, the given matrix is positive definite.

Note that $\mathbf{A}$ is Hermitian, strictly diagonally dominant ($3 > |-2i|$, $4 > |2i|$) with positive real diagonal entries. Therefore, $\mathbf{A}$ is positive definite (see Remark 26(a).)

(c) $Q = \bar{\mathbf{x}}^T\mathbf{A}\mathbf{x} = [\bar{x}_1, \bar{x}_2, \bar{x}_3]\begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [\bar{x}_1, \bar{x}_2, \bar{x}_3]\begin{bmatrix} x_1 + ix_3 \\ x_2 \\ -ix_1 + 3x_3 \end{bmatrix}$

$$= x_1\bar{x}_1 + i\bar{x}_1 x_3 + x_2\bar{x}_2 - ix_1\bar{x}_3 + 3x_3\bar{x}_3$$
$$= |x_1|^2 + |x_2|^2 + 3|x_3|^2 + i(\bar{x}_1 x_3 - x_1\bar{x}_3)$$

Taking $x_1 = p_1 + iq_1$, $x_2 = p_2 + iq_2$, $x_3 = p_3 + iq_3$ and simplifying, we obtain

$$Q = (p_1^2 + q_1^2) + (p_2^2 + q_2^2) + 3(p_3^2 + q_3^2) - 2(p_1 q_3 - p_3 q_1)$$
$$= (p_1 - q_3)^2 + (p_3 + q_1)^2 + (p_2^2 + q_2^2) + 2(p_3^2 + q_3^2) > 0.$$

Therefore, the matrix $\mathbf{A}$ is positive definite. It can be verified that the eigenvalues of $\mathbf{A}$ are 1, 2, 2 which are all positive.

Example 3.44 Let $\mathbf{A}$ be a real square matrix. Show that the matrix $\mathbf{A}^T\mathbf{A}$ has real and positive eigenvalues.

Solution Since $\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T\mathbf{A}$, the matrix $\mathbf{A}^T\mathbf{A}$ is symmetric. Therefore, the eigenvalues of $\mathbf{A}^T\mathbf{A}$ are all real. Now,

$$\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \mathbf{y}^T\mathbf{y}, \text{ where } \mathbf{A}\mathbf{x} = \mathbf{y}.$$

Since $\mathbf{y}^T\mathbf{y} > 0$ for any vector $\mathbf{y} \neq \mathbf{0}$, the matrix $\mathbf{A}^T\mathbf{A}$ is positive definite and hence all the eigenvalues of $\mathbf{A}^T\mathbf{A}$ are positive. Therefore, all the eigenvalues of $\mathbf{A}^T\mathbf{A}$ are real and positive.

Nature, rank, index and signature of a quadratic form

Let **A** be the matrix of the quadratic form $Q = \mathbf{x}^T \mathbf{A}\mathbf{x}$, where **A** is a symmetric matrix. The quadratic form Q is said to be

positive definite if all the eigenvalues of **A** are real and positive,
negative definite if all the eigenvalues of **A** are real and negative,
semi positive definite if all the eigenvalues of **A** are real and non-negative,
semi negative definite if all the eigenvalues of **A** are real and non-positive,
indefinite if some eigenvalues of **A** are positive and some are negative.

This defines the nature of the quadratic form.

Rank of the quadratic form The rank r of **A** is called the rank of the quadratic form, that is, the number of non-zero eigenvalues.

Index of a quadratic form The number of positive eigenvalues is called the index of the quadratic form and is denoted by k.

Signature of a quadratic form We define

Signature = (Number of positive eigenvalues) – (Number of negative eigenvalues)

$$= k - (r - k) = 2k - r.$$

Signature can be a negative integer.

We now, prove the invariance of a quadratic form under non-singular linear transformations.

Theorem 3.15 Under a non-singular linear transformation, a quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$, remains a quadratic form.

Proof Let $\mathbf{x}^T\mathbf{A}\mathbf{x}$ be a quadratic form, where **A** is symmetric and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$. Let $\mathbf{x} = \mathbf{P}\mathbf{y}$ be a non-singular linear transformation, which transforms the quadratic form from the variables $x_1, x_2, \ldots, x_n$ to $y_1, y_2, \ldots, y_n$. Then,

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = (\mathbf{P}\mathbf{y})^T\mathbf{A}(\mathbf{P}\mathbf{y}) = \mathbf{y}^T\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{y} = \mathbf{y}^T\mathbf{B}\mathbf{y}$$

where $\mathbf{B} = \mathbf{P}^T\mathbf{A}\mathbf{P}$. Now,

$$\mathbf{B}^T = (\mathbf{P}^T\mathbf{A}\mathbf{P})^T = \mathbf{P}^T\mathbf{A}^T\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{B}.$$

Hence, **B** is symmetric and $\mathbf{y}^T\mathbf{B}\mathbf{y}$ is also a quadratic form.

This proves the invariance of a quadratic form under a non-singular linear transformation.

3.6.1 Canonical Form of a Quadratic Form

A quadratic form $Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$ is said to be in *canonical form* if all the mixed terms such as x_1x_2, x_1x_3, ... are absent, that is, $a_{ij} = 0$, $i \neq j$. We may also say that a canonical form is a *sum of squares form*. The canonical form is, therefore, given by

$$Q = a_1 y_1^2 + a_2 y_2^2 + \ldots + a_r y_r^2 \tag{3.57}$$

if rank (**A**) $= r < n$

and

$$Q = a_1 y_1^2 + a_2 y_2^2 + \ldots + a_n y_n^2 \tag{3.58}$$

if rank (**A**) $= n$, where $a_1, a_2, \ldots, a_n$ are any real numbers.

For example, $Q = 6x_1^2 + 5x_2^2$, $Q = 3x_1^2 - 4x_2^2$ are canonical forms.

Remark 27

Since the matrix **A** is symmetric, it is diagonalizable. Hence, every quadratic form $Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$ can be reduced to a sum of squares form. The number of square terms is equal to the rank r.

Theorem 3.16 An orthogonal transformation $\mathbf{x} = \mathbf{P}\,\mathbf{y}$, where **P** is an orthogonal matrix, transforms a quadratic form $Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$ to the sum of squares form $Q = \mathbf{y}^T\,\mathbf{D}\,\mathbf{y}$ where **D** is the diagonal matrix, $\mathbf{D} = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_r)$.

Proof Let the rank of the quadratic form $Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$ be r. Now, **A** is a symmetric matrix. Let **P** be the normalised modal matrix of **A**. Therefore,

$$\mathbf{P}^{-1}\,\mathbf{A}\,\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \text{ where } \mathbf{D} = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_r).$$

If $r = n$, then $\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D}$, where $\mathbf{D} = \text{diag}\,(\lambda_1, \lambda_2 \ldots, \lambda_n)$.

Under the orthogonal transformation $\mathbf{x} = \mathbf{P}\,\mathbf{y}$, we obtain

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = (\mathbf{P}\,\mathbf{y})^T\,\mathbf{A}(\mathbf{P}\,\mathbf{y}) = \mathbf{y}^T\,\mathbf{A}\mathbf{P}\,\mathbf{y} = \mathbf{y}^T\begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{y}$$

$$= [y_1, y_2 \ldots, y_n]\begin{bmatrix} \lambda_1 & & & & & 0 \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_r & & \\ 0 & & & & 0 & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_r y_r^2. \tag{3.59}$$

If rank $= r = n$, then we get $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$.

Let k coeffiecients in the sum of squares in Eq. (3.57) be positive and $r - k$ coefficients be negative. Arrange the terms in Eq. (3.57) such that the terms with positive coefficients appear first and then the terms with negative coefficients. That is, Eq. (3.57) is arranged as

$$Q = a_1 y_1^2 + a_2 y_2^2 + \ldots + a_k y_k^2 - a_{k+1} y_{k+1}^2 - \ldots - a_r y_r^2. \tag{3.60}$$

where $a_i > 0$.

...and signature of a quadratic form The number of positive terms, k, is called the *index*. The difference between the number of positive and negative terms, that is, $2k - r$, is called the signature of the quadratic form.

Sylvester's law of inertia The rank r and index k of a real quadratic form Q are invariants under all real, non-singular transformations.

Reduction of a quadratic form to a canonical form

We give below two methods for reducing a quadratic form to a canonical form.

1. *Lagrange reduction* Let the quadratic form contain the variables x_1, x_2, x_3. We write the non singular transformation as

$$\left.\begin{aligned} y_1 &= x_1 + px_2 + qx_3,\\ y_2 &= \quad\;\; x_2 + rx_3,\\ y_3 &= \qquad\quad\;\; x_3,\end{aligned}\right\}, \text{ or } \mathbf{y} = \begin{bmatrix} 1 & p & q\\ 0 & 1 & r\\ 0 & 0 & 1\end{bmatrix}\mathbf{x}, \text{ or } \mathbf{y} = \mathbf{P}\mathbf{x}. \tag{3.61}$$

Let the sum of squares form be $ay_1^2 + by_2^2 + cy_3^2$. Substitute (3.61) in $ay_1^2 + by_2^2 + cy_3^2$, simplify and compare the terms with the terms in the given quadratic form. Solve for a, p, q, b, r and c.

2. *Orthogonalisation method* Find the eigenvalues and eigenvectors of $\mathbf{A}$. Obtain the normalized modal matrix $\mathbf{P}$. Under the transformation $\mathbf{x} = \mathbf{P}\mathbf{y}$, the quadratic form reduces to $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_r y_r^2$ or $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$, depending on the rank of $\mathbf{A}$, where λ_i's are the eigenvalues of $\mathbf{A}$.

Example 3.45 Reduce the quadratic form $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1$ to canonical form through an orthogonal transformation. Find the index and signature.

Solution The matrix of the quadratic form is

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}.$$

Eigenvalues of $\mathbf{A}$ are 0, 3, 3.

Eigenvector corresponding to $\lambda = 0$, is $\mathbf{v}_1 = [1\ 1\ 1]^T$.

Corresponding to the repeated eigenvalue $\lambda = 3$, we have the equation for a finding the eigevector as $x_1 + x_2 + x_3 = 0$. One eigenvector can be taken as $\mathbf{v}_2 = [1\ 0\ -1]^T$. Since the model matrix should be orthogonal, the third eigenvector must satisfy the above equation and also be orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. Assuming $\mathbf{v}_3 = [a\ b\ c]^T$ and orthogonalising with $\mathbf{v}_1$ and $\mathbf{v}_2$, we obtain $\mathbf{v}_3 = [1\ -2\ 1]^T$.

The normalized modal matrix is given by

$$\mathbf{P} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\ 1/\sqrt{3} & 0 & -2/\sqrt{6}\\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6}\end{bmatrix}.$$

The orthogonal transformation is $\mathbf{x} = \mathbf{P}\mathbf{y}$, and canonical form $= \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{y} = \mathbf{y}^T\mathbf{D}\mathbf{y} = (0)\,y_1^2 + 3y_2^2 + 3y_3^2 = 3y_2^2 + 3y_3^2$.

index = 2, signature = 2.

3.7 Condition Number of a Matrix

Norm of a matix Let $\mathbf{A}$ be a real or a complex matrix. Then, the norm of a matrix denoted by $\|\mathbf{A}\|$, is defined as follows:

(i) *Euclidean norm:* $\quad \|\mathbf{A}\| = \sqrt{\sum_{i,j} |a_{ij}|^2}$.

(ii) *Spectral norm or Hilbert norm:* Compute $\mathbf{A}^*\mathbf{A} = (\overline{\mathbf{A}})^T\mathbf{A}$. Define

λ = spectral radius (largest eigenvalue in magnitude) of $\mathbf{A}^*\mathbf{A}$.

Then, $\quad \|\mathbf{A}\| = \sqrt{\lambda}$.

If $\mathbf{A}$ is an Hermitian matrix ($\mathbf{A}^* = \mathbf{A}$) or $\mathbf{A}$ is a real symmetric matrix, then λ = spectral radius of $\mathbf{A}^*\mathbf{A}$ = spectral radius of $\mathbf{A}^2$ = (spectral radius of $\mathbf{A}$)^2.

Therefore, $\|\mathbf{A}\| = \sqrt{\lambda}$ = spectral radius of $\mathbf{A}$.

For most engineering applications, we use the spectral norm.

Condition number of a matrix

Condition number of a matrix is an important concept in the theory of solution of linear algebraic equations. In engineering applications, we often require to solve a large system of linear algebraic equations. Since the system is large, we solve it by iterative methods. Naturally, iterative methods produce round off errors. The round off errors should not magnify during iteration. Condition number of the coefficient matrix $\mathbf{A}$ of the system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, gives a measure of the sensitivity of the system to round off errors.

Using the spectral norm, we define the *condition number* of a matrix $\mathbf{A}$ as

$$\text{cond}(\mathbf{A}) = \kappa(\mathbf{A}) = \|\mathbf{A}\|\,\|\mathbf{A}^{-1}\| = \sqrt{\frac{\lambda}{\mu}} \tag{3.62}$$

where λ = largest eigenvalue in magnitude of $\mathbf{A}^*\mathbf{A}$,

and μ = smallest eigenvalue in magnitude of $\mathbf{A}^*\mathbf{A}$.

If $\mathbf{A}$ is an Hermitian matrix or $\mathbf{A}$ is a real and symmetric matrix, then (3.62) simplifies to

$$\text{cond}(\mathbf{A}) = \kappa(\mathbf{A}) = \|\mathbf{A}\|\,\|\mathbf{A}^{-1}\| = \frac{\lambda_1}{\mu_1} \tag{3.63}$$

where λ_1 = largest eigenvalue in magnitude of $\mathbf{A}$,

and μ_1 = smallest eigenvalue in magnitude of $\mathbf{A}$.

The larger the value of the condition number more is the sensitivity of the system to round off errors. For example, when the condition number is large, the solution obtained using say, four decimal places arithmetic may differ completely from the solution obtained by say, six decimal places arithmetic.

We can use other norms also to compute the condition number.

Example 3.46 Find the condition numbers of the following matrices.

(i) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$. (iii) $\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$.

Solution

(i) **A** is real. We have

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 10 \\ 10 & 20 \end{bmatrix}.$$

The characteristic equation of $\mathbf{A}^T\mathbf{A}$ is given by

$$|\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 10-\lambda & 10 \\ 10 & 20-\lambda \end{vmatrix} = \lambda^2 - 30\lambda + 100 = 0.$$

The eigenvalues are given by $\lambda = [15 + 5\sqrt{5}], [15 - 5\sqrt{5}]$.

Largest eigenvalue in magnitude of $\mathbf{A}^T\mathbf{A} = \lambda_1 = [15 + 5\sqrt{5}]$.

Smallest eigenvalue in magnitude of $\mathbf{A}^T\mathbf{A} = \mu_1 = [15 - 5\sqrt{5}]$.

Hence, $$\text{cond}(\mathbf{A}) = \sqrt{\frac{\lambda_1}{\mu_1}} = \sqrt{\frac{15+5\sqrt{5}}{15-5\sqrt{5}}} = 2.618.$$

(ii) **A** is real and symmetric. The characteristic equation of **A** is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{vmatrix} = \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0.$$

The eigenvalues are given by $\lambda = 3, 6, 9$.

Hence, $$\text{cond}(\mathbf{A}) = \frac{\text{Largest eigenvalue in magnitude of } \mathbf{A}}{\text{Smallest eigenvalue in magnitude of } \mathbf{A}} = \frac{9}{3} = 3.$$

(iii) **A** is a complex (Hermitian) matrix. We have

$$\mathbf{A}^*\mathbf{A} = (\bar{\mathbf{A}})^T\mathbf{A} = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix} = \begin{bmatrix} 29 & 12+16i \\ 12-16i & 29 \end{bmatrix}.$$

The characteristic equation of $\mathbf{A}^*\mathbf{A}$ is given by

$$|\mathbf{A}^*\mathbf{A} - \lambda\mathbf{I}| = \begin{bmatrix} 29-\lambda & 12+16i \\ 12-16i & 29-\lambda \end{bmatrix} = (29-\lambda)^2 - 400 = 0.$$

The eigenvalues are given by $\lambda = 49, 9$.

Largest eigenvalue in magnitude of $\mathbf{A}^*\mathbf{A} = \lambda_1 = 49$.

Smallest eigenvalue in magnitude of $\mathbf{A}^*\mathbf{A} = \mu_1 = 9$.

Hence, $$\text{cond}(\mathbf{A}) = \sqrt{\frac{\lambda_1}{\mu_1}} = \sqrt{\frac{49}{9}} = \frac{7}{3}.$$

Note that since **A** is a Hermitian matrix, we could have used the formula

$$\text{cond}(\mathbf{A}) = \frac{\text{Largest eigenvalue in magnitude of A}}{\text{Smallest eigenvalue in magnitude of A}} = \frac{7}{3}.$$

3.8 Singular Value Decomposition

Singular value decomposition is an important concept which has applications in many areas of engineering, computer science etc. In an earlier section, we discussed the diagonalization of a square matrix. We now discuss diagonalization of a rectangular matrix **A**. **A** may be a real or a complex matrix. We shall discuss in detail the case when **A** is a real matrix.

Theorem 3.17 An arbitrary $m \times n$ real matrix **A** can be decomposed as $\mathbf{A} = \mathbf{PDQ}$, where **D** is a generalized diagonal matrix of order $m \times n$ and **P** and **Q** are orthogonal matrices of orders $m \times m$ and $n \times n$ respectively.

Proof Consider the $n \times n$ matrix $\mathbf{B} = \mathbf{A}^T\mathbf{A}$. Now, **B** is a symmetric and positive semi-definite matrix, since

$$\mathbf{B}^T = (\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T\mathbf{A} = \mathbf{B} \tag{3.64}$$

and

$$\mathbf{x}^T\mathbf{B}\mathbf{x} = \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x}\|^2 \geq 0, \tag{3.65}$$

where **x** is an $n \times 1$ arbitrary vector. Therefore, the eigenvalues of **B** are either positive or zero. Denote the eigenvalues of **B** as $\lambda_1 = \mu_1^2, \lambda_2 = \mu_2^2, \ldots, \lambda_n = \mu_n^2$. If the eigen values are repeated, we count its multiplicity also. If some of the eigenvalues are zero, we order the eigenvalues such that the non-zero eigenvalues are taken first and then the zero eigenvalues. If rank $(\mathbf{B}) = r$, then we take the r positive eigenvalues as $\mu_1^2, \mu_2^2, \ldots, \mu_r^2$ and $n - r$ zero eigenvalues as $\mu_{r+1}^2, \mu_{r+2}^2, \ldots, \mu_n^2$. Define the generalized diagonal matrix **D** of order $m \times n$, with $\mu_1, \mu_2, \ldots, \mu_r$ on its diagonal and zeros elsewhere.

Find the normalised eigenvectors of **B** and denote them as $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. The normalised eigenvectors form an orthonormal system. That is, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are solutions of

$$\mathbf{B}\mathbf{u}_i = \mathbf{A}^T\mathbf{A}\mathbf{u}_i = \mu_i^2\, \mathbf{u}_i.$$

and

$$\mathbf{u}_i^T \mathbf{u}_j = 0, \quad i \neq j,$$
$$= 1, \; i = j. \tag{3.66}$$

Now, form the orthogonal matrix **Q** with $\mathbf{u}_1^T, \mathbf{u}_2^T, \ldots, \mathbf{u}_n^T$ as its rows.

Now, $\|\mathbf{A}\mathbf{u}_i\|^2 = (\mathbf{A}\mathbf{u}_i)^T(\mathbf{A}\mathbf{u}_i) = \mathbf{u}_i^T\mathbf{A}^T\mathbf{A}\mathbf{u}_i = \mu_i^2(\mathbf{u}_i^T\mathbf{u}_i) = \mu_i^2$.

Therefore, $\|\mathbf{A}\mathbf{u}_i\|^2 > 0$, for $i = 1, 2, \ldots, r$, and $\|\mathbf{A}\mathbf{u}_i\|^2 = 0$, for $i = r+1, r+2, \ldots, n$.

This implies that $\mathbf{A}\mathbf{u}_i = \mathbf{0}$, for $i = r+1, r+2, \ldots, n$. (3.67)

Define the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ such that

$$\mathbf{v}_i = \frac{1}{\mu_i}\mathbf{A}\mathbf{u}_i, \quad i = 1, 2, \ldots, r. \tag{3.68}$$

Now,

$$\mathbf{v}_i^T\mathbf{v}_j = \frac{1}{\mu_i\mu_j}(\mathbf{A}\mathbf{u}_i)^T(\mathbf{A}\mathbf{u}_j) = \frac{1}{\mu_i\mu_j}\,\mathbf{u}_i^T(\mathbf{A}^T\mathbf{A}\mathbf{u}_j) = \frac{\mu_j^2}{\mu_i\mu_j}\,\mathbf{u}_i^T\mathbf{u}_j$$

That is, $\mathbf{v}_i^T\mathbf{v}_j = 1$, for $i = j = 1, 2, \ldots, r$ and $\mathbf{v}_j^T\mathbf{v}_j = 0$, $i \neq j$. (3.69)

Therefore, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ form an orthonormal system. Since $r \le m$, we select the vectors $\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \ldots, \mathbf{v}_m$ such that they form an orthonormal system with the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. This choice of $\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \ldots, \mathbf{v}_m$ is arbitrary. Define the $m \times m$ matrix **P**, such that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ form its columns. That is, $\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m]$. Note that **P** is an orthogonal matrix.

Now,

$$\mathbf{P}^T\mathbf{A}\mathbf{Q}^T = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \cdots \\ \mathbf{v}_m^T \end{bmatrix} \mathbf{A}[\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n] = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \cdots \\ \mathbf{v}_m^T \end{bmatrix} [\mathbf{A}\mathbf{u}_1, \mathbf{A}\mathbf{u}_2, \ldots, \mathbf{A}\mathbf{u}_n] = (c_{ij}).$$

Therefore, $\mathbf{P}^T\mathbf{A}\mathbf{Q}^T$ defines an $m \times n$ matrix whose elements are $c_{ij} = \mathbf{v}_i^T\mathbf{A}\mathbf{u}_j$.

From (3.67), $\mathbf{A}\mathbf{u}_j = \mathbf{0}$, for $j = r+1, r+2, \ldots, n$. Therefore, $c_{ij} = 0$, for $j = r+1, r+2, \ldots, n$.

From (3.68) and (3.69), $c_{ij} = \mathbf{v}_j^T\mathbf{A}\mathbf{u}_j = \mu_j\,\mathbf{v}_i^T\mathbf{v}_j = \mu_j$, for $i = j = 1, 2, \ldots, r$, and 0 for $i \neq j$.

Therefore, $\mathbf{P}^T\mathbf{A}\mathbf{Q}^T$ defines the $m \times n$ generalized diagonal matrix **D**, that is $\mathbf{P}^T\mathbf{A}\mathbf{Q}^T = \mathbf{D}$.

Pre-multiplying by **P** and post multiplying by **Q**, we obtain

$$\mathbf{P}(\mathbf{P}^T\mathbf{A}\mathbf{Q}^T)\,\mathbf{Q} = \mathbf{P}\mathbf{D}\mathbf{Q}, \quad \text{or} \quad \mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{Q}, \tag{3.70}$$

since **P** and **Q** are orthogonal matrices.

The decomposition $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{Q}$, is called a *singular value decomposition*, and the numbers $\mu_1, \mu_2, \ldots, \mu_r$, which are the positive square roots of $\mu_1^2, \mu_2^2, \ldots, \mu_r^2$ are called the *singular values* of **A**. If rank (**B**) = n, the singular values are $\mu_1, \mu_2, \ldots, \mu_n$. Since, the ordering of the non-zero eigenvalues is arbitrary and the choice of $\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \ldots, \mathbf{v}_m$ is arbitrary, singular value decomposition of a matrix is not unique.

For example, we have the following decompositions:

$$\mathbf{A}_{4\times3} = \mathbf{P}_{4\times4}\,\mathbf{D}_{4\times3}\,\mathbf{Q}_{3\times3}, \quad \mathbf{A}_{3\times3} = \mathbf{P}_{3\times3}\,\mathbf{D}_{3\times3}\,\mathbf{Q}_{3\times3}, \quad \mathbf{A}_{2\times3} = \mathbf{P}_{2\times2}\,\mathbf{D}_{2\times3}\,\mathbf{Q}_{3\times3}$$

where **D** is suitably defined.

Remark 28

We can determine $\mathbf{v}_i$ in an alternate way. From (3.68), we have $\mathbf{A}\mathbf{u}_i = \mu_i\,\mathbf{v}_i$.

But from (3.66), we get

$$\mathbf{A}^T\mathbf{A}\mathbf{u}_i = \mu_i^2\,\mathbf{u}_i, \text{ or } \mathbf{A}^T(\mu_i\,\mathbf{v}_i) = \mu_i^2\,\mathbf{u}_i,$$

or

$$\mathbf{A}(\mathbf{A}^T\mathbf{v}_i) = \mu_i(\mathbf{A}\mathbf{u}_i), \quad \text{or} \quad (\mathbf{A}\mathbf{A}^T)\mathbf{v}_i = \mu_i^2\,\mathbf{v}_i. \tag{3.71}$$

Therefore, μ_i^2 are also eigenvalues of $\mathbf{A}\mathbf{A}^T$ and $\mathbf{v}_i$ are the normalised eigenvectors of $\mathbf{A}\mathbf{A}^T$. That is, we can determine $\mathbf{v}_i$ by solving (3.71). If rank (**B**) = r, we determine $\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \ldots, \mathbf{v}_m$ as described above.

Remark 29

If **A** is a $n \times n$ real symmetric matrix, we have $\mathbf{B} = \mathbf{A}^T\mathbf{A} = \mathbf{A}^2$. The eigenvalues of **B** are squares of eigenvalues of **A**. The eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$; and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are identical. Therefore, $\mathbf{P}^T = \mathbf{Q}$, or $\mathbf{P} = \mathbf{Q}^T$. In this case, singular value decomposition is given by $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$.

Remark 30

In the case of a complex matrix, we have the following result:

An arbitrary $m \times n$ complex matrix **A** can be decomposed as $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{Q}$, where **D** is a generalized diagonal matrix of order $m \times n$ and **P** and **Q** are unitary matrices of orders $m \times m$ and $n \times n$ respectively.

Example 3.47 Find singular value decompositions of the following matrices.

(i) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. (ii) $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$. (iii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution

(i) We have $\mathbf{B} = \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix}$.

Eigenvalues of **B** are given by $\lambda^2 - 15\lambda + 1 = 0$.
We obtain $\lambda_1 = \mu_1^2 = 14.93303$, $\lambda_2 = \mu_2^2 = 0.066966$, $\mu_1 = 3.86433$, $\mu_2 = 0.25878$.

The diagonal matrix **D** is given by

$$\mathbf{D} = \begin{bmatrix} 3.86433 & 0 \\ 0 & 0.25878 \end{bmatrix}.$$

The eigenvectors are obtained as the following:

$$\lambda_1 = 14.93303:\ \mathbf{x}_1 = [0.38661,\ 1]^T;\ \mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = [0.36060,\ 0.93272]^T.$$

$$\lambda_2 = 0.066966:\ \mathbf{x}_2 = [1,\ -0.386607]^T;\ \mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = [0.93272,\ -0.36060]^T.$$

The matrix **Q** is given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \end{bmatrix} = \begin{bmatrix} 0.36060 & 0.93272 \\ 0.93272 & -0.36060 \end{bmatrix}.$$

Note that **Q** is an orthogonal matrix.
Define the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ as the following.

$$\mathbf{v}_1 = \frac{1}{\mu_1}\mathbf{A}\mathbf{u}_1 = \frac{1}{3.86433}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0.36060 \\ 0.93272 \end{bmatrix} = \begin{bmatrix} 0.57605 \\ 0.81741 \end{bmatrix}.$$

$$\mathbf{v}_2 = \frac{1}{\mu_2}\mathbf{A}\mathbf{u}_2 = \frac{1}{0.25878}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0.93272 \\ -0.36060 \end{bmatrix} = \begin{bmatrix} 0.81737 \\ -0.57609 \end{bmatrix}.$$

The matrix **P** is given by

$$\mathbf{P} = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{bmatrix} 0.57605 & 0.81737 \\ 0.81741 & -0.57609 \end{bmatrix}.$$

Note that **P** is also an orthogonal matrix.

Alternate way to find $\mathbf{v}_i$

The vectors $\mathbf{v}_i$ are solutions of $(\mathbf{A}\mathbf{A}^T)\,\mathbf{v}_i = \mu_i^2\,\mathbf{v}_i$. We have

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix}.$$

The eigenvectors are obtained as the following.

$$\lambda_1 = \mu_1^2 = 14.93303:\ \ \mathbf{x}_1 = [1,\ 1.41900]^T;\ \mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = [0.57605,\ 0.81741]^T.$$

$$\lambda_2 = \mu_2^2 = 0.066966:\ \ \mathbf{x}_2 = [1,\ -0.70472]^T;\ \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = [0.81741,\ -0.57605]^T.$$

The singular value decomposition is **A = PDQ**.

(ii) **The given matrix is real and symmetric.** The eigenvalues of $\mathbf{B} = \mathbf{A}^T\mathbf{A}$ are squares of eigenvalues **of A. The singular value decomposition is given by** $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T = \mathbf{Q}^T\mathbf{D}\mathbf{Q}$. The eigenvalues of A **are 10, 0. Eigenvalues of B** are 100, 0.
The diagonal matrix D is given by

$$\mathbf{D} = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now,

$$\mathbf{B} = \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 10 & 30 \\ 30 & 90 \end{bmatrix}.$$

The eigenvectors are obtained as the following:

$$\lambda_1 = 100:\ \mathbf{x}_1 = [1,\ 3]^T;\ \mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = [1/\sqrt{10},\ 3/\sqrt{10}]^T.$$

$$\lambda_2 = 0:\ \mathbf{x}_2 = [-3,\ 1]^T;\ \mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = [-3/\sqrt{10},\ 1/\sqrt{10}]^T.$$

The matrix **Q** is given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}.$$

Note that **Q** is an orthogonal matrix. We have

$$\mathbf{P} = \mathbf{Q}^T = \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}.$$

The singular value decomposition is $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$.

iii) We have the decomposition as $\mathbf{A}_{3\times 2} = \mathbf{P}_{3\times 3}\,\mathbf{D}_{3\times 2}\,\mathbf{Q}_{2\times 2}$.

We have

$$\mathbf{B} = \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}.$$

Eigenvalues of **B** are given by

$$\begin{vmatrix} 3-\lambda & 3 \\ 3 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda = 0.\ \ \lambda_1 = \mu_1^2 = 6,\ \lambda_2 = \mu_2^2 = 0,\ \mu_1 = \sqrt{6},\ \mu_2 = 0.$$

The matrix **D** is given by

$$\mathbf{D} = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvectors are obtained as the following:

$$\lambda_1 = 6: \ \mathbf{x}_1 = [1, 1]^T; \ \mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = [1/\sqrt{2}, 1/\sqrt{2}]^T.$$

$$\lambda_2 = 0: \ \mathbf{x}_2 = [1, -1]^T; \ \mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = [1/\sqrt{2}, -1/\sqrt{2}]^T.$$

The matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Note that $\mathbf{Q}$ is an orthogonal matrix.

Define the vector $\mathbf{v}_1$ as the following.

$$\mathbf{v}_1 = \frac{1}{\mu_1} \mathbf{A}\mathbf{u}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{i}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The vectors $\mathbf{v}_2$, $\mathbf{v}_3$ are arbitrary, but $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, should form an orthonormal system.

Choose $\mathbf{v}_2$, and $\mathbf{x}_3$ as $\quad \mathbf{v}_2 = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$

Making $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{x}_3$ orthogonal, we get the equations $a + b + c = 0$, $a - b = 0$. The solution is $a = b$, $c = -2b$. We obtain

$$\mathbf{x}_3 = \begin{bmatrix} b \\ b \\ -2b \end{bmatrix}, \ \mathbf{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

The matrix $\mathbf{P}$ is given by

$$\mathbf{P} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}.$$

Note that P is an orthogonal matrix.
The singular value decomposition is A = PDQ.

Exercise 3.4

Verify the Cayley-Hamilton theorem for the matrix A. Find $\mathbf{A}^{-1}$, if it exists, where A is as given in Problems 1 to 6.

1. $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$. **2.** $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \\ 3 & 1 & -1 \end{bmatrix}$.

4. $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$. **5.** $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix}$. **6.** $\begin{bmatrix} 1 & i & i \\ i & 1 & i \\ i & i & 1 \end{bmatrix}$.

Find all the eigenvalues and the corresponding eigenvectors of the matrices given in Problems 7 to 18. Which of the matrices are diagonalizable?

7. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$. **8.** $\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. **9.** $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

10. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{bmatrix}$. **11.** $\begin{bmatrix} 1 & 1 & i \\ 1 & 0 & i \\ -i & -i & 1 \end{bmatrix}$. **12.** $\begin{bmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{bmatrix}$.

13. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. **14.** $\begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 1 & -2 & 1 \\ -1 & 1 & -2 & 1 \end{bmatrix}$. **15.** $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

16. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{bmatrix}$. **17.** $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$. **18.** $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Show that the matrices given in Problems 19 to 24 are diagonalizable. Find the matrix P such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

19. $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$. **20.** $\begin{bmatrix} -3 & -2 & 1 \\ -2 & 0 & 4 \\ -6 & -3 & 5 \end{bmatrix}$. **21.** $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix}$.

22. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$. **23.** $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. **24.** $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

Find the matrix **A** whose eigenvalues and the corresponding eigenvectors are as given in Problems 25 to 30.

25. Eigenvalues: 2, 2, 4; Eigenvectors: $(-2, 1, 0)^T$, $(-1, 0, 1)^T$, $(1, 0, 1)^T$.

26. Eigenvalues: 1, −1, 2; Eigenvectors: $(1, 1, 0)^T$, $(1, 0, 1)^T$, $(3, 1, 1)^T$.

27. Eigenvalues: 1, 2, 3; Eigenvectors: $(1, 2, 1)^T$, $(2, 3, 4)^T$, $(1, 4, 9)^T$.

28. Eigenvalues: 1, 1, 1; Eigenvectors: $(-1, 1, 1)^T$, $(1, -1, 1)^T$, $(1, 1, -1)^T$.

29. Eigenvalues: 0, −1, 1; Eigenvectors: $(-1, 1, 0)^T$, $(1, 0, -1)^T$, $(1, 1, 1)^T$.

30. Eigenvalues: 0, 0, 3; Eigenvectors: $(1, 2, -1)^T$, $(-2, 1, 0)^T$, $(3, 0, 1)^T$.

31. Let a 4×4 matrix **A** have eigenvalues 1, −1, 2, −2. Find the value of the determinant of the matrix $\mathbf{B} = 2\mathbf{A} + \mathbf{A}^{-1} - \mathbf{I}$.

32. Let a 3×3 matrix **A** have eigenvalues 1, 2, −1. Find the trace of the matrix $\mathbf{B} = \mathbf{A} - \mathbf{A}^{-1} + \mathbf{A}^2$.

33. Show that the matrices **A** and $\mathbf{P}^{-1}\mathbf{AP}$ have the same eigenvalues.

34. Let **A** and **B** be square matrices of the same order. Then, show that **AB** and **BA** have the same eigenvalues but different eigenvectors.

35. Show that the matrices $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{BA}^{-1}$ have the same eigenvlaues but different eigenvectors.

36. An $n \times n$ matrix **A** is *nilpotent* if for some positive integer k, $\mathbf{A}^k = \mathbf{0}$. Show that all the eigenvalues of a nilpotent matrix are zero.

37. If **A** is an $n \times n$ diagonalizable matrix and $\mathbf{A}^2 = \mathbf{A}$, then show that each eigenvalue of **A** is 0 or 1.

38. Show that the matrix $\mathbf{A} = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$, $a \neq b$, is transformed to a diagonal matrix $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, where

P is of the form $\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\tan 2\theta = \dfrac{2h}{a-b}$.

39. Let **A** be similar to **B**. Then show that (i) $\mathbf{A}^{-1}$ is similar to $\mathbf{B}^{-1}$, (ii) $\mathbf{A}^m$ is similar to $\mathbf{B}^m$ for any positive integer m, (iii) $|\mathbf{A}| = |\mathbf{B}|$.

40. Let **A** and **B** be symmetric matrices of the same order. Then, show that **AB** is symmetric if and only if **AB** = **BA**.

41. For any square matrix **A**, show that $\mathbf{A}^T\mathbf{A}$ is symmetric.

42. Let **A** be a non-singular matrix, show that $\mathbf{A}^T\mathbf{A}^{-1}$ is symmetric if and only if $\mathbf{A}^2 = (\mathbf{A}^T)^2$.

43. If **A** is a symmetric matrix and $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, then show that **P** is an orthogonal matrix.

44. Show that the product of two orthogonal matrices of the same order is also an orthogonal matrix.

45. Find the conditions that a matrix $\mathbf{A} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is orthogonal.

46. If **A** is an orthogonal matrix, show that $|\mathbf{A}| = \pm 1$.

47. Prove that the eigenvectors of a symmetric matrix corresponding to distinct eigenvalues are orthogonal.

48. A matrix **A** is called a ***normal matrix*** if $\mathbf{A}\overline{\mathbf{A}}^T = \overline{\mathbf{A}}^T\mathbf{A}$. Show that the Hermitian, skew-Hermitian and unitary matrices are normal.

49. If a matrix **A** can be diagonalized using an orthogonal matrix, then show that **A** is symmetric.

50. Suppose that a matrix **A** is both unitary and Hermitian. Then, show that $\mathbf{A} = \mathbf{A}^{-1}$.

51. If **A** is a symmetric matrix and $\mathbf{x}^T\mathbf{Ax} > 0$ for every real vector $\mathbf{x} \neq \mathbf{0}$, then show that $\overline{\mathbf{z}}^T\mathbf{Az}$ is real and positive for any complex vector $\mathbf{z} \neq \mathbf{0}$.

52. Show that an unitary transformation $\mathbf{y} = \mathbf{Ax}$, where **A** is an unitary matrix preserves the value of inner product.

53. Do the following matrices satisfy *property A*?

(i) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, (ii) $\begin{bmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 0 & 1 & -4 \end{bmatrix}$.

54. Prove that a real 2×2 symmetric matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite if and only if $a > 0$ (1×1 leading minor) and $ac - b^2 > 0$ (2×2 leading minor).

55. Show that the matrix $\begin{bmatrix} 2 & 1 & 3 \\ -3 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}$ is positive definite.

56. Show that the matrix $\begin{bmatrix} -3 & -2 & 1 \\ -2 & 0 & 4 \\ -6 & -3 & 5 \end{bmatrix}$ is not positive definite.

Find the symmetric or the Hermitian matrix **A** for the quadratic forms given in Problems 57 to 61.

57. $x_1^2 - 2x_1x_2 + 4x_2x_3 - x_2^2 + x_3^2$.

58. $3x_1^2 + 2x_1x_2 - 4x_1x_3 + 8x_2x_3 + x_2^2$.

59. $x_1^2 + 2ix_1x_2 - 8x_1x_3 + 4ix_2x_3 + 4x_3^2$.

60. $x_1^2 - (2 + 4i)\, x_1x_2 - (4 - 6i)x_2x_3 + x_2^2$.

61. $2x_1^2 - 3x_2^2 + (6 + 8i)x_1x_2 + (4 - 2i)x_2x_3$.

Reduce the quadratic form in Problems 62 and 63 to canonical form using Lagrange reduction.

62. $x_1^2 + 7x_2^2 + 7x_3^2 + 4x_1x_2 - 18x_2x_3 - 6x_3x_1$.

63. $x_1^2 + 7x_2^2 + 26x_3^2 + 4x_1x_2 - 22x_2x_3 - 2x_3x_1$.

Reduce the quadratic form in Problems 64 and 65 to canonical form using orthogonal reduction.

64. $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$.

65. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$.

Find the condition numbers of the following matrices.

66. $\begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$. **67.** $\begin{bmatrix} 3 & 5 \\ 5 & 4 \end{bmatrix}$. **68.** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. **69.** $\begin{bmatrix} 3 & 5-i \\ 5+i & 4 \end{bmatrix}$.

70. $\begin{bmatrix} 1 & 1 \\ 6 & 2 \end{bmatrix}$

71. $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

72. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

73. $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

74. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

75. $\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

3.9 More on Vector Spaces

We have earlier defined a subspace S of V as any vector space inside V. This subspace includes the zero vector. We also take V itself as a subspace of V. For example: (i) In R^2, the zero vector, any line through origin and R^2 itself are possible subspaces: (ii) In the space V of all $n \times n$ matrices, symmetric matrices of order n from a subspace.

3.9.1 Sum of Subspaces

Let V be a vector space and let S_1, S_2 be subspaces of V. We define the sum $S_1 + S_2$ as

$$S_1 + S_2 = \{\mathbf{s}_1 + \mathbf{s}_2 \mid \mathbf{s}_1 \in S_1,\ \mathbf{s}_2 \in S_2\}.$$

that is, space of all $[(\mathbf{s}_1 \text{ in } S_1) + (\mathbf{s}_2 \text{ in } S_2)]$. Also, $S = S_1 + S_2$ is a subspace of V. The proof is straight forward.

Let $\mathbf{x}_1$, $\mathbf{y}_1$ be two elements in S_1 and $\mathbf{x}_2$, $\mathbf{y}_2$ be two elements in S_2. Then, there are elements $\mathbf{x}$ and $\mathbf{y}$ in V such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$.

Now,
$$\mathbf{x} + \mathbf{y} = (\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + \mathbf{y}_2) = (\mathbf{x}_1 + \mathbf{y}_1) + (\mathbf{x}_2 + \mathbf{y}_2)$$
is an element of S, since $(\mathbf{x}_1 + \mathbf{y}_1) \in S_1$ and $(\mathbf{x}_2 + \mathbf{y}_2) \in S_2$

Also,
$$\alpha\mathbf{x} = \alpha(\mathbf{x}_1 + \mathbf{x}_2) = \alpha\mathbf{x}_1 + \alpha\mathbf{x}_2,\ (\alpha \text{ is a scalar})$$
is an element of S, since $\alpha\mathbf{x}_1 \in S_1$ and $\alpha\mathbf{x}_2 \in S_2$. Therefore, $S = S_1 + S_2$ is a subspace of V.

Two simple examples are the following:

(i) Let $S_1 = \{(x, 0) \mid x \in R\}$ and $S_2 = \{(0, y) \mid y \in R\}$ be in R^2. Then, $S_1 + S_2 \subset R^2$.

(ii) Let S_1 be the x-axis and S_2 be the y-axis in R^3. Then, $S_1 + S_2$ is x-y plane in R^3.

If $S_1, S_2, \ldots, S_r$ are subspaces of V, then $S_1 + S_2 + \ldots + S_r = \{\mathbf{s}_1 + \mathbf{s}_2 + \ldots + \mathbf{s}_r\}$, where $\mathbf{s}_i \in S_i$, is also a subspace of V.

Theorem 3.18 If S_1, S_2 are subspaces of a vector space V, then

$$\dim(S_1 + S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2). \tag{3.72}$$

Proof Let $\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ be a basis of $S_1 \cap S_2$. Since, $(S_1 \cap S_2) \subset S_1$, $\mathbf{x}$ can be extended to a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_q\}$ of S_1. Similarly, $\mathbf{x}$ can be extended to a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_r\}$ of S_2. Therefore, the set of vectors

$$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_q, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_r\} \tag{3.73}$$

spans the vector space $S = S_1 + S_2$. We shall show that the set of $p + q + r$ vectors (3.73) is linearly independent. If they are not independent, then

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_p\mathbf{x}_p + \beta_1\mathbf{y}_1 + \beta_2\mathbf{y}_2 + \ldots + \beta_q\mathbf{y}_q + \gamma_1\mathbf{z}_1 + \gamma_2\mathbf{z}_2 + \ldots + + \gamma_r\mathbf{z}_r = 0.$$

$$\text{or, } \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_p\mathbf{x}_p + \beta_1\mathbf{y}_1 + \beta_2\mathbf{y}_2 + \ldots + \beta_q\mathbf{y}_q = -(\gamma_1\mathbf{z}_1 + \gamma_2\mathbf{z}_2 + \ldots + \gamma_r\mathbf{z}_r) \tag{3.74}$$

for some scalars α_i, β_i, γ_i. The left hand side is an element of S_1 and the right hand side is an element of S_2. Therefore, both of them must belong to $S_1 \cap S_2$. Hence,

$$\gamma_1\mathbf{z}_1 + \gamma_2\mathbf{z}_2 + \ldots + \gamma_r\mathbf{z}_r = \delta_1\mathbf{x}_1 + \delta_2\mathbf{x}_2 + \ldots + \delta_r\mathbf{x}_r$$

for some scalars δ_i. Since the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_r$ are linearly independent, it follows that $\delta_i = 0$, $\gamma_i = 0$ for all i. From (3.74), we get

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_p\mathbf{x}_p + \beta_1\mathbf{y}_1 + \beta_2\mathbf{y}_2 + \ldots + \beta_q\mathbf{y}_q = 0.$$

Since the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_q$ are also linearly independent, it follows that $\alpha_i = 0$, $\beta_i = 0$ for all i. Therefore, the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_q, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_r\}$ forms a basis of $S = S_1 + S_2$.

We obtain
$$\dim(S_1 + S_2) = p + q + r = (p + q) + (p + r) - p$$
$$= \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2).$$

3.9.2 Direct Sum of Subspaces

Theorem 3.19 A vector space V is called the direct sum of its subspaces S_1 and S_2 if and only if (i) $V = S_1 + S_2$, and (ii) $S_1 \cap S_2 = \{\mathbf{0}\}$. The direct sum is denoted by $V = S_1 \oplus S_2$.

Proof

(i) By definition $S_1 + S_2 = \{\mathbf{s}_1 + \mathbf{s}_2 \mid \mathbf{s}_1 \in S_1, \mathbf{s}_2 \in S_2\}$. Therefore, every vector in V can be expressed in the form $\mathbf{s}_1 + \mathbf{s}_2$, if and only if $V = S_1 + S_2$.

(ii) Since S_1 and S_2 are subspaces of V, $\mathbf{0} \in S_1$ and $\mathbf{0} \in S_2$, that is $\mathbf{0} \in S_1 \cap S_2$. Let, there exist an arbitrary vector $\mathbf{x}$ such that $\mathbf{x} \in S_1 \cap S_2$. Then, $\mathbf{x} \in S_1$ and $\mathbf{x} \in S_2$. We can write the vector $\mathbf{x}$ in two ways as $\mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x}$, as $\mathbf{x} \in S_1$, $\mathbf{0} \in S_1$, and $\mathbf{0} \in S_2$, $\mathbf{x} \in S_2$. By (i), there is exactly one way to write $\mathbf{x}$ as the sum of a vector in S_1 and a vector in S_2. Hence, $\mathbf{x} = \mathbf{0}$ and $\mathbf{0} = \mathbf{x}$. Since $\mathbf{x}$ is arbitrary, we get $S_1 \cap S_2 = \{\mathbf{0}\}$.

Now, let $S_1 \cap S_2 = \{\mathbf{0}\}$. Let $\mathbf{s}_1 \in S_1$, $\mathbf{s}_1^* \in S_1$, and $\mathbf{s}_2 \in S_2$, $\mathbf{s}_2^* \in S_2$. Suppose that we can write an arbitrary vector $\mathbf{x}$ in V as $\mathbf{x} = \mathbf{s}_1 + \mathbf{s}_2$ and $\mathbf{x} = \mathbf{s}_1^* + \mathbf{s}_2^*$. Therefore, $\mathbf{s}_1 + \mathbf{s}_2 = \mathbf{s}_1^* + \mathbf{s}_2^*$, or $\mathbf{s}_1 - \mathbf{s}_1^* = \mathbf{s}_2^* - \mathbf{s}_2$. But, $\mathbf{s}_1 \in S_1$, $\mathbf{s}_1^* \in S_1$, and $\mathbf{s}_2 \in S_2$, $\mathbf{s}_2^* \in S_2$. That is, $(\mathbf{s}_1 - \mathbf{s}_1^*) \in S_1$ and $(\mathbf{s}_2^* - \mathbf{s}_2) \in S_2$. Therefore, $\mathbf{s}_1 - \mathbf{s}_1^* = \mathbf{s}_2^* - \mathbf{s}_2 \in S_1 \cap S_2$. Since, $S_1 \cap S_2 = \{\mathbf{0}\}$, we get $\mathbf{s}_1 = \mathbf{s}_1^*$ and $\mathbf{s}_2 = \mathbf{s}_2^*$. Therefore, each vector $\mathbf{x}$ in V can be uniquely written as $\mathbf{x} = \mathbf{s}_1 + \mathbf{s}_2$, where $\mathbf{s}_1 \in S_1$ and $\mathbf{s}_2 \in S_2$.

For example, let V be a vector space of real $n \times n$ matrices $\mathbf{A}_i$. Let S_1 be the subspace of all symmetric matrices $\mathbf{B}_i = [\{\mathbf{A}_i + (\mathbf{A}_i)^T\}/2]$ of order n, and S_2 be the subspace of all skew-symmetric matrices $\mathbf{C}_i = [\{\mathbf{A}_i - (\mathbf{A}_i)^T\}/2]$ of order n. Then, V is the direct sum of S_1 and S_2, that is $V = S_1 \oplus S_2$.

Remark 3.31

When a vector space V is the direct sum of its subspaces S_1 and S_2, then

$$\dim(S_1) + \dim(S_2) = \dim(V), \quad \text{and} \quad \text{basis}\,(S_1) + \text{basis}\,(S_2) = \text{basis}\,(V). \tag{3.75}$$

Remark 3.32

The vector space V is called the direct sum of its subspaces $S_1, S_2, \ldots, S_r$, if and only if

$$\text{(i) } V = S_1 + S_2 + \ldots + S_r \quad \text{and} \quad \text{(ii) } S_i \cap \left(\sum_{j \neq i} S_j\right) = \{\mathbf{0}\},\ i = 1, 2, \ldots, r. \tag{3.76}$$

It is written as $V = S_1 \oplus S_2 \oplus \ldots \oplus S_r$. Further,

$$\dim(V) = \sum_{i=1}^{r} \dim(S_i), \quad \text{and} \quad \text{basis}\,(V) = \sum_{i=1}^{r} \text{basis}\,(S_i). \tag{3.77}$$

Remark 3.33

A collection of subspaces $\{S_1, S_2, \ldots, S_r\}$ is independent if no non-zero vector from any S_i can be written as a linear combination of vectors from the other subspaces $S_1, S_2, \ldots, S_{i-1}, S_{i+1}, \ldots, S_r$.

Remark 3.34

A vector space V is the direct sum of its subspaces $S_1, S_2, \ldots, S_r$ if $V = S_1 + S_2 + \ldots + S_r$ and the subspaces $S_1, S_2, \ldots, S_r$ are independent.

Example 3.48 Let $V = R^3$, and

$$S_1 = \text{span}\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}, \text{ and } S_2 = \text{span}\left\{\begin{pmatrix}1\\3\\1\end{pmatrix}\right\}.$$

Show that $V = S_1 \oplus S_2$.

Solution It is easy to verify that the vectors $\mathbf{x} = [1, -1\ 1]^T$, $\mathbf{y} = [0, 1, 2]^T$, $\mathbf{z} = [1, 3, 1]^T$ are linearly independent (det[$\mathbf{x}, \mathbf{y}, \mathbf{z}$] $\neq 0$), and hence form a basis for R^3. Every vector in R^3 can be uniquely expressed as

$$\left[\alpha\begin{pmatrix}1\\-1\\1\end{pmatrix} + \beta\begin{pmatrix}0\\1\\2\end{pmatrix}\right] + \gamma\left[\begin{pmatrix}1\\3\\1\end{pmatrix}\right] = \mathbf{s}_1 + \mathbf{s}_2 \in S_1 + S_2.$$

Thus, $R^3 = S_1 + S_2$. Now, we shall show that the subspaces S_1, S_2 are independent. That is, no non-zero vector from S_1 or S_2 can be written as a linear combination of vectors from the other subspaces. Suppose that the subspaces are dependent. Then, we can write.

$$a\begin{pmatrix}1\\-1\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\2\end{pmatrix} = c\begin{pmatrix}1\\3\\1\end{pmatrix}, \text{ or } a = c, -a + b = 3c, a + 2b = c.$$

The solution of this system is $a = b = c = 0$. We conclude that the subspaces S_1, S_2 are independent, that is $S_1 \cap S_2 = \{\mathbf{0}\}$. Therefore, $R^3 = S_1 \oplus S_2$.

Example 3.49 Let $V = R^4$, and

$$S_1 = \text{span}\left\{\begin{pmatrix}1\\2\\3\\-1\end{pmatrix}\right\}, S_2 = \text{span}\left\{\begin{pmatrix}0\\1\\-1\\2\end{pmatrix}, \begin{pmatrix}1\\5\\1\\8\end{pmatrix}\right\}, \text{ and } S_3 = \text{span}\left\{\begin{pmatrix}1\\-7\\8\\3\end{pmatrix}\right\}.$$

Show that $V = S_1 \oplus S_2 \oplus S_3$.

Solution It is easy to verify that the vectors $\mathbf{x} = [1, 2, 3, -1]^T$, $\mathbf{y} = [0, 1, -1, 2]^T$, $\mathbf{z} = [1, 5, 1, 8]^T$, $\mathbf{w} = [1, -7, 8, 3]^T$, are linearly independent (det[$\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$] $\neq 0$), and hence form a basis for R^4. Every vector in R^4 can be uniquely written as

$$\alpha\begin{pmatrix}1\\2\\3\\-1\end{pmatrix} + \left[\beta\begin{pmatrix}0\\1\\-1\\2\end{pmatrix} + \gamma\begin{pmatrix}1\\5\\1\\8\end{pmatrix}\right] + \delta\begin{pmatrix}1\\-7\\8\\3\end{pmatrix} = \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3 \in S_1 + S_2 + S_3.$$

Thus, $R^4 = S_1 + S_2 + S_3$. Now, we shall show that the subspaces S_1, S_2, S_3 are independent, that is no non-zero vector from any S_i can be written as a linear combination of vectors from the other subspaces. Suppose that the subspaces are dependent. Let us try to express the vector in S_3, in terms of the vectors in S_1 and S_2. Then, we can write

$$a\begin{pmatrix}1\\2\\3\\-1\end{pmatrix} + \left[b\begin{bmatrix}0\\1\\-1\\2\end{bmatrix} + c\begin{pmatrix}1\\5\\1\\8\end{pmatrix}\right] = d\begin{pmatrix}1\\-7\\8\\3\end{pmatrix}.$$

that is $a + c = d$, $2a + b + 5c = -7d$, $3a - b + c = 8d$, $-a + 2b + 8c = 3d$. The system is inconsistent and the solution of the system is $a = b = c = d = 0$. Hence, $S_3 \cap (S_1 + S_2) = \{\mathbf{0}\}$. Similarly, we find that

$S_1 \cap (S_2 + S_3) = \{\mathbf{0}\}$. The vectors in S_2 cannot be written in terms of vectors in S_1 and S_3. We conclude that the subspaces S_1, S_2, S_3 are independent. Therefore, $V = S_1 \oplus S_2 \oplus S_3$.

3.9.3 Complementary Subspaces

If the vector space V is the direct sum of its subspaces S_1 and S_2, that is $V = S_1 \oplus S_2$, then S_1 and S_2 are called complementary subspaces of each other. We note that a vector space can have more than one pair of complementary subspaces, that is, a complement of a subspace may not be unique.

Consider the following examples:

(i) In R^2 : (a) x-axis and y-axis are complementary subspaces. (b) Any pair of straight lines through origin are also complementary subspaces.

(ii) In R^3 : (a) x-axis, y-axis and z-axis are complementary subspaces. (b) The complement of x-axis is yz-plane, or the complement of yz-plane is x-axis. (c) The complement of z-axis is xy-plane, or the complement of xy-plane is z-axis. (d) Any line through the origin and any plane containing the origin but not containing the above line are complementary subspaces.

(iii) The space of all real 2×2 matrices can be written as a direct sum as

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\} = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}\right\}$$

$$= S_1 \oplus S_2 \oplus S_3 \oplus S_4.$$

The subspaces S_1, S_2, S_3 and S_4 are complementary to each other.

Theorem 3.20 Every subspace of a vector space V has a complement.

Proof Let S_1 be a subspace of V and let B_1 be a basis of S_1. Extend this basis B_1 to a basis B of V. Define the subspace S_2 as $S_2 = \text{span}\,(B - B_1)$. Let $\mathbf{x} \in S_1$, that is $\mathbf{x}$ is a linear combination of vectors from B_1, and let $\mathbf{y} = \in S_2$, that is $\mathbf{y}$ is a linear combination of vectors from $(B - B_1)$. Suppose that $\mathbf{0} = \mathbf{x} + \mathbf{y}$. Since B is linearly independent, we obtain $\mathbf{x} = \mathbf{y} = \mathbf{0}$. Therefore, $S_1 + S_2$ is direct and $S_1 \cap S_2 = \{\mathbf{0}\}$. Since $B_1 \cup B$ generates V, we get $V = S_1 \oplus S_2$. Hence, S_2 is a complement of S_1.

Example 3.50 Consider the vector space $V = R^3$. Let $S = \{(x, y, z) \mid x + y + z = 0\}$. Determine the complement of S.

Solution It is easy to verify that the vectors $\mathbf{x} = [1, -2, 1]^T$, $\mathbf{y} = [1, 0, -1]^T$ form a basis to S, that is, $B_1 = \{\mathbf{x}, \mathbf{y}\}$. Extend the basis B_1 to a basis of R^3. Consider the standard basis of R^3, that is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Now, consider the extension of the basis B_1 as $B = \{\mathbf{x}, \mathbf{y}, \mathbf{e}_1\}$. If it is not a basis, then $\mathbf{e}_1$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$. Then, there exist non-zero scalars α and β such that $\alpha\mathbf{x} + \beta\mathbf{y} = \mathbf{e}_1$. Comparing the elements, we get $\alpha + \beta = 1, -2\alpha = 0$, and $\alpha - \beta = 0$. The system of equations is inconsistent, and hence $\mathbf{e}_1$ is not a linear combination of $\mathbf{x}$ and $\mathbf{y}$. We conclude that $B = \{\mathbf{x}, \mathbf{y}, \mathbf{e}_1\}$ is a basis. The size of B is $\dim(R^3) = 3$. We find that $\mathbf{e}_2 = -[(\mathbf{x} + \mathbf{y})/2] + \mathbf{e}_1$ and $\mathbf{e}_3 = -\mathbf{y} + \mathbf{e}_1$ are linear combinations of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{e}_1$. Therefore, B is a basis of $V = R^3$. The complementary subspace of S is given by

$$S = \text{span}\,(B - B_1) = \{\mathbf{e}_1\} = \{(\eta, 0, 0)\}, \text{ where } \eta \in R.$$

We may also obtain other complementary subspaces.

Now, consider the extension of the basis B_1 as $B = \{\mathbf{x}, \mathbf{y}, \mathbf{e}_2\}$. If it is not a basis, then $\mathbf{e}_2$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$. Then, there exists scalars α and β such that $\alpha\mathbf{x} + \beta\mathbf{y} = \mathbf{e}_2$. Comparing the elements, we get $\alpha + \beta = 0, -2\alpha = 1$, and $\alpha - \beta = 0$. The system of equations is inconsistent, and hence $\mathbf{e}_2$ is not a linear combination of $\mathbf{x}$ and $\mathbf{y}$. We conclude that $B = \{\mathbf{x}, \mathbf{y}, \mathbf{e}_2\}$ is a basis. The size of B is $\dim(R^3) = 3$. We find that $\mathbf{e}_1 = [(\mathbf{x} + \mathbf{y})/2] + \mathbf{e}_2$, and $\mathbf{e}_3 = [(\mathbf{x} - \mathbf{y})/2] + \mathbf{e}_2$ are linear combinations of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{e}_2$. Therefore, B is a basis of $V = R^3$. The complementary subspace of S is given by

$$S = \text{span}\,(B - B_1) = \{\mathbf{e}_2\} = \{(0, \eta, 0)\}, \text{ where } \eta \in R.$$

Similarly, we can show that $B = \{\mathbf{x}, \mathbf{y}, \mathbf{e}_3\}$ is a basis of $V = R^3$. In this case, the complementary subspace of S is given by

$$S = \text{span}(B - B_1) = \{\mathbf{e}_3\} = \{(0, 0, \eta)\}, \text{ where } \eta \in R.$$

This verifies the remark that a complement of a subspace may not be unique.

3.9.4 Inner Product Spaces and Gram-Schmidt Orthogonalization Process

Consider a real vector space V. An inner product denoted by $\langle\ ,\ \rangle$ on V satisfies the following properties:

(i) $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}\rangle = \alpha\langle \mathbf{x}, \mathbf{z}\rangle + \beta\langle \mathbf{y}, \mathbf{z}\rangle$.

(ii) $\langle \mathbf{z}, \alpha\mathbf{x} + \beta\mathbf{y}\rangle = \alpha\langle \mathbf{z}, \mathbf{x}\rangle + \beta\langle \mathbf{z}, \mathbf{y}\rangle$.

(iii) $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{y}, \mathbf{x}\rangle$.

(iv) $\langle \mathbf{x}, \mathbf{x}\rangle = \|\mathbf{x}\|^2 \geq 0, \mathbf{x} \neq 0$. $\langle \mathbf{x}, \mathbf{x}\rangle = 0$, if and only if $\mathbf{x} = \mathbf{0}$.

An *inner product space* is a real or a complex space together with a specified inner product on the space.

The following are some inner product spaces.

(i) Consider R^n. Let $\mathbf{u} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{w} = (y_1, y_2, \ldots, y_n)$. The *standard inner product* on R^n is defined by $\langle \mathbf{u}, \mathbf{w}\rangle = \sum_{i=1}^{n} x_i y_i$.

(ii) Let the vector space be $V = C\,[a, b]$. Then, the inner product of two functions f and g is defined by $\langle f, g\rangle = \int_a^b f(x)g(x)dx$.

We shall consider only the real inner product spaces and the standard inner product.

Remark 3.35

Every orthogonal set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ in an inner product space is linearly independent. If they are not linearly independent, then for some $a_i \neq 0$, $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_n\mathbf{x}_n = \mathbf{0}$. Taking the inner product on both sides with $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ successively and using the orthogonal property, that is, $\langle \mathbf{x}_i, \mathbf{x}_j\rangle = 0$, $i \neq j$ and $\langle \mathbf{x}_i, \mathbf{x}_i\rangle > 0$, for all i, we get $a_i = 0$ for all i. Hence, the given orthogonal set of vectors is linearly independent.

Given an arbitrary basis (a set of linearly independent vectors) in an inner product space, it is always possible to derive an orthonormal basis to the inner product space by using the *Gram-Schmidt orthogonalization* process.

Theorem 3.21 Let $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n\}$ be an arbitrary basis to an inner product space V. Then, there exists an orthonormal basis $\mathbf{Y} = \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n\}$ of V. The translation matrix $\mathbf{T}$ from $\mathbf{X}$ to $\mathbf{Y}$, $\mathbf{X} = \mathbf{TY}$, is a lower triangular matrix.

Proof We describe the Gram-Schmidt orthogonalization procedure as follows:

Define

$$\mathbf{u}_1 = \mathbf{x}_1, \quad \mathbf{y}_1 = \mathbf{u}_1/\|\mathbf{u}_1\|,$$

$$\mathbf{u}_2 = t_{21}\mathbf{u}_1 + \mathbf{x}_2, \quad \mathbf{y}_2 = \mathbf{u}_2/\|\mathbf{u}_2\|,$$

$$\mathbf{u}_3 = t_{31}\mathbf{u}_1 + t_{32}\mathbf{u}_2 + \mathbf{x}_3, \quad \mathbf{y}_3 = \mathbf{u}_3/\|\mathbf{u}_3\|,$$

$$\cdots\cdots\cdots$$

$$\mathbf{u}_n = t_{n1}\mathbf{u}_1 + t_{n2}\mathbf{u}_2 + t_{n3}\mathbf{u}_3 + \dots + t_{n,n-1}\mathbf{u}_{n-1} + \mathbf{x}_n, \quad \mathbf{y}_n = \mathbf{u}_n/\|\mathbf{u}_n\|,$$

where t_{ij}' s are scalars to be determined such that $\mathbf{u}_j$, $j = 1, 2, ..., n$ are mutually orthogonal.

Requiring $\mathbf{u}_2$ to be orthogonal to $\mathbf{u}_1$, we get

$$\langle \mathbf{u}_2, \mathbf{u}_1\rangle = t_{21}\langle \mathbf{u}_1, \mathbf{u}_1\rangle + \langle \mathbf{x}_2, \mathbf{u}_1\rangle = 0, \quad t_{21} = -\frac{\langle \mathbf{x}_2, \mathbf{u}_1\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle},$$

$$\mathbf{u}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{u}_1\rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{u}_1\rangle}{\|\mathbf{u}_1\|}\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \mathbf{x}_2 - \langle \mathbf{x}_2, \mathbf{y}_1\rangle\,\mathbf{y}_1.$$

$\mathbf{y}_2 = \mathbf{u}_2/\|\mathbf{u}_2\|$. Hence, $\mathbf{y}_1, \mathbf{y}_2$ are orthonormal vectors.

Requiring $\mathbf{u}_3$ to be orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$, we get

$$\langle \mathbf{u}_3, \mathbf{u}_1\rangle = t_{31}\langle \mathbf{u}_1, \mathbf{u}_1\rangle + t_{32}\langle \mathbf{u}_2, \mathbf{u}_1\rangle + \langle \mathbf{x}_3, \mathbf{u}_1\rangle = 0, \quad t_{31} = -\frac{\langle \mathbf{x}_3, \mathbf{u}_1\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle},$$

$$\langle \mathbf{u}_3, \mathbf{u}_2\rangle = t_{31}\langle \mathbf{u}_1, \mathbf{u}_2\rangle + t_{32}\langle \mathbf{u}_2, \mathbf{u}_2\rangle + \langle \mathbf{x}_3, \mathbf{u}_2\rangle = 0, \quad t_{32} = -\frac{\langle \mathbf{x}_3, \mathbf{u}_2\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle},$$

$$\mathbf{u}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{u}_1\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle \mathbf{x}_3, \mathbf{u}_2\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle}\mathbf{u}_2 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{u}_1\rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{\langle \mathbf{x}_3, \mathbf{u}_2\rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2$$

$$= \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{u}_1\rangle}{\|\mathbf{u}_1\|}\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} - \frac{\langle \mathbf{x}_3, \mathbf{u}_2\rangle}{\|\mathbf{u}_2\|}\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \mathbf{x}_3 - \langle \mathbf{x}_3, \mathbf{y}_1\rangle\mathbf{y}_1 - \langle \mathbf{x}_3, \mathbf{y}_2\rangle\mathbf{y}_2.$$

$\mathbf{y}_3 = \mathbf{u}_3/\|\mathbf{u}_3\|$. Hence, $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ are orthonormal vectors.

We prove the result by induction. Repeating the above procedure, we get

$$\mathbf{u}_{n-1} = \mathbf{x}_{n-1} - \langle \mathbf{x}_{n-1}, \mathbf{y}_1\rangle\mathbf{y}_1 - \langle \mathbf{x}_{n-1}, \mathbf{y}_2\rangle\mathbf{y}_2 - \dots - \langle \mathbf{x}_{n-1}, \mathbf{y}_{n-2}\rangle\mathbf{y}_{n-2}.$$

$\mathbf{y}_{n-1} = \mathbf{u}_{n-1}/\|\mathbf{u}_{n-1}\|$, where $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_{n-1}$ are orthonormal vectors.

Requiring $\mathbf{u}_n$ to be orthogonal to $\mathbf{u}_{n-1}, \mathbf{u}_{n-2}, ..., \mathbf{u}_1$ and simplifying, we get

$$\mathbf{u}_n = \mathbf{x}_n - \langle \mathbf{x}_n, \mathbf{y}_1\rangle\mathbf{y}_1 - \langle \mathbf{x}_n, \mathbf{y}_2\rangle\mathbf{y}_2 - \dots - \langle \mathbf{x}_n, \mathbf{y}_{n-1}\rangle\mathbf{y}_{n-1}.$$

$$\mathbf{y}_n = \mathbf{u}_n/\|\mathbf{u}_n\|.$$

Hence, $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n$ are orthonormal vectors. The set of orthonormal vectors $\mathbf{y} = \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n\}$ form a basis of V. The procedure of construction implies that the matrix of transformation is lower triangular.

Example 3.51 Using the Gram-Schmidt orthogonalization procedure, obtain an orthonormal basis for R^3 for the set of linearly independent vectors $\mathbf{x}_1 = (2, 2, 0)^T$, $\mathbf{x}_2 = (3, 0, 2)^T$, $\mathbf{x}_3 = (2, -2, 2)^T$.

Solution Since $\det(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \neq 0$, the given vectors are linearly independent. We obtain

$$\mathbf{u}_1 = \mathbf{x}_1 = (2, 2, 0)^T, \mathbf{y}_1 = (2, 2, 0)^T/\sqrt{8}, \langle \mathbf{x}_2, \mathbf{y}_1\rangle = 6/\sqrt{8},$$

$$\mathbf{u}_2 = \mathbf{x}_2 - \langle \mathbf{x}_2, \mathbf{y}_1\rangle\mathbf{y}_1 = (3, 0, 2)^T - (6/8)(2, 2, 0)^T = (3/2, -3/2, 2)^T.$$

$$\mathbf{y}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \sqrt{\frac{2}{17}}\left(\frac{3}{2}, -\frac{3}{2}, 2\right)^T, \langle \mathbf{x}_3, \mathbf{y}_1\rangle = 0, \langle \mathbf{x}_3, \mathbf{y}_2\rangle = 10\sqrt{\frac{2}{17}}.$$

$$\mathbf{u}_3 = \mathbf{x}_3 - \langle \mathbf{x}_3, \mathbf{y}_1\rangle\mathbf{y}_1 - \langle \mathbf{x}_3, \mathbf{y}_2\rangle\mathbf{y}_2$$

$$= (2, -2, 2)^T - \frac{20}{17}\left(\frac{3}{2}, -\frac{3}{2}, 2\right)^T = \frac{1}{17}(4, -4, -6)^T.$$

$$\mathbf{y}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \left(\sqrt{\frac{17}{2}}\right)\frac{2}{17}(2, -2, -3)^T = \frac{1}{\sqrt{17}}(2, -2, -3)^T.$$

The set $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ forms an orthonormal basis for R^3.

Exercise 3.5

1. In R^3, the subspaces S_1, S_2, S_3 whose elements are row vectors are defined as $S_1 = \{(a, b, c) | a + b + 2c = 0\}$, $S_2 = \{(a, b, c) | a = b\}$, $S_3 = \{(a, b, c) | a - 2c = 0, b + 4c = 0\}$, $a, b, c \in R$. Determine $S_1 \cap S_2$, $S_1 \cap S_3$ and $S_2 \cap S_3$.
2. Let $S_1 = \{(x, y, z) | 2x - y + z = 0\}$, $S_2 = \{(x, y, z) | x + 2y - z = 0\}$, be subspaces in R^3. Find a basis for $S_1 \cap S_2$.
3. Let $S_1 = \{(x, y, z) | 2x + y - z = 0\}$, $S_2 = \{(x, y, z) | x - y + 2z = 0\}$, be subspaces in R^3. Find $\dim(S_1 \cap S_2)$ and $\dim(S_1 + S_2)$.

In problems 4 to 6, test whether $R^3 = S_1 \oplus S_2$.

4. $S_1 = \{(a, b, 0) : a, b \in R\}$, $S_2 = \{(0, c, d) : c, d \in R\}$.
5. $S_1 = \{(a, b, 0) : a, b \in R\}$, $S_2 = \{(c, c, c) : c \in R\}$.
6. $S_1 = \{(a, 0, b) : a, b \in R\}$, $S_2 = \{(0, c, d) : c, d \in R\}$.
7. Test whether $R^4 = S_1 \oplus S_2 \oplus S_3$, where $S_1 = \text{span}[(1, 0, -1, 2)^T, (0, 1, 2, 3)^T]$, $S_2 = \text{span}[(1, 2, 3, 4)^T]$, $S_3 = \text{span}[(1, -2, 3, 2)^T]$.

In Problems 8 and 9, obtain the complementary subspaces of S_1 in R^3.

8. $S_1 = \{(x, y, z) | 2x - y + z = 0\}$. **9.** $S_1 = \{(x, y, z) | x + y + z = 0\}$.

Using the Gram-Schmidt orthogonalization procedure, obtain an orthonormal basis for R^3 for the set of linearly independent vectors given in problems 10 to 13.

10. $\mathbf{x}_1 = (1, -1, 0)^T, \mathbf{x}_2 = (0, 1, -1)^T, \mathbf{x}_3 = (0, 2, 1)^T$.

11. $\mathbf{x}_1 = (1, 1, 1)^T, \mathbf{x}_2 = (1, -1, 1)^T, \mathbf{x}_3 = (2, -4, -2)^T$.

12. $\mathbf{x}_1 = (0, 1, -1)^T, \mathbf{x}_2 = (-1, 0, -1)^T, \mathbf{x}_3 = (3, 1, 3)^T$

13. $\mathbf{x}_1 = (1, 0, 1)^T, \mathbf{x}_2 = (1, 1, 0)^T, \mathbf{x}_3 = (3, 2, 0)^T$.

Using the Gram-Schmidt orthogonalization procedure, obtain an orthonormal basis for R^4 for the set of linearly independent vectors given in problems 14 to 15.

14. $\mathbf{x}_1 = (1, 1, 0, 1)^T, \mathbf{x}_2 = (1, 1, 1, 1)^T, \mathbf{x}_3 = (4, 4, 1, 1)^T, \mathbf{x}_4 = (1, 0, 0, 1)^T$.

15. $\mathbf{x}_1 = (2, 1, 0, 1)^T, \mathbf{x}_2 = (1, 0, 1, 2)^T, \mathbf{x}_3 = (0, 2, 2, 1)^T, \mathbf{x}_4 = (1, 0, 2, 1)^T$.

3.10 Answers and Hints

Exercise 3.1

3. $\mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$ **4.** $\mathbf{A}^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

8. (i) $|\mathbf{A}\ adj\ (\mathbf{A})| = \text{diag}\ (|\mathbf{A}|, |\mathbf{A}|, \ldots |\mathbf{A}|) = |\mathbf{A}|^n$. Therefore, $|\ adj\ (\mathbf{A})| = |\mathbf{A}|^{n-1}$.

(ii) Let $\mathbf{B} = adj\ (\mathbf{A})$. Since $\mathbf{B}^{-1} = adj\ (\mathbf{B})/|\mathbf{B}|$, we have $\mathbf{B}\ adj\ (\mathbf{B}) = |\mathbf{B}|\ \mathbf{I}$. Therefore, $adj\ (\mathbf{A})\ adj\ (adj(\mathbf{A})) = |adj\ (\mathbf{A})|\ \mathbf{I} = |\mathbf{A}|^{n-1}\ \mathbf{I}$.

Pre-multiplying by $\mathbf{A}$ and using $adj\ (\mathbf{A}) = \mathbf{A}^{-1}\ |\mathbf{A}|$, we get

$\mathbf{A}[\mathbf{A}^{-1}\ |\mathbf{A}|]\ adj\ (adj\ (\mathbf{A})) = |\mathbf{A}|^{n-1}\ \mathbf{A}\mathbf{I}$ or $adj\ (adj\ (\mathbf{A})) = |\mathbf{A}|^{n-2}\ \mathbf{A}$.

9. $|\mathbf{A}\mathbf{A}^{-1}| = |\mathbf{A}||\mathbf{A}^{-1}| = |\mathbf{I}|$ or $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$.

10. $(\mathbf{B}\mathbf{A}\mathbf{B}^T)^T = \mathbf{B}\mathbf{A}^T\mathbf{B}^T = \mathbf{B}\mathbf{A}\mathbf{B}^T$.

13. $\mathbf{AB} = \mathbf{BA} \Rightarrow \mathbf{B}^{-1}\mathbf{AB} = \mathbf{A} \Rightarrow \mathbf{B}^{-1}\mathbf{A} = \mathbf{AB}^{-1}$. Similarly, $\mathbf{A}^{-1}\mathbf{B} = \mathbf{BA}^{-1}$.

(i) $(\mathbf{AB}^{-1})^T = (\mathbf{B}^{-1})^T\mathbf{A}^T = (\mathbf{B}^T)^{-1}\mathbf{A}^T = \mathbf{B}^{-1}\mathbf{A} = \mathbf{AB}^{-1}$.

(ii) $(\mathbf{A}^{-1}\mathbf{B})^T = \mathbf{B}^T(\mathbf{A}^{-1})^T = \mathbf{B}^T\ (\mathbf{A}^T)^{-1} = \mathbf{BA}^{-1} = \mathbf{A}^{-1}\mathbf{B}$.

(iii) $(\mathbf{A}^{-1}\mathbf{B}^{-1})^T = [(\mathbf{BA})^{-1}]^T = [(\mathbf{AB})^{-1}]^T = (\mathbf{A}^T)^{-1}\ (\mathbf{B}^T)^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$.

14. Pre-multiply both sides by (i) $\mathbf{I} - \mathbf{A}$, (ii) $\mathbf{I} + \mathbf{A}$.

15. $(\mathbf{PAQ})^{-1} = \mathbf{Q}^{-1}\mathbf{A}^{-1}\mathbf{P}^{-1} = \mathbf{I} \Rightarrow \mathbf{A}^{-1}\mathbf{P}^{-1} = \mathbf{Q} \Rightarrow \mathbf{A}^{-1} = \mathbf{QP}$.

16. Use $(\mathbf{I} - \mathbf{A})\ (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots) = \mathbf{I}$.

17. $(\mathbf{ABC})\ (\mathbf{ABC})^{-1} = \mathbf{I}$. Pre-multiply successively by $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$ and $\mathbf{C}^{-1}$.

18. 1, 2, 3 **19.** 1, 1, 1 **20.** 1, 1, 1

21. 1, 2, 1 **22.** (i) $k \neq 2$ and $k \neq -3$, (ii) $k = 2$, or $k = -3$.

23. $\theta = \pi/6$, or $\theta = \sin^{-1}[(9 - \sqrt{161})/4]$.

25. (i) $\lambda \neq 3$, μ arbitrary, (ii) $\lambda = 3$, $\mu = 10$, (iii) $\lambda = 3$, $\mu \neq 10$.

26. 2 **27.** 1. **28.** 2.

29. $|\mathbf{A}| = (p - q)\ (q - r)\ (r - p)\ (p + q + r)$; rank (A) is

(i) 3, if $p \neq q \neq r$ and $p + q + r \neq 0$;

(ii) 2, if $p \neq q \neq r$ and $p + q + r = 0$,

(iii) 2, if exactly two of p, q and r are identical;

(iv) 1, if $p = q = r$.

30. (a) 2; (b) $|\mathbf{A}| = (a_1a_2 + b_1b_2 + c_1c_2)^2$, rank (A) is

(i) 4, if $a_1a_2 + b_1b_2 + c_1c_2 \neq 0$;

(ii) 2, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$, since all determinants of third order have the value zero.

32. Consider $(\mathbf{I} + \mathbf{A})\ (\mathbf{I} - \mathbf{A} + \mathbf{A}^2 - \ldots + (-1)^{n-1}\ \mathbf{A}^{n-1}) = \mathbf{I} + (-1)^{n-1}\mathbf{A}^n$. In the limit $n \to \infty$, $\mathbf{A}^n \to 0$. Therefore, $(\mathbf{I} + \mathbf{A})\ (\mathbf{I} - \mathbf{A} + \mathbf{A}^2 - \ldots) = \mathbf{I}$.

33. (i) $Trace\ (\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha\sum_{i=1}^{n} a_{ii} + \beta\sum_{i=1}^{n} b_{ii} = \alpha\ Trace\ (\mathbf{A}) + \beta\ Trace\ (\mathbf{B})$.

(ii) $Trace\ (\mathbf{AB}) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{i=1}^{n}\sum_{m=1}^{n} b_{im}a_{mi} = Trace\ (\mathbf{BA})$.

(iii) If the result is true, then Trace (AB – BA) = Trace (I) which gives $0 = n$ which is not possible.

34. Result is true for $p = 0$ and 1. Let it be true for $p = k$ and show that it is true for $p = k + 1$. Note that when $\mathbf{BC} = \mathbf{CB}$ and $\mathbf{C}^2 = \mathbf{0}$, we have $\mathbf{CB}^{k+1} = \mathbf{B}^{k+1}\ \mathbf{C}$ and $\mathbf{CB}^k\mathbf{C} = \mathbf{0}$.

35. Apply the operation $C_1 \leftarrow C_1 + C_2 + \ldots + C_n$ and then the operation $R_i \leftarrow R_i - R_1$, $i = 2, 3, \ldots, n$.

36. None. **37.** Symmetric. **38.** Skew-symmetric.

39. Hermitian. **40.** None. **41.** Skew-Hermitian.

42. None. **43.** Skew–Hermitian. **44.** Hermitian.

45. None.

Exercise 3.2

1. Yes. **2.** No, 1, 4, 5, 6. **3.** No, 1, 4, 5, 6.

4. No, when the scalar α is irrational, Property 6 is not satisfied. If the field of scalars is taken only as rationals, then it defines a veactor space.

5. Yes, since $1 + \mathbf{x} = 1\mathbf{x} = \mathbf{x} = \mathbf{x}$ and $\mathbf{x} + 1 = \mathbf{x}1 = \mathbf{x} = \mathbf{x}$, the zero vector $\mathbf{0}$ is $1 = 1$. Define $-\mathbf{x} = 1/\mathbf{x}$. Then, $\mathbf{x} + (-\mathbf{x}) = \mathbf{x}(1/\mathbf{x}) = 1 = 1 = \mathbf{0}$. Therefore, negative vector is its reciprocal.

6. No, 8, 10. **7.** No, 2, 3, 8, 10.

8. Yes (same arguments as in Problem 5). $(\alpha + \beta)x = x^{\alpha+\beta} = x^\alpha x^\beta = x^\alpha + x^\beta = \alpha x + \beta x$.

9. (i) Yes, (ii) No, 1, 6.

10. (i) Yes, (ii) No, 1, 4, 6.

11. (i) Yes, (ii) No, when $\mathbf{x}, \mathbf{y} \in W$, $\mathbf{x} + \mathbf{y} \notin W$.

(iii) No, when $\mathbf{x}, \mathbf{y} \in W$, $\mathbf{x} + \mathbf{y} \notin W$. (iv) Yes.

12. (i) No, when $\mathbf{A} \in W$, $\alpha\mathbf{A} \notin W$ for α negative,

(ii) No, sum of two non-singular matrices need not be non-singular,

(iii) Yes,

(iv) No, $\alpha\mathbf{A}$ and $\mathbf{A} + \mathbf{B}$ need not belong to W, ($\mathbf{A} = \mathbf{I}$, $\mathbf{A}^2 = \mathbf{I} = \mathbf{A}$ but $2\mathbf{A} \neq (2\mathbf{A})^2$).

13. (i) Yes, (ii) No; let $\alpha = i$, Then, $\alpha\mathbf{A} = i\mathbf{A} \notin W$.

14. (i) No, for $P, Q \in W$, $P + Q \notin W$. (ii) Yes
(iii) No, for $P, Q \in W$, $\alpha P \notin W$ and also $P + Q \notin W$.
(iv) No, for $P, Q \in W$ having real roots, $P + Q$ need not have real roots. For example, take $P = 2t^2 - 1$, $Q = -t^2 + 3$.

15. (i) Yes,
(ii) No, $\mathbf{x}, \mathbf{y} \in W$, $\mathbf{x} + \mathbf{y} \notin W$. For example, if $\mathbf{x} = (x_1, x_1, x_1 - 1)$, $\mathbf{y} = (y_1, y_1, y_1 - 1)$, $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_1 + y_1, x_1 + y_1 - 2) \notin W$,
(iii) No, $\mathbf{x} \in W$, $\alpha \mathbf{x} \notin W$, for α negative,
(iv) No, $\mathbf{x} \in W$, $\alpha \mathbf{x} \notin W$. (v) No, $\mathbf{x} \in W$, $\alpha \mathbf{x} \notin W$, (for α a rational number)

16. (i) $\mathbf{u} + 2\mathbf{v} - \mathbf{w}$, (ii) $2\mathbf{u} + \mathbf{v} - \mathbf{w}$,
(iii) $(-33\mathbf{u} - 11\mathbf{v} + 23\mathbf{w})/16$.

17. (i) $\mathbf{u} - 2\mathbf{v} + 2\mathbf{w}$, (ii) $3\mathbf{u} + \mathbf{v} - \mathbf{w}$, (iii) not possible.

18. (i) $3P_1(t) - 2P_2(t) - P_3(t)$. (ii) $4P_1(t) - P_2(t) + 3P_3(t)$.

19. Let $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. Then, $\mathbf{x} = (a, b, c)^T = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$, where $\alpha = (a + b)/2$, $\beta = (a - b)/2$ and $\gamma = c$.

20. Let $S = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$. Then, $\mathbf{E} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C} + \delta\mathbf{D}$, where $\alpha = (-a - b + 2c - 2d)/3$, $\beta = (5a + 2b - 4c + 4d)/3$, $\gamma = (-4a - b + 5c - 2d)/3$ and $\delta = (-2a + b + c - d)/3$.

21. (i) independent, (ii) dependent, (iii) dependent,
(iv) independent, (v) dependent,

22. (i) independent, (ii) dependent, (iii) dependent,
(iv) independent, (v) independent,

24. $(-4, 7, 9) = (1, 2, 3) + 2(-1, 3, 4) - (3, 1, 2)$. The vectors in S are linearly dependent.

25. $t^2 + t + 1 = [-t + (t^2 - 1) + 2(t^2 + 2t + 2)]/3$. The elements in S are linearly independent.

26. (i) dimension: 2, a basis : $\{(1, 0, 0, -1), (0, 1, -1, 0)\}$,
(ii) dimension: 3, a basis: $\{(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1)\}$,
(iii) dimension: 3, a basis: $\{(1, 0, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$,
(iv) dimension: 1, a basis: $\{(1, 1, 1, 1)\}$.

27. **The given vectors must be linearly independent.**
(i) $k \neq 0, 1 - 4/3$, (ii) $k \neq 0$. (iii) $k \neq 0$, (iv) $k \neq -8$.

28. **(i) dimension: 4, basis:** $\{\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{21}, \mathbf{E}_{22}\}$ **where $\mathbf{E}_{rs}$ is the standard basis of order 2,**
(ii) dimension: 3, basis: $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$.
(iii) dimension: 1, basis: $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$,

(iv) a 2 × 2 skew-Hermitian matrix (diagonal elements are 0 or pure imaginary) is given by

$$\mathbf{A} = \begin{pmatrix} ia_1 & b_1 + ib_2 \\ -b_1 + ib_2 & ia_2 \end{pmatrix} = \begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix} + i\begin{pmatrix} a_1 & b_2 \\ b_2 & a_2 \end{pmatrix} = \mathbf{B} + i\mathbf{C}$$

where B is a skew-symmetric and C is a symmetric matrix,

dimension: 4, basis: $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$.

(v) dimension: 3, basis: $\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$.

(vi) dimension: 3, basis: $\left\{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

29. (i) dimension: 3, basis: $\{\mathbf{E}_{11}, \mathbf{E}_{22}, \mathbf{E}_{33}\}$,
(ii) dimension: 6, basis: $\{\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{13}\ \mathbf{E}_{22}, \mathbf{E}_{23}, \mathbf{E}_{33}\}$,
(iii) dimension: 6, basis: $\{\mathbf{E}_{11}, \mathbf{E}_{21}, \mathbf{E}_{22}\ \mathbf{E}_{31}, \mathbf{E}_{32}, \mathbf{E}_{33}\}$,
where $\mathbf{E}_{rs}$ is the standard basis of order 3.

30. (i) n^2, (ii) n, (iii) $n(n + 1)/2$, (iv) $n(n - 1)/2$.

31. Not linear, $T(\mathbf{x}) + T(\mathbf{y}) \neq T(\mathbf{x} + \mathbf{y})$. **32.** Linear.

33. Not linear, $T(\mathbf{x}) + T(\mathbf{y}) \neq T(\mathbf{x} + \mathbf{y})$.

34. Not linear, $T(1, 0) = 3$, $T(0, 1) = 2$, $T(1, 1) = 0 \neq T(1, 0) + T(0, 1)$.

35. Not linear, $T(\mathbf{x}) + T(\mathbf{y}) \neq T(\mathbf{x} + \mathbf{y})$.

36. $ker(T) = (0, 0, 0)^T$, $ran(T) = x(1, 0, 1)^T + y(1, 0, -1)^T + z(0, 1, 0)^T$.
$dim(ker(T)) = 0$, $dim(ran(T)) = 3$.

37. $ker(T) = (0, 0)^T$, $ran(T) = x(2, -1, 3)^T + y(1, 1, 4)^T$. $dim(ker(T)) = 0$, $dim(ran(T)) = 2$.

38. $ker(T) = w(1, -2, 0, 1)^T$,
$ran(T) = x(1, 0, 0)^T + y(1, 0, 1)^T + z(0, 1, 0)^T + w(1, 0, 2)^T$
$= r(1, 0, 0)^T + s(1, 0, 1)^T + z(0, 1, 0)^T$,
where $r = x - w$, $s = y + 2w$. $dim(ker(T)) = 1$, $dim(ran(T)) = 3$.

39. $ker(T) = x(-3, 1)^T$, $ran(T)$ = real number. $dim(ker(T)) = 1$, $dim(ran(T)) = 1$.

40. $ker(T) = x(1, -3, 0)^T + z(0, 0, 1)^T$, $ran(T)$ = real number. $dim(ker(T)) = 2$, $dim(ran(T)) = 1$.

41. $ker(T) = x(1, 1)^T$, $ran(T) = x(1, 1)^T - y(1, 1)^T = r(1, 1)^T$, where $r = x - y$.
$dim(ker(T)) = 1$, $dim(ran(T)) = 1$.

42. $ker(T) = x(1, 2, -3)^T$, $ran(T) = x(2, 3)^T + y(-1, 0)^T + z(0, 1)^T$ or $ran(T) = r(-1, 0)^T + s(0, 1)^T$, where $r = y + 2x$, $s = z + 3x$, $dim(ker(T)) = 1$, $dim(ran(T)) = 2$.

43. $\mathbf{A} = \begin{bmatrix} -5 & -8 & -7 \\ 3 & 5 & 4 \end{bmatrix}$.

44. $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & -1/2 \\ 1 & 1 & 1/2 \end{bmatrix}$.

45. $\mathbf{A} = \begin{bmatrix} -1/2 & -1/2 & -3/2 \\ -1/2 & -3/2 & -1/2 \\ 0 & -1 & -1 \end{bmatrix}$.

46. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

47. We have $T[\mathbf{v}_1, \mathbf{v}_2] = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]\,\mathbf{A} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}$.

Now, any vector $\mathbf{x} = (x_1, x_2)^T$ in $\mathbb{R}^2$ with respect to the given basis can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

We obtain $\alpha = (-4x_1 + 3x_2)/2$, $\beta = (2x_1 - x_2)/2$. Hence, we have

$$T\mathbf{x} = \alpha\, T\mathbf{v}_1 + \beta T\mathbf{v}_2 = \alpha \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4\alpha + 5\beta \\ 2\alpha + 3\beta \\ \beta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -6x_1 + 7x_2 \\ -2x_1 + 3x_2 \\ 2x_1 - x_2 \end{bmatrix}$$

48. $T\mathbf{x} = \begin{bmatrix} -x_1 + 2x_2 + 8x_3 \\ -2x_1 + 3x_2 + 12x_3 \end{bmatrix}$.

49. $TP_1(t) = (4x_2 - 5x_1) + 7(x_2 - x_1)t + (2x_1 - x_2)t^2$.

50. (i) Two degrees of freedom, dimension is 2, a basis is {[3, 1, 0], [−2, 0, 1]}.

(ii) One degree of freedom, dimension is 1, a basis is {(−5, 4, 23)}.

Exercise 3.3

1. 3. 2. 2. 3. 3. 4. 2.
5. 2. 6. 2. 7. 2. 8. 3.
9. 4. 10. 2. 11. 2. 12. 3.
13. 2. 14. 2. 15. 2.
16. Independent, 3. 17. Independent, 3. 18. Dependent, 3.
19. Independent, 3. 20. Dependent, 2. 21. Dependent, 3.
22. Dependent, 2. 23. Dependent, 2. 24. Independent, 4.
25. Dependent, 2. 26. [1, 2, 2].
27. $[1 + \alpha, -2\alpha, \alpha]$, α arbitrary. 28. Inconsistent.
29. [1, 1, 1]. 30. [1, 3, 3]. 31. [3/2, 3/2, 1]. 32. [−1, −1/2, 3/4].
33. $[(5 + \alpha - 4\beta)/3, (1 + 2\alpha + \beta)/3, \alpha, \beta]$, α, β arbitrary.

34. $[2 - \alpha, 1, \alpha, 1]$, α arbitrary. 35. [−1/4, 1/4, 1/4, 1/4].
36. $[-\alpha, \alpha, \alpha]$, α arbitrary. 37. $[-15\alpha, 13\alpha, \alpha]$, α arbitrary.
38. [0, 0, 0]. 39. $[-2\alpha/3, 7\alpha/3, -8\alpha/3, \alpha]$, α arbitrary.
40. $[2(\beta - \alpha)/3, -(5\beta + \alpha)/3, \beta, \alpha]$, α, β arbitrary.
41. [0, 0, 0, 0].
42. $[(2\beta - 5\alpha)/4, -(10\beta + \alpha)/4, \beta, \alpha]$, α, β arbitrary.
43. $[(\alpha + 5\beta)/3, (4\beta - 7\alpha)/3, \beta, \alpha]$, α, β arbitrary.
44. $[(3\beta - 5\alpha)/3, (3\beta - 4\alpha)/3, \beta, \alpha]$, α, β arbitrary.
45. $[\alpha - 3\beta, 5\beta, \beta, \alpha]$ α, β arbitrary.

46. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$.

47. $\begin{bmatrix} 3 & -5 & 6 \\ -2 & 4 & -5 \\ 1 & -2 & 3 \end{bmatrix}$.

48. $\frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

49. $\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

50. $\begin{bmatrix} -1 & -1/3 & 1/3 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & 1/3 & -1/3 & 0 \end{bmatrix}$.

Exercise 3.4

1. $P(\lambda) = \lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0$; $\mathbf{A}^{-1} = \frac{1}{81}\begin{bmatrix} 1 & 16 & -20 \\ 16 & 13 & 4 \\ -20 & 4 & -5 \end{bmatrix}$.

2. $P(\lambda) = \lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$; $\mathbf{A}^{-1} = \frac{1}{16}\begin{bmatrix} 6 & -4 & -2 \\ 0 & 8 & 0 \\ -2 & -4 & 6 \end{bmatrix}$

3. $P(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda = 0$; Inverse does not exist.

4. $P(\lambda) = \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$; $\mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 4 & -4 \\ -2 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix}$.

5. $P(\lambda) = \lambda^3 - 5\lambda^2 + 9\lambda - 13 = 0$; $\mathbf{A}^{-1} = \frac{1}{13}\begin{bmatrix} 2 & -3 & -7 \\ 1 & 5 & 3 \\ 5 & -1 & 2 \end{bmatrix}$.

6. $P(\lambda) = \lambda^3 - 3\lambda^2 + 6\lambda - 4 + 2i = 0$; $\mathbf{A}^{-1} = -\frac{1+3i}{10}\begin{bmatrix} i-1 & 1 & 1 \\ 1 & i-1 & 1 \\ 1 & 1 & i-1 \end{bmatrix}$

7. $\lambda = 1$: $(1, 1, -1)^T$; $\lambda = 2, 2$: $(2, 1, 0)^T$, not diagonalizable.

8. $\lambda = 1$: $(0, -1, 1)^T$, $\lambda = i$: $(1 + i, 1, 1)^T$, $\lambda = -i$: $(1 - i, 1, 1)^T$; diagonlizable

9. $\lambda = 1, 1, 1$: $[0, 3, -2]^T$; not diagonlizable

10. $\lambda = 1, 1$: $[0, 1, -1]^T$, $\lambda = 7$: $(6, 7, 5)^T$, not diagonalizable.

11. $\lambda = 0$: $[i, 0, -1]^T$; $\lambda = 1 + \sqrt{3}$: $[1, \sqrt{3} - 1, -i]^T$;
$\lambda = 1 - \sqrt{3}$: $[1, -(\sqrt{3} + 1), -i]^T$, diagonalizable

12. $\lambda = -i, -i$: $[1, 0, -1]^T$, $[1, -1, 0]^T$; $\lambda = 2i$: $[1, 1, 1]^T$, diagonalizable

13. $\lambda = 0, 0, 0, 0$: $[1, 0, 0, 0]^T$, not diagonlizable

14. $\lambda = 0, 0$: $[1, 0, 0, 1]^T$, $[1, -1, -1, 0]^T$, $\lambda = 2$: $[1, 1, 0, 0]^T$,
$\lambda = -2$: $[1, 0, 1, 1]^T$, diagonalizable.

15. $\lambda = -1, -1, -1$: $[1, -1, 0, 0]^T$, $[1, 0, -1, 0]^T$, $[1, 0, 0, -1]^T$,
$\lambda = 3$: $[1, 1, 1, 1]^T$; diagonalizable.

16. $\lambda = -4$: $[1, 1, -1, -1]^T$, $\lambda = 10$: $[1, 1, 1, 1]^T$, $\lambda = \sqrt{2}$: $[\sqrt{2} - 1, 1 - \sqrt{2}, -1, 1]^T$,
$\lambda = -\sqrt{2}$: $[-(1 + \sqrt{2}), 1 + \sqrt{2}, -1, 1]^T$; diagonalizable

17. $\lambda = -1, -1$: $[1, 0, 0, 0, -1]^T$, $[0, 1, 0, -1, 0]^T$; $\lambda = 1, 1, 1$: $[1, 0, 0, 0, 1]^T$, $[0, 1, 0, 1, 0]^T$,
$[0, 0, 1, 0, 0]^T$, diagonlizable

18. $\lambda = 1, w, w^2, w^3, w^4$, w is fifth root of unity. Let $\xi_j = w^j$, $j = 0, 1, 2, 3, 4$
$\lambda = \xi_j$: $[1, \xi_j, \xi_j^2, \xi_j^3, \xi_j^4]^T$, $j = 0, 1, 2, 3, 4$, diagonalizable

19. $\lambda = 2, 2$: $[1, 0, -1]^T$, $[-2, 1, 0]^T$; $\lambda = 4$: $[1, 0, 1]^T$.

$$\mathbf{P} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

20. $\lambda = 1$: $[1, -2, 0]^T$; $\lambda = -1$: $[3, -2, 2]^T$; $\lambda = 2$: $[-1, 3, 1]^T$.

$$\mathbf{P} = \begin{bmatrix} 1 & 3 & -1 \\ -2 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} -8 & -5 & 7 \\ 2 & 1 & -1 \\ -4 & -2 & 4 \end{bmatrix}.$$

21. $\lambda = 0$: $[3, 1, -2]^T$; $\lambda = 2i$: $[3 + i, 1 + 3i, -4]^T$; $\lambda = -2i$: $[3 - i, 1 - 3i, -4]^T$.

$$\mathbf{P} = \begin{bmatrix} 3 & 3+i & 3-i \\ 1 & 1+3i & 1-3i \\ -2 & -4 & -4 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{32}\begin{bmatrix} 24 & -8 & 16 \\ 2i-6 & 2-6i & -8 \\ -2i-6 & 2+6i & -8 \end{bmatrix}$$

22. $\lambda = 0$: $[1, 0, -1]^T$; $\lambda = 1$: $[-1, -1, 1]^T$; $\lambda = 2$: $[1, 1, 0]^T$.

$$\mathbf{P} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}; \mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

23. $\lambda = 1$: $[3, -1, 3]^T$; $\lambda = 2, 2$: $[2, 0, 1]^T$, $[2, 1, 0]^T$.

$$\mathbf{P} = \begin{bmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}; \mathbf{P}^{-1} = \begin{bmatrix} -1 & 2 & 2 \\ 3 & -6 & -5 \\ -1 & 3 & 2 \end{bmatrix}.$$

24. $\lambda = 1$: $[1, -1, -1]^T$; $\lambda = 2$: $[0, 1, 1]^T$; $\lambda = -2$: $[8 - 5, 7]^T$.

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 8 \\ -1 & 1 & -5 \\ -1 & 1 & 7 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{12}\begin{bmatrix} 12 & 8 & -8 \\ 12 & 15 & -3 \\ 0 & -1 & 1 \end{bmatrix}.$$

25. $$\mathbf{P} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 1 & 2 & 1 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

26. $$\mathbf{P} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} 6 & -5 & -7 \\ 1 & 0 & -1 \\ 3 & -3 & -4 \end{bmatrix}.$$

27. $$\mathbf{P} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 9 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{12}\begin{bmatrix} -11 & 14 & -5 \\ 14 & -8 & 2 \\ -5 & 2 & 1 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \frac{1}{12}\begin{bmatrix} 30 & -12 & 6 \\ 2 & 4 & 14 \\ -34 & 4 & 38 \end{bmatrix}.$$

28. $$\mathbf{P} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

29. $$\mathbf{P} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \frac{1}{3}\begin{bmatrix} 0 & 0 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}.$$

30. $$\mathbf{P} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}; \mathbf{P}^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 2 & -3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix}; \mathbf{A} = \mathbf{PDP}^{-1} = \frac{1}{8}\begin{bmatrix} 9 & 18 & 45 \\ 0 & 0 & 0 \\ 3 & 6 & 15 \end{bmatrix}.$$

31. **Eigenvalues of B are $2\lambda_j + (1/\lambda_j) - 1$, $j = 1, 2, 3, 4$ or $2, -4, 7/2, -11/2$. $|\mathbf{B}|$ = product of eigenvalues of B = 154.**

32. **Eigenvalues of B are $\lambda_j + \lambda_j^2 - (1/\lambda_j)$, $j = 1, 2, 3$, or $1, 11/2, 1$. Trace of B = sum of eigenvalues of B = 15/2.**

33. **Premultiply $\mathbf{Ax} = \lambda\mathbf{x}$ by $\mathbf{P}^{-1}$ and substitute $\mathbf{x} = \mathbf{Py}$.**

34. Let λ be an eigenvalue and $\mathbf{x}$ be the corresponding eigenvector of $\mathbf{AB}$, that is $\mathbf{ABx} = \lambda\mathbf{x}$. Pre-multiply by $\mathbf{A}^{-1}$ and substitute $\mathbf{x} = \mathbf{Ay}$. We get $\mathbf{BAy} = \lambda\mathbf{y}$. Therefore, λ is also an eigenvalue of $\mathbf{BA}$ and eigenvectors are related by $\mathbf{x} = \mathbf{Ay}$.

35. Let λ be an eigenvalue and $\mathbf{x}$ be the corresponding eigenvector of $\mathbf{A}^{-1}\mathbf{B}$, that is $\mathbf{A}^{-1}\mathbf{Bx} = \lambda\mathbf{x}$. Premultiply by $\mathbf{A}$ and set $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$. We obtain $\mathbf{BA}^{-1}\mathbf{y} = \lambda\mathbf{y}$. Therefore, λ is also an eigenvalue of $\mathbf{BA}^{-1}$ with the corresponding eigenvector $\mathbf{y} = \mathbf{Ax}$.

36. From $\mathbf{Ax} = \lambda\mathbf{x}$, we obtain $\mathbf{A}^k\mathbf{x} = \lambda^k\mathbf{x} = \mathbf{0}$. Therefore, $\lambda^k = 0$ or $\lambda = 0$, since $\mathbf{x} \neq \mathbf{0}$.

37. Since $\mathbf{A}$ is a diagonalizable matrix, there exists a non-singular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ and the eigenvalues of $\mathbf{A}$ and $\mathbf{D}$ are same. We have $\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} = \mathbf{D}^2$. Since $\mathbf{A}^2 = \mathbf{A}$, we get $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}^2$. Therefore, we obtain $\mathbf{D}^2 - \mathbf{D} = \mathbf{0}$. Thus $\mathbf{D} = \mathbf{0}$ or $\mathbf{D} = \mathbf{I}$. Hence, the eigenvalues of $\mathbf{A}$ are 0 or 1.

38. Simplify the right hand side and set the off-diagonal element to zero.

39. Since $\mathbf{A}$ and $\mathbf{B}$ are similar, we have $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$. From this equation, show that $\mathbf{A}^{-1} = \mathbf{P}^{-1}\mathbf{B}^{-1}\mathbf{P}$ and $\mathbf{A}^m = \mathbf{P}^{-1}\mathbf{B}^m\mathbf{P}$. Also $|\mathbf{A}| = |\mathbf{P}^{-1}||\mathbf{B}||\mathbf{P}| = |\mathbf{B}|$.

40. We have $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{B} = \mathbf{B}^T$. Therefore, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T = \mathbf{BA}$.

41. $(\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T\mathbf{A}$.

42. Let $\mathbf{A}^T\mathbf{A}^{-1}$ be a symmetric matrix. We have $(\mathbf{A}^T\mathbf{A}^{-1})^T = (\mathbf{A}^{-1})^T\mathbf{A} = \mathbf{A}^T\mathbf{A}^{-1}$, or $(\mathbf{A}^{-1})^T\mathbf{A}^2 = \mathbf{A}^T$ or $\mathbf{A}^2 = (\mathbf{A}^T)^2$. Now, let $\mathbf{A}^2 = (\mathbf{A}^T)^2$. We have $\mathbf{AA} = \mathbf{A}^T\mathbf{A}^T \Rightarrow \mathbf{A} = \mathbf{A}^{-1}\mathbf{A}^T\mathbf{A}^T \Rightarrow \mathbf{A}(\mathbf{A}^T)^{-1} = \mathbf{A}^{-1}\mathbf{A}^T$, or $\mathbf{A}(\mathbf{A}^{-1})^T = (\mathbf{A}^{-1}\mathbf{A}^T)^T = \mathbf{A}^{-1}\mathbf{A}^T$. Therefore, $\mathbf{A}^T\mathbf{A}^{-1}$ is symmetric.

43. Since $\mathbf{A}$ is symmetric, we have
$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{-1}\mathbf{A}^T = (\mathbf{PDP}^{-1})^{-1}(\mathbf{PDP}^{-1})^T = (\mathbf{PD}^{-1}\mathbf{P}^{-1})[(\mathbf{P}^{-1})^T\mathbf{DP}^T]$, since $\mathbf{D}^T = \mathbf{D}$. This result is true only when $\mathbf{P}^{-1}(\mathbf{P}^{-1})^T = \mathbf{I}$, or $\mathbf{P}^{-1} = \mathbf{P}^T$.

44. Let $\mathbf{A}$ and $\mathbf{B}$ be the orthogonal matrices, that is $\mathbf{A}^{-1} = \mathbf{A}^T$ and $\mathbf{B}^{-1} = \mathbf{B}^T$. Then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T = \mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$.

45. $\mathbf{A}^{-1} = \mathbf{A}^T$ gives $\mathbf{AA}^T = \mathbf{I}$. We obtain conditions as $l_i^2 + m_i^2 + n_i^2 = 1$, $i = 1, 2, 3$ and $l_1l_2 + m_1m_2 + n_1n_2 = 0$, $l_1l_3 + m_1m_3 + n_1n_3 = 0$, $l_2l_3 + m_2m_3 + n_2n_3 = 0$.

46. Since $\mathbf{A}$ is an orthogonal matrix, we have $\mathbf{A}^{-1} = \mathbf{A}^T$. Hence, $|\mathbf{A}^{-1}| = |\mathbf{A}^T| = |\mathbf{A}|$ or $1/|\mathbf{A}| = |\mathbf{A}| \Rightarrow |\mathbf{A}|^2 = 1$ or $|\mathbf{A}| = \pm 1$.

47. Let λ and μ be two distinct eigenvalues and $\mathbf{x}$, $\mathbf{y}$ be the corresponding eigenvectors. We have $\mathbf{Ax} = \lambda\mathbf{x}$ and $\mathbf{Ay} = \mu\mathbf{y}$. From the first equation, we get $\mathbf{x}^T\mathbf{A}^T = \lambda\mathbf{x}^T$ or $\mathbf{x}^T\mathbf{A} = \lambda\mathbf{x}^T$. Postmultiplying by $\mathbf{y}$, we obtain $\mathbf{x}^T\mathbf{Ay} = \lambda\mathbf{x}^T\mathbf{y}$. From the second equation, we get $\mathbf{x}^T\mathbf{Ay} = \mu\mathbf{x}^T\mathbf{y}$. Subtracting the two results, we obtain $(\lambda - \mu)\mathbf{x}^T\mathbf{y} = 0$, which gives $\mathbf{x}^T\mathbf{y} = 0$ since $\lambda \neq \mu$.

49. There exists an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. Now, $\mathbf{A} = \mathbf{PDP}^{-1} = \mathbf{PDP}^T$, since $\mathbf{P}$ is orthogonal. We have $\mathbf{A}^T = (\mathbf{PDP}^T)^T = \mathbf{PD}^T\mathbf{P}^T = \mathbf{PDP}^T = \mathbf{A}$, since a diagonal matrix is always symmetric.

51. Let $\mathbf{z} = \mathbf{U} + i\mathbf{V}$, where $\mathbf{U} \neq \mathbf{0}$, $\mathbf{V} \neq \mathbf{0}$ be real vectors. Then
$\bar{\mathbf{z}}^T\mathbf{Az} = (\mathbf{U}^T\mathbf{AU} + \mathbf{V}^T\mathbf{AV}) + i(\mathbf{U}^T\mathbf{AV} - \mathbf{V}^T\mathbf{AU}) = \mathbf{U}^T\mathbf{AU} + \mathbf{V}^T\mathbf{AV} > 0$
since $\mathbf{U}^T\mathbf{AV} = (\mathbf{U}^T\mathbf{AV})^T = \mathbf{V}^T\mathbf{A}^T\mathbf{U} = \mathbf{V}^T\mathbf{AU}$.

52. Let the vectors $\mathbf{a}$, $\mathbf{b}$ be transformed to vectors $\mathbf{u}$, $\mathbf{v}$ respectively. Then
$(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} = \bar{\mathbf{u}}^T \cdot \mathbf{v} = (\bar{\mathbf{A}}\bar{\mathbf{a}})^T(\mathbf{Ab}) = \bar{\mathbf{a}}^T\bar{\mathbf{A}}^T\mathbf{Ab} = \bar{\mathbf{a}}^T\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$.

53. (i) No. (ii) Yes. (interchange rows 2 and 3 followed by interchange of columns 2 and 3). $U_1 = \{1, 3\}$, $U_2 = \{2, 4\}$.

54. $\mathbf{x}^T\mathbf{Ax} = [x_1, x_2]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2$

$= a[(x_1 + bx_2/a)^2 + x_2^2(ac - b^2)/a^2] > 0$, for all x_1, x_2.
Therefore, $a > 0$, $ac - b^2 > 0$.

55. $\mathbf{x}^T\mathbf{Ax} = [x_1, x_2, x_3]\begin{bmatrix} 2 & 1 & 3 \\ -3 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= 2x_1^2 - 2x_1x_2 + 2x_1x_3 + 4x_2^2 + 2x_3^2 = (x_1 - x_2)^2 + (x_1 + x_3)^2 + 3x_2^2 + x_3^2 > 0$.

56. All the leading minors are not positive. It can also be verified that all the eigenvalues are not positive.

57. $\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ 58. $\begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 4 \\ -2 & 4 & 0 \end{bmatrix}$. 59. $\begin{bmatrix} 1 & i & -4 \\ -i & 0 & 2i \\ -4 & -2i & 4 \end{bmatrix}$.

60. $\begin{bmatrix} 1 & -1-2i & 0 \\ -1+2i & 1 & -2+3i \\ 0 & -2-3i & 0 \end{bmatrix}$. 61. $\begin{bmatrix} 2 & 3+4i & 0 \\ 3-4i & -3 & 2-i \\ 0 & 2+i & 0 \end{bmatrix}$ 62. $y_1^2 + 3y_2^2 - 5y_3^2$.

63. $y_1^2 + 3y_2^2 - 2y_3^2$. 64. $y_1^2 + 2y_2^2 + 4y_3^2$. 65. $8y_1^2 + 2y_2^2 + 2y_3^2$.

66. 6.1713. 67. 5.59. 68. 2.

69. 5.312.

70. $\mathbf{P} = \begin{bmatrix} 0.19795 & -0.98021 \\ 0.98022 & 0.19795 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 6.45101 & 0 \\ 0 & 0.62006 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 0.94237 & 0.33458 \\ 0.33458 & -0.94237 \end{bmatrix}$.

71. $\mathbf{P} = \begin{bmatrix} 0.52573 & 0.87065 \\ -0.87065 & -0.52573 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 3.61803 & 0 \\ 0 & 1.38197 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} -0.08981 & 0.99596 \\ 0.99596 & 0.08981 \end{bmatrix}$.

72. $\mathbf{P} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$.

73. $\mathbf{P} = \mathbf{Q}^T$, $\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} -2/3 & -2/3 & 1/3 \\ 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}$.

74. $\mathbf{P} = \mathbf{Q}^T, \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

75. $\mathbf{P} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 1/\sqrt{14} & 3/\sqrt{14} & 2/\sqrt{14} \\ -3/\sqrt{10} & 1/\sqrt{10} & 0 \\ 1/\sqrt{35} & 3/\sqrt{35} & -5/\sqrt{35} \end{bmatrix}$

Exercise 3.5

1. $\{(a, b, c) \mid a = b = -c\}$, $\{(a, b, c) \mid a - 2c = 0, b + 4c = 0\}$, $\{(0, 0, 0)\}$.

2. $\{(-1, 3, 5)^T\}$.

3. Basis of $(S_1 \cap S_2) = \{(-1, 5, 3)^T\}$; 1, 3.

4. $R^3 = S_1 + S_2$. $(S_1 \cap S_s) = \{(0, \alpha, 0)^T\}$. Not a direct sum.

5. $(S_1 \cap S_2) = \{(0, 0, 0)^T\}$. $R^3 = S_1 \oplus S_2$.

6. $R^3 = S_1 + S_2$. $(S_1 \cap S_2) = \{(0, 0, \alpha)^T\}$. Not a direct sum.

7. $S_i \cap \left(\sum_{j \neq i} S_j\right) = \{\mathbf{0}\}$. $R^4 = S_1 \oplus S_2 \oplus S_3$.

8. A basis of S_1 : $\{(\mathbf{x}, \mathbf{y})\} = \{(1, 1, -1), (0, 1, 1)\}$. A basis of R^3 : $\{(\mathbf{x}, \mathbf{y}, \mathbf{e}_1)\}$. Complemetary subspace is S = span $\{\mathbf{e}_1\}$ = span $\{(\alpha, 0, 0)\}$, α arbitrary. We can also take S = span $\{\mathbf{e}_2\}$, or S = span$\{\mathbf{e}_3\}$ as complementary subspaces.

9. A basis S_1 : $\{(\mathbf{x}, \mathbf{y})\} = \{(1, 0, -1), (0, -1, 1)\}$. A basis of R^3 : $\{(\mathbf{x}, \mathbf{y}, \mathbf{e}_1)\}$. Complementary subspace is S = span$\{\mathbf{e}_1\}$ = span $\{(\alpha, 0, 0)\}$, α arbitrary. We can also take S = span$\{\mathbf{e}_2\}$, or S = span$\{\mathbf{e}_3\}$ as complementary subspaces.

10. $(1/\sqrt{2})(1, -1, 0)^T)$, $(1/\sqrt{6})(1, 1, -2)^T$, $(1/\sqrt{3})(1, 1, 1)^T$.

11. $(1/\sqrt{3})(1, 1, 1)^T$, $(1/\sqrt{6})(1, -2, 1)^T$, $(1/\sqrt{2})(1, 0, -1)^T$.

12. $(1/\sqrt{2})(0, 1, -1)^T$, $(1/\sqrt{6})(-2, -1, -1)^T$, $(1/\sqrt{3})(-1, 1, 1)^T$

13. $(1/\sqrt{2})(1, 0, 1)^T$, $(1/\sqrt{6})(1, 2, -1)^T$, $(1/\sqrt{3})(1, -1, -1)^T$.

14. $(1/\sqrt{3})(1, 1, 0, 1)^T$, $(0, 0, 1, 0)^T$, $(1/\sqrt{6})(1, 1, 0, -2)^T$, $(1/\sqrt{2})(1, -1, 0, 0)^T$.

15. $(1/\sqrt{6})(2, 1, 0, 1)^T$, $(1/\sqrt{30})(-1, -2, 3, 4)^T$, $(1/\sqrt{630})(-8, 19, 14, -3)^T$, $(1/\sqrt{21294})(65, -52, 91, -78)^T$.

(ii) $\begin{pmatrix} 3i \\ 4i \\ 0 \end{pmatrix}, \begin{pmatrix} -4i \\ 3i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1+i \end{pmatrix}$ form an orthogonal set in $\mathbb{C}^3$ and $\begin{pmatrix} 3i/5 \\ 4i/5 \\ 0 \end{pmatrix}, \begin{pmatrix} -4i/5 \\ 3i/5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ (1+i)/\sqrt{2} \end{pmatrix}$

form an orthonormal set in $\mathbb{C}^3$.

Orthonormal and unitary system of vectors Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n vectors in $\mathbb{R}^n$. Then, this set of vectors forms an *orthonormal system* of vectors, if

$$\mathbf{x}_i \cdot \mathbf{x}_j = \mathbf{x}_i^T \mathbf{x}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n vectors in $\mathbb{C}^n$. Then, this set of vectors forms an *unitary system* of vectors, if

$$\mathbf{x}_i \cdot \mathbf{x}_j = \mathbf{x}_i^T \overline{\mathbf{x}}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

In section 3.2.2, we have defined symmetric, skew-symmetric, Hermitian and skew-Hermitian matrices. We now define a few more special matrices.

Orthogonal matrices A real matrix $\mathbf{A}$ is *orthogonal* if $\mathbf{A}^{-1} = \mathbf{A}^T$. A simple example is

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

A linear transformation in which the matrix of transformation is an orthogonal matrix is called an *orthogonal transformation*.

Unitary matrices A complex matrix $\mathbf{A}$ is *unitary* if $\mathbf{A}^{-1} = (\overline{\mathbf{A}})^T$, or $(\overline{\mathbf{A}})^{-1} = \mathbf{A}^T$. If $\mathbf{A}$ is real, then unitary matrix is same as orthogonal matrix.

A linear transformation in which the matrix of transformation is a unitary matrix is called a *unitary tranformation*.

We note the following:

1. If $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices, then $\alpha\mathbf{A} + \beta\mathbf{B}$ is also Hermitian for any real scalars α, β, since
$$\left(\overline{\alpha\mathbf{A} + \beta\mathbf{B}}\right)^T = (\alpha\overline{\mathbf{A}} + \beta\overline{\mathbf{B}})^T = \alpha\overline{\mathbf{A}}^T + \beta\overline{\mathbf{B}}^T = \alpha\mathbf{A} + \beta\mathbf{B}.$$
2. Eigenvalues and eigenvectors of $\overline{\mathbf{A}}$ are the conjugates of the eigenvalues and eigenvectors of $\mathbf{A}$, since
$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \text{ gives } \overline{\mathbf{A}}\,\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}.$$
3. The inverse of a unitary (orthogonal) matrix is unitary (orthogonal). We have $\mathbf{A}^{-1} = \overline{\mathbf{A}}^T$. Let $\mathbf{B} = \mathbf{A}^{-1}$. Then
$$\mathbf{B}^{-1} = \mathbf{A} = (\overline{\mathbf{A}}^T)^{-1} = [(\overline{\mathbf{A}})^{-1}]^T = [\overline{(\mathbf{A}^{-1})}]^T = \overline{\mathbf{B}}^T.$$

Diagonally dominant matrix A matrix $\mathbf{A} = (a_{ij})$ is said to be diagonally dominant, if

$$|a_{ii}| \geq \sum_{j=1, i \neq j}^{n} |a_{ij}|, \text{ for all } i.$$

The system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, is called a *diagonally dominant system*, if the above conditions are satisfied and the strict inequality is satisfied for at least one i. If the strict inequality is satisfied for all i, then it is called a *strictly diagonally dominant system*.

Permutation matrix A matrix $\mathbf{P}$ is called a *permutation matrix* if it has exactly one 1 in each row and column and all other elements are 0.

***Property A* of a matrix** Let $\mathbf{B}$ be a sparse matrix. Then, the matrix $\mathbf{B}$ is said to satisfy the *property A*, if and only if there exists a permutation matrix $\mathbf{P}$ such that

$$\mathbf{P}\mathbf{B}\mathbf{P}^T = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix},$$

where $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are diagonal matrices. The similarity transformation performs row interchanges followed by corresponding column interchanges in $\mathbf{B}$ such that $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ become diagonal matrices. The following procedure is a simple way of testing whether $\mathbf{B}$ can be reduced to the required form. It finds the locations of the non-zero elements and tests whether the interchanges of rows and corresponding interchanges of columns are possible to bring $\mathbf{B}$ to the required form. Let n be the order of the matrix $\mathbf{B}$ and $b_{ii} \neq 0$. Denote the set $U = \{1, 2, 3, \ldots, n\}$. Let there exist disjoint subsets U_1 and U_2 such that $U = U_1 \cup U_2$, where the suffixes of the non-zero off diagonal elements $b_{ik} \neq 0$, $i \neq k$, can be grouped as either $(i \in U_1, k \in U_2)$ or $(i \in U_2, k \in U_1)$. Then, the matrix $\mathbf{B}$ satisfies *property A*.

Consider, for example the matrix $\mathbf{B} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$.

Let the permutation matrix be taken as $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Then,

$$\mathbf{P}\mathbf{B}\mathbf{P}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \begin{matrix} -2 & 0 \\ 0 & -2 \end{matrix} & \begin{matrix} [1] \\ [1] \end{matrix} \\ \begin{matrix} [1 & 1] \end{matrix} & (-2) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$